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Testing for Unit Roots in Time Series Data

Sastry G. Pantula
 North Carolina State University
 Raleigh, NC 27695-8203

ABSTRACT

Let Y_t satisfy the stochastic difference equation $Y_t = \sum_{j=1}^p \alpha_j Y_{t-j} + e_t$, for $t = 1, 2, \dots$, where e_t are independent and identically distributed random variables and the initial conditions (Y_{-p+1}, \dots, Y_0) are fixed constants. It is assumed that the true, but unknown, roots m_1, m_2, \dots, m_p of $m^p - \sum_{j=1}^p \alpha_j m^{p-j} = 0$ satisfy the hypothesis $H_d: m_1 = \dots = m_d = 1$ and $|m_j| < 1$ for $j = d+1, \dots, p$. We present a reparameterization of the model for Y_t that is convenient for testing the hypothesis H_d . We consider the asymptotic properties of (i) a likelihood ratio type "F-statistic" for testing the hypothesis H_d , (ii) a least squares regression t-statistic, and (iii) a likelihood ratio type t-statistic for testing the hypothesis H_d against the alternative H_{d-1} . Using these asymptotic results, we obtain two sequential testing procedures that are asymptotically consistent. Extensions to autoregressive moving average processes are also presented.

Key Words and Phrases: Asymptotic Distributions, Differencing, Least Squares, Reparameterizations

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1. Introduction: Autoregressive integrated moving average (ARIMA) models are used to represent economic, meteorological and physical processes. The hypothesis regarding the number of unit roots in the model has important consequences for the interpretation of economic data. Methods for detecting the presence of unit roots and their applications have appeared in various journals. For example, Altonji and Ashenfelter (1980) use the unit root tests to investigate an equilibrium hypothesis for wage movements; Meese and Singleton (1983) apply the unit root tests to exchange rates and discuss the importance of unit root testing in the theory of linearized expectations; Nelson and Plosser (1982) discuss the relationship of unit roots to the effect that monetary disturbances have on macroeconomic series; and Schwert (1987) analyses several macroeconomic variables and compares different methods for detecting the presence of a single unit root. Schwert (1987) also considers ARIMA models with two unit roots for the variables: Consumer Price Index, Gross National product, Monetary Base and Total Civilian Noninstitutional Population. (All of the variables are transformed using natural logarithm.) Models with more than two unit roots, though very rare in economics, have some applications in physical sciences. Box and Pallezen (1978) analysed the x-coordinate of a missile trajectory using an ARIMA model (ARIMA (0,3,5)) with three unit roots.

In this paper we present methods for testing the hypothesis that the process has a specified number of unit roots. We begin the discussion with pure autoregressive processes.

Let the time series $\{Y_t\}$ satisfy

$$Y_t = \sum_{j=1}^p \alpha_j Y_{t-j} + e_t, \quad t = 1, 2, \dots, \quad (1.1)$$

where $\{e_t\}_{t=1}^{\infty}$ is a sequence of iid random variables with mean zero and variance σ^2 . It is assumed that $E[|e_t|^{4+\delta}] < M$ for some $\delta > 0$. It is further assumed that the initial conditions (Y_{-p+1}, \dots, Y_0) are known constants. The time series is said to be an autoregressive process of order p . Let

$$m^p - \sum_{j=1}^p \alpha_j m^{p-j} = 0 \quad (1.2)$$

be the characteristic equation of the process. The roots of (1.2), denoted by m_1, \dots, m_p are called the characteristic roots of the process. Assume that $1 \geq |m_1| \geq |m_2| \geq \dots \geq |m_p|$.

Let the observations Y_1, Y_2, \dots, Y_n be available. It is assumed that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)'$ and σ^2 are unknown. The least squares estimator $\hat{\alpha}$ of α is obtained by regressing Y_t on Y_{t-1}, \dots, Y_{t-p} . Lai and Wei (1983) established the strong consistency of $\hat{\alpha}$ under a very general set of conditions, and Chan and Wei (1986) obtained the asymptotic distribution of $\hat{\alpha}$. If $|m_j| < 1, j = 1, \dots, p$, then Y_t converges to a weakly stationary process as $t \rightarrow \infty$. In the stationary case, the asymptotic properties of $\hat{\alpha}$ and of the related likelihood ratio type "F-statistic" are well known. See Mann and Wald (1943), Anderson (1971) and Hannan and Heyde (1972).

Dickey and Fuller (1979) and Fuller (1979) have considered testing for a single unit root in a p -th order autoregressive process (i.e., $m_1 = 1$ and $|m_j| < 1, j=2, \dots, p$). Phillips (1987) and Phillips and Perron (1986) present alternative methods for testing a single unit root in a general time series

setting. Their approach is nonparametric with respect to nuisance parameters and hence allows for a wide class of ARIMA models with a single unit root. Hasza and Fuller (1979) considered a p -th order autoregressive process with $m_1 = m_2 = 1$ and $|m_j| < 1$ for $j = 3, 4, \dots, p$. They characterized the asymptotic distribution of the likelihood ratio type "F-statistic" for testing $m_1 = m_2 = 1$. Yajima (1985) proposed a method of estimating the number of unit roots using $\hat{\sigma}_n^2(k, d)$, k large and for reasonable choices of d , where $\hat{\sigma}_n^2(k, d)$ is the residual mean square error obtained by fitting a k th order autoregression to the d th difference of the data. For a second order process, Sen (1985) studied the asymptotic distribution of the regression t -statistic for testing $H_1: m_1 = 1$ and $|m_2| < 1$, under the assumption $H_2: m_1 = m_2 = 1$. He observed empirically that the probability of the Dickey and Fuller (1979) criterion rejecting the hypothesis H_1 , in favor of the hypothesis $H_0: |m_1| < 1$ and $|m_2| < 1$, is higher under H_2 than under H_1 . That is, it is more likely to conclude incorrectly that the process is stationary when there are really two unit roots present than when there is just one unit root present. Pantula (1985) observed empirically that the probability of rejecting H_1 increases with the number of unit roots present. It is not clear how the modified procedure of Phillips (1987) behaves under the hypothesis H_d , $d > 1$.

In this paper we (i) present a reparameterization of the model (1.1) that is convenient for testing the hypothesis $H_d: m_1 = \dots = m_d = 1$ and $|m_j| < 1$, $j = d+1, \dots, p$, (ii) characterize the asymptotic distribution of the likelihood ratio type "F-statistic," the regression t -statistic and the likelihood ratio type t -statistic under the hypothesis H_d for $d = 0, 1, \dots, p$, and (iii) present two asymptotically consistent sequential procedures to test H_d versus H_{d-1} .

The procedures are such that, asymptotically, the chance of rejecting the hypothesis H_d in favor of H_{d-1} is smaller than the specified level (α) when the process has d or more unit roots and is equal to one when the process has less than d unit roots. The procedures presented here successfully answer the question "how many times should one difference the series to attain stationarity?" A proof of the main results is presented in Appendix A. Extensions of the procedures to autoregressive moving average processes is included in Appendix B.

2. Notation and the Main Results: Consider the following reparameterization of the model (1.1),

$$Y_{p,t} = \sum_{i=1}^p \beta_i Y_{i-1,t-1} + e_t \quad (2.1)$$

where $Y_{i,t} = (1-B)^i Y_t = i$ th difference of Y_t ; and B is the back shift operator.

Comparing (1.1) with (2.1) it is easy to show that the vector of parameters $\alpha = (\alpha_1, \dots, \alpha_p)'$ is linearly related to the parameters $\beta = (\beta_1, \dots, \beta_p)'$ and the relation is given by

$$\alpha = T \beta + c \quad (2.2)$$

where T is a $p \times p$ nonsingular upper triangular matrix with $T_{ij} = (-1)^{i-1} \binom{j-1}{i-1}$ for $j \geq i$ and $c = (c_1, \dots, c_p)'$ with $c_i = (-1)^{i-1} \binom{p}{i}$. We now show that the hypothesis $H_d: m_1 = \dots = m_d = 1$ can be tested using a test for $\beta_1 = \dots = \beta_d = 0$ in the model (2.1).

Lemma 1: Let Y_t satisfy the model (1.1). Then, $m_1 = \dots = m_d = 1$ if and only if $\beta_1 = \dots = \beta_d = 0$.

Proof: In terms of the characteristic roots, the p th order autoregressive model (1.1) can be written as

$$\pi_{i=1}^p [(1-m_i) + m_i(1-B)]Y_t = e_t \quad (2.3)$$

Also, from (2.1) we have

$$[(1-B)^p - \sum_{i=1}^p \beta_i (1-B)^{i-1} B]Y_t = e_t,$$

which can also be written as

$$[(1-B)^p (1+\beta_p) - \sum_{i=1}^{p-1} (1-B)^i (\beta_i - \beta_{i+1}) - \beta_1]Y_t = e_t \quad (2.4)$$

Now, comparing (2.3) with (2.4) we get

$$\beta_1 = -\pi_{i=1}^p (1-m_i),$$

$$\beta_k = \beta_{k-1} - \sum_{i_1 \dots i_{k-1}} m_{i_1} \dots m_{i_{k-1}} (1-m_{j_1}) \dots (1-m_{j_{p-k+1}}),$$

and

$$j = 2, \dots, p-1$$

$$\beta_p = \pi_{i=1}^p m_i - 1, \quad (2.5)$$

where the summation is over all possible indices $i_{k-1} = \{i_1, \dots, i_{k-1}\}$ and $\{j_1, \dots, j_{p-k+1}\} = \{1, \dots, p\} - i_{k-1}$. Note that if $m_1 = \dots = m_d = 1$, then $\beta_1 = \dots = \beta_d = 0$ and $\beta_{d+1} = -\pi_{i=d+1}^p (1-m_i)$. We prove the converse by induction. Since $1 \geq |m_1| \geq \dots \geq |m_p|$ and $\beta_1 = -\pi_{i=1}^p (1-m_i)$, we have that $\beta_1 = 0$ implies $m_1 = 1$. Suppose $\beta_1 = \dots = \beta_d = 0$ implies that $m_1 = \dots = m_d = 1$. Now, suppose $\beta_1 = \dots = \beta_{d+1} = 0$. Then, clearly $m_1 = \dots = m_d = 1$, $\beta_{d+1} = -\pi_{i=d+1}^p (1-m_i) = 0$ and hence $m_{d+1} = 1$. []

Note that the additional assumption $|m_j| < 1$ for $j = d+1, \dots, p$ in the hypothesis H_d imposes extra conditions on β_j . In particular,

$$\beta_{d+1} = -\pi_{i=d+1}^p (1-m_i) < 0.$$

Let $\hat{\beta}$ denote the ordinary least squares estimator of β obtained by regressing $Y_{p,t}$ on $Y_{0,t-1}, Y_{1,t-1}, \dots, Y_{p-1,t-1}$. That is

$$\hat{\beta} = (\Phi' \Phi)^{-1} \Phi' Y_{(p)} \quad (2.6)$$

where

$$\Phi_t = (Y_{0,t-1}, \dots, Y_{p-1,t-1})'$$

$$\Phi = (\Phi_1, \Phi_2, \dots, \Phi_n)'$$

and

$$Y_{(p)} = (Y_{p,1}, Y_{p,2}, \dots, Y_{p,n})'$$

Define the regression "F-statistic" for testing $\beta_1 = \beta_2 = \dots = \beta_i = 0$ by

$$F_{i,n}(p) = (i \hat{\sigma}_n^2)^{-1} \hat{\beta}'_{(i)} C_{(i)}^{-1} \hat{\beta}_{(i)} \quad (2.7)$$

where $C_{(i)}$ is the $(i \times i)$ submatrix consisting of the first i rows and i columns of

$(\Phi' \Phi)^{-1}$, $\hat{\beta}_{(i)} = (\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_i)'$; and

$$\hat{\sigma}_n^2 = (n-p)^{-1} \sum_{t=1}^n [Y_{p,t} - \Phi_t' \hat{\beta}_{(p)}]^2$$

Define also the regression t-statistic for testing the hypothesis $\beta_i = 0$ by

$$t_{i,n}(p) = \left[\hat{\sigma}_n^2 \hat{\sigma}_{p,n}^{ii} \right]^{-1/2} \hat{\beta}_i \quad (2.8)$$

where $\hat{\sigma}_{p,n}^{ii}$ is the (i,i) th element of $(\Phi' \Phi)^{-1}$. Following Lemma 1, we use the

"F-statistic" $F_{i,n}(p)$ to test the hypothesis H_i . We now present a lemma that

will be useful in obtaining the asymptotic distribution of $F_{i,n}(p)$ and $t_{i,n}(p)$.

For the statistics considered in this paper, without loss of generality, we

assume that $Y_{-p+1} = \dots = Y_0 = 0$ and $\sigma^2 = 1$. We use the notation " \xrightarrow{D} " to

indicate convergence in distribution and the notation " \xrightarrow{P} " to indicate

convergence in probability of a statistic as the sample size tends to infinity.

Lemma 2: Define,

$$S_{0,0} = 0$$

$$S_{0,t} = e_t$$

$$(1-B)S_{i,t} = S_{i-1,t}, \text{ for } i = 1, 2, \dots, d,$$

where e_t is a sequence of iid random variables with $E[|e_t|^{4+\delta}] < M < \infty$ for some $\delta > 0$. Then, as n tends to infinity,

$$n^{-i} \sum_{t=1}^n S_{i,t} S_{0,t+1} \xrightarrow{D} \xi_i = \int_0^1 H_{i-1}(t) dH_0(t)$$

and

$$n^{-(i+j)} \sum_{t=1}^n S_{i,t} S_{j,t} \xrightarrow{D} \sigma_{ij} = \int_0^1 H_{i-1}(t) H_{j-1}(t) dt$$

where $H_0(t)$ is a standard Brownian motion and

$$H_i(t) = \int_0^t H_{i-1}(s) ds .$$

Proof: See Chan and Wei (1986) . []

Pantula (1985) obtained an alternate representation, involving quadratic forms, for the limiting distributions of Lemma 2. (See also Yajima (1985).) The representations are similar to those of the W variables presented in Hasza and Fuller (1979) and are useful in obtaining the percentiles of the limiting distribution.

We now present the joint limiting distributions of the "F-statistic" $F_{i,n}(p)$ and the regression t-statistic $t_{i,n}(p)$, under the hypothesis H_d .

Theorem 1: Assume that Y_t is defined by (1.1). Then, under the hypothesis H_d ,

$$F_{i,n}(p) \xrightarrow{D} \begin{cases} F_i(d) & \text{if } i \leq d \\ \infty & \text{if } i > d \end{cases} \quad (2.9)$$

$$t_{i,n}(p) \xrightarrow{D} \begin{cases} t_i(d) & \text{if } i \leq d \end{cases} \quad (2.10)$$

and

$$n^{-1/2} t_{i,n}(p) \xrightarrow{P} f \beta_i \quad \text{if } i > d \quad (2.11)$$

where f is a finite positive constant,

$$F_i(d) = i^{-1} [d F_d(d) - (d-i) F_{d-i}(d-i)],$$

$$F_k(k) = k^{-1} \xi_k \Sigma_k^{-1} \xi_k,$$

$$t_i(d) = \bar{\mu}_{d-i+1,d} \sigma_d^{d-i+1,d-i+1}^{-1/2},$$

$$\bar{\mu}_{i,d} = \text{ith element of } \Sigma_d^{-1} \xi_d,$$

$$\sigma_d^{ii} = (i,i)\text{th element of } \Sigma_d^{-1},$$

$$\xi_k = (\xi_1, \xi_2, \dots, \xi_k)',$$

$$\Sigma_k = ((\sigma_{ij}))_{k \times k}, \quad k = 1, \dots, d,$$

and ξ_i and σ_{ij} as defined in Lemma 2.

Proof: See Appendix A. []

Notice that the limiting distribution of $F_{i,n}(p)$ is independent of the order p . It depends only on i (the number of unit roots being tested) and d (the true number of unit roots present). Pantula (1985) used the quadratic form type representations to obtain the empirical percentiles of $F_i(d)$. Let

$F_i(d, \alpha)$ denote the $100(1-\alpha)\%$ percentile of the distribution of $F_i(d)$. Pantula (1985) obtained $F_i(d, \alpha)$ for $1 \leq i \leq d \leq 5$ and $\alpha = 0.5, 0.2, 0.1, 0.05, 0.025$ and 0.01 . Extending the criterion suggested by Hasza and Fuller (1979) we may reject the hypothesis H_d of exactly d unit roots (in favor of the hypothesis H_{d-1} of exactly $d-1$ unit roots) if $F_{d,n}(p) > F_d(d, \alpha)$. Pantula (1985) observed that the empirical percentiles $F_i(d, \alpha)$ increase in d . That is, $F_d(d, \alpha) < F_d(d+1, \alpha) < F_d(d+2, \alpha) < \dots$. Therefore, the probability of rejecting H_d (in favor of H_{d-1}) increases with the number of unit roots ($\geq d$) present. For example, if we use the criterion of rejecting the hypothesis H_1 that there is exactly one unit root (in favor of H_0 : stationary process) whenever $F_{1,n}(p) > F_1(1, \alpha)$, then the higher the number of unit roots present, the more likely we are to conclude that the process is stationary. Intuitively, however, we hope that our criterion should strongly indicate nonstationarity when more than one unit root is present.

Suppose we have a prior knowledge that there are at most $s \leq p$ unit roots present. (If no such prior knowledge is available then take $s = p$.) In practice $s \leq 3$. We now suggest a sequential criterion to test the hypothesis H_d for $d \leq s$. The criterion is: "Reject the hypothesis H_d (in favor of H_{d-1}) if $F_{i,n}(p) > F_i(i, \alpha)$ for $i = d, d+1, \dots, s$." Since under the hypothesis H_d , $F_{i,n}(p)$ converges in probability to infinity for $i > d$ we get

$$\lim_{n \rightarrow \infty} P_{H_j} [\text{Rejecting } H_d] = \begin{matrix} \alpha & , & j=d \\ \leq \alpha & , & j>d \\ 1 & , & j<d \end{matrix} .$$

That is, if we use our sequential procedure, asymptotically, the chance that we reject the hypothesis that there are exactly d unit roots is (i) α , if there are exactly d unit roots, (ii) less than or equal to α , if there are more than

d unit roots and (iii) one, if there are less than d unit roots. Thus, for example, if we use the sequential procedure, the chance of concluding that the process is stationary is smaller than α when there is one or more unit roots present. In this sense, it is advantageous to use the F-statistics sequentially rather than individually.

Let us now interpret the results from the hypothesis testing problem in the context of estimating $0 \leq d_0 \leq s$, the unknown number of unit roots that are actually present. Suppose for a given α we use the estimator \hat{d}_α which takes the value d if $F_{d,n}(p) \leq F_d(d,\alpha)$ and $F_{i,n}(p) > F_i(i,\alpha)$ for $d < i \leq s$ and takes the value zero if $F_{i,n}(p) > F_i(i,\alpha)$ for $1 \leq i \leq s$. Then our results imply that, for $d_0 > 0$,

$$\lim_{n \rightarrow \infty} P_{H_{d_0}}[\hat{d}_\alpha = d] = \begin{cases} 1-\alpha & , \quad d = d_0 \\ \leq \alpha & , \quad d < d_0 \\ 0 & , \quad d > d_0 \end{cases}$$

and for $d_0 = 0$, $P_{H_0}[\hat{d}_\alpha = 0]$ converges to one. In this sense, our procedure is asymptotically consistent. Pantula (1986) compared the power of the sequential procedure with that of the traditional methods and concluded that the loss in power is minimal even for samples of size 25.

Note that the asymptotic distribution of $t_{i,n}(p)$ also depends on the number of unit roots that are actually present. A natural question is to see whether the t-statistic $t_{i,n}(p)$ can be used in a sequential manner to test the relevant hypotheses. Consider for example an AR(3) process. As mentioned earlier, Sen (1985) observed that the Dickey and Fuller (1979) criterion based on $t_{1,n}(3)$ rejects the hypothesis H_1 of exactly one unit root in favor of the hypothesis H_0 of no unit roots more often under H_2 and H_3 than under H_1 . So we cannot hope to obtain a consistent procedure by first testing $\beta_1 = 0$ (for a single unit

root), then testing $\beta_2 = 0$ (for two unit roots) and then testing $\beta_3 = 0$. Also the sequential procedure that first tests $\beta_3 = 0$ using $t_{3,n}(3)$ and then tests $\beta_2 = 0$ using $t_{2,n}(3)$ will not be consistent. This is because it is possible to have $\beta_2 \geq 0$ even though the process is stationary. For example, if $m_1 = m_2 = m_3 = -0.5$, then the process is stationary with $\beta_2 = 0$ and the t-statistic $t_{2,n}(3)$ converges in distribution to $N(0,1)$. Also, if $m_1 = m_2 = m_3 = -0.75$, then the process is stationary with $\beta_2 = 0.5(1.75)^2$ and the t-statistic $t_{2,n}(3)$ diverges to positive infinity.

Now consider the problem of testing the hypothesis $H_d: \beta_1 = \beta_2 = \dots = \beta_d = 0$ against the alternative $H_{d-1}: \beta_1 = \beta_2 = \dots = \beta_{d-1} = 0, \beta_d < 0$. Note that under the null (H_d) and the alternative (H_{d-1}) hypotheses we have $\beta_1 = \dots = \beta_{d-1} = 0$ and the alternative hypothesis is one sided in nature ($\beta_d < 0$). This suggests that a more powerful (likelihood ratio type) test for testing the hypothesis H_d against the alternative H_{d-1} is obtained by regressing $Y_{p,t}$ on $Y_{d-1,t-1}, \dots, Y_{p-1,t-1}$ (i.e., set $\beta_1 = \dots = \beta_{d-1} = 0$ in (2.1)). Let $t_{d,n}^*(p)$ denote the regression t-statistic for testing the coefficient of $Y_{d-1,t-1}$ is zero in the regression of $Y_{p,t}$ on $Y_{d-1,t-1}, \dots, Y_{p-1,t-1}$. Note that $t_{1,n}^*(p) = t_{1,n}(p)$.

We now obtain the asymptotic distribution of the likelihood ratio type t-statistic $t_{i,n}^*(p)$ under the hypothesis H_d .

Theorem 2: Suppose the process Y_t satisfies the conditions of Theorem 1.

Then, under the hypothesis H_d ,

$$t_{i,n}^*(p) \xrightarrow{D} \begin{cases} \hat{\tau} & , i=d \\ t_i^*(d) & , i < d \\ L = -\infty & , i > d \end{cases} .$$

where

$$\hat{\tau} = t_1(1) = \xi_1 \sigma_{11}^{-1/2}$$

$$t_i^*(d) = \bar{\mu}_{d-i+1, d-i+1} \left[\sigma_{d-i+1}^{d-i+1, d-i+1} \right]^{-1/2}$$

and ξ_1 , σ_{11} , $\bar{\mu}_{i,d}$ and σ_d^{ii} are as defined in Theorem 1.

Proof: See Appendix A. []

Note that for any d , under the hypothesis H_d , the asymptotic distribution of the t-statistic $t_{d,n}^*(p)$ is the distribution of $\hat{\tau}$ obtained by Dickey and Fuller (1979) for the single unit root case. Therefore, we do not have to tabulate different percentiles for $t_d^*(d)$ as we have done for the F-statistic. We now suggest a sequential procedure based on the t-statistic $t_{d,n}^*(p)$. For a given α , the second procedure is: "Reject H_d in favor of H_{d-1} if $t_{i,n}^*(p) < \hat{\tau}_\alpha$ for $i=d, d+1, \dots, s$, where $\hat{\tau}_\alpha$ is the lower α percentile of the $\hat{\tau}$ distribution given in Table 8.5.2 of Fuller (1976)." Also, let \tilde{d}_α be an estimator of d_0 (the number of unit roots actually present) which takes the value d if $t_{d,n}^*(p) \geq \hat{\tau}_\alpha$ and $t_{i,n}^*(p) < \hat{\tau}_\alpha$, $i = d+1, \dots, s$ and takes the value zero when $t_{i,n}^*(p) < \hat{\tau}_\alpha$ for $i = 1, \dots, s$.

From Theorem 2 it follows that

$$\lim_{n \rightarrow \infty} P_{H_j} [\text{Rejecting } H_d] = \begin{cases} \alpha & , j=d \\ \leq \alpha & , j>d \\ 1 & , j<d \end{cases} .$$

Also, for $d_0 > 0$,

$$\lim_{n \rightarrow \infty} P_{H_{d_0}} [\tilde{d}_\alpha = d] = \begin{cases} 1-\alpha & , d=d_0 \\ \leq \alpha & , d<d_0 \\ 0 & , d>d_0 \end{cases} .$$

and for $d_0 = 0$, $P_{H_0}[\tilde{d}_\alpha = 0]$ converges to one.

The sequential procedure based on the t-statistic $t_{d,n}^*(p)$ has several advantages over the sequential procedure based on the F-statistic $F_{d,n}(p)$. The sequential t^* -procedure requires only one set of tables given in Fuller (1976). Also, the t^* -procedure makes use of the one sided nature of the alternatives and hence is expected to be more powerful than the F-procedure. On the other hand, the F-procedure is a two-tailed test and hence is expected to be robust in the presence of mildly explosive roots. Of course, the t^* -procedure can be modified to a two-tailed procedure by considering $t_{d,n}^{*2}(p)$ and comparing it with the percentiles of $\hat{\tau}^2 = F_1(1)$. Using simulations, Dickey and Pantula (1987) compared the t^* -procedure and the F-procedure and recommend the use of the t^* -procedure. Dickey and Pantula (1987) also present a numerical example (Real Estate Loans) of a third order autoregressive process with two unit roots.

3. Concluding Remarks: We have given here a reparameterization of an autoregressive model that is convenient for testing the number of unit roots. We have derived the asymptotic distributions of the regression t-statistic $t_{i,n}(p)$ and the likelihood ratio type F-statistic, $F_{i,n}(p)$, under various hypotheses.

We have indicated why the t-statistic $t_{i,n}(p)$ may not be useful in testing or estimating the number of unit roots present. We have obtained a sequential procedure based on the F-statistic $F_{i,n}(p)$ for testing the hypothesis H_d and estimating d_0 the actual number of unit roots present. We also obtained the asymptotic distribution of the likelihood ratio type t-statistic $t_{d,n}^*(p)$ for testing the hypothesis H_d against the alternative hypothesis H_{d-1} . We

developed a sequential procedure based on the t^* -statistics that is also asymptotically consistent. The t^* -procedure takes into account the one-sided nature of the alternatives and hence is superior to the F-procedure.

We have presented the results for an autoregressive process with mean zero. Our procedure can be extended to autoregressive processes where an intercept or a time trend is included. The percentiles for the F-statistic when an intercept is included in the regression are given in Pantula (1985). (See also Phillips and Perron (1986).) Also, the assumption of e_t being a sequence of independent random variables can be weakened to e_t being a sequence of martingale differences as in Chan and Wei (1986) and Phillips (1987). In Appendix B, following the approach of Said and Dickey (1984), the sequential procedures are extended to autoregressive moving average processes.

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Appendix A

We prove Theorem 1 in three steps: (i) we consider a d th order process with d unit roots and obtain the limiting representation $F_i(d)$ of the "F-statistic" $F_{i,n}(d)$ for $i=1,2,\dots,d$; (ii) for a p th order process with exactly d unit roots and remaining $p-d$ roots inside the unit circle we will show that $F_{i,n}(p)$ converges in distribution to $F_i(d)$ for $i = 1,2,\dots,d$, and (iii) diverges to infinity for $i > d$. The results for the t -statistic $t_{i,n}(p)$ follow in the process. We will use the following fact from regression theory repeatedly: The F (or t) - statistic for testing a linear hypothesis involving the parameters of a linear model is identical to the F (or t) - statistic of an equivalent hypothesis under a (linearly) reparameterized model.

Consider a d th order autoregressive process with d unit roots given by

$$(1-B)^d W_t = e_t, \quad (\text{A.1})$$

where $\{e_t\}$ is a sequence of iid random variables given in Lemma 2. Notice that $W_t = S_{d,t}$ where $S_{d,t}$ is defined in Lemma 2. In the parameterization of (2.1) we can write W_t as

$$W_{d,t} = \sum_{i=1}^d \beta_i W_{i-1,t-1} + e_t \quad (\text{A.2})$$

where

$$W_{i,t} = (1-B)^i W_t = S_{d-i,t},$$

and

$$\beta_1 = \beta_2 = \dots = \beta_d = 0.$$

Note that we can rewrite (A.2) in terms of $S_{i,t}$ as

$$S_{0,t} = \sum_{i=1}^d \mu_i S_{i,t-1} + e_t \quad (\text{A.3})$$

where $\mu_i = \beta_{d-i+1}$.

Therefore, the hypothesis $\beta_1 = \dots = \beta_i = 0$ is the same as $\mu_{d-i+1} = \dots = \mu_d = 0$. Consider the regression of $S_{0,t}$ on $S_{1,t-1}, S_{2,t-1}, \dots, S_{d,t-1}$. Then, the regression F-statistic $F_{i,n}(d)$ for testing exactly $i (\leq d)$ unit roots in a d th order process is given by

$$F_{i,n}(d) = [i \hat{\sigma}_n^2(d)]^{-1} [R_n(\mu_1, \dots, \mu_d) - R_n(\mu_1, \dots, \mu_{d-i})] \quad (\text{A.4})$$

where $R_n(\mu_1, \dots, \mu_i)$ denotes the regression sums of squares for the regression of $S_{0,t}$ on $S_{1,t-1}, \dots, S_{i,t-1}$ and $\hat{\sigma}_n^2(d)$ denotes the mean square error in the regression of $S_{0,t}$ on $S_{1,t-1}, \dots, S_{d,t-1}$. Let $t_{d-i+1,n}(d)$ denote the regression t-statistic for testing $\mu_i = 0$. Note that, for $k = 1, \dots, d$,

$$\hat{\mu}_d = G_{d,n}^{-1} \mathbf{g}_{d,n} = (\hat{\mu}_1, \dots, \hat{\mu}_d)',$$

$$t_{d-i+1,n}(d) = [\hat{\sigma}_n^2(d) G_{d,n}^{ii}]^{-1/2} \hat{\mu}_i,$$

$$R_n(\mu_1, \dots, \mu_k) = \mathbf{g}'_{k,n} G_{k,n}^{-1} \mathbf{g}_{k,n},$$

where $G_{k,n} = ((G_{ij,k,n}))$ is a $k \times k$ matrix; $\mathbf{g}_{k,n} = (g_{1,k,n}, \dots, g_{k,k,n})'$; $G_{ij,k,n} = \sum_{t=1}^n S_{i,t-1} S_{j,t-1}$; $g_{i,k,n} = \sum_{t=1}^n S_{i,t-1} S_{0,t}$ and $G_{d,n}^{ii}$ is the (i,i) th element of $G_{d,n}^{-1}$.

We now obtain the limiting distributions of $F_{i,n}(d)$ and $t_{i,n}(d)$.

Lemma A.1: Consider the model (A.3) where e_t satisfy the conditions of Lemma 2.

Then,

$$(i) \quad R_n(\mu_1, \dots, \mu_k) \xrightarrow{\mathcal{D}} \xi_k' \Sigma_k^{-1} \xi_k = R_k, \quad k=1, \dots, d$$

$$(ii) \quad \hat{\sigma}_n^2(d) \xrightarrow{P} 1$$

$$(iii) \quad F_{i,n}(d) \xrightarrow{\mathcal{D}} F_i(d) = i^{-1} [R_d - R_{d-i}], \quad i \leq d$$

and

$$(iv) \quad t_{d-i+1,n}(d) \xrightarrow{\mathcal{D}} t_{d-i+1}(d), \quad i \leq d$$

where $F_{i,n}(d)$ is defined in (A.4), ξ_k , Σ_k , and $t_{d-i+1}(d)$ are as defined in Theorem 1 and $R_0 = 0$.

Proof: From Lemma 2, we know that for $k \leq d$,

$$D_{k,n}^{-1} g_{k,n} \xrightarrow{\mathcal{D}} \xi_k \tag{A.5}$$

and

$$D_{k,n}^{-1} G_{k,n} D_{k,n}^{-1} \xrightarrow{\mathcal{D}} \Sigma_k$$

where $D_{k,n} = \text{diagonal } \{n, n^2, \dots, n^k\}$.

Pantula (1985) and Chan and Wei (1986) showed that Σ_k is nonsingular.

Therefore,

$$R_n(\mu_1, \dots, \mu_k) \xrightarrow{\mathcal{D}} \xi_k' \Sigma_k^{-1} \xi_k. \tag{A.6}$$

Now,

$$\hat{\sigma}_n^2(d) = (n-d)^{-1} \sum_{t=1}^n S_{0,t}^2 - R_n(\mu_1, \dots, \mu_d)$$

$$= (n-d)^{-1} \sum_{t=1}^n e_t^2 + o_p((n-d)^{-1})$$

$$\xrightarrow{P} 1.$$

The result (iii) follows immediately from (i) and (ii). The result (iv) follows from (A.5), (A.6) and (ii). []

Note that, from Lemma A.1,

$$F_d(d) = d^{-1} \xi_d' \Sigma_d^{-1} \xi_d$$

and hence for $i \leq d$,

$$F_i(d) = i^{-1} [d F_d(d) - (d-i)F_{d-i}(d-i)] .$$

Now we will obtain the asymptotic distribution of $F_{i,n}(p)$, the F-statistic for testing i unit roots in a p th order autoregressive process, under the hypothesis H_d .

Consider Y_t a p th order autoregressive process with d unit roots and $p-d$ stationary roots. Then, Y_t can be written as

$$Y_{p,t} = \sum_{i=1}^d \beta_i Y_{i-1,t-1} + \sum_{i=d+1}^p \beta_i Y_{i-1,t-1} + e_t \quad (A.7)$$

where $Y_{i,t} = (1-B)^i Y_t$, $\beta_1 = \dots = \beta_d = 0$ and $\beta_{d+1} < 0$.

Let

$$W_t = Y_{p-d,t} - \sum_{i=d+1}^p \beta_i Y_{i-1-d,t-1} \quad (A.8)$$

and

$$Z_t = (1-B)^d Y_t = Y_{d,t} . \quad (A.9)$$

Then W_t is a d th order autoregressive process with d unit roots and Z_t is a $(p-d)$ th order autoregressive process with $(p-d)$ stationary roots. Following the arguments used in Lemma 5.1 of Hasza and Fuller (1979) and Lemmas 2.4 and 2.5 of Tiao and Tsay (1983) it can be shown that, for $0 \leq i, j \leq d-1$,

$$n^{-(2d-i-j)} \sum_{t=1}^n Y_{i,t-1} Y_{j,t-1} = \beta_{d+1}^{-2} n^{-(2d-i-j)} \sum_{t=1}^n W_{i,t-1} W_{j,t-1} + o_p(n^{-1}) \quad (A.10)$$

and

$$n^{-(d-i)} \sum_{t=1}^n Y_{i,t-1} Y_{p,t} = -\beta_{d+1}^{-1} n^{-(d-i)} \sum_{t=1}^n W_{i,t-1} e_t + o_p(n^{-1}), \quad (\text{A.11})$$

where $W_{i,t} = (1-B)^i W_t$.

(See also Lemma A.1 and Proposition A.1 of Yajima (1985).)

We now obtain the limiting distributions of $F_{i,n}(p)$ and $t_{i,n}(p)$ under the hypothesis H_d .

Lemma A.2: Let Y_t be a p th order process with d unit roots and $(p-d)$ stationary roots. Let $F_{i,n}(p)$ be the regression F -statistic for testing $\beta_1 = \dots = \beta_i = 0$ in the model (A.7). Then, as n tends to infinity,

$$(i) \quad F_{i,n}(p) \xrightarrow{\mathcal{D}} F_i(d) \quad \text{for } i \leq d,$$

$$(ii) \quad F_{d+1,n}(p) \text{ diverges to infinity,}$$

$$(iii) \quad t_{i,n}(p) \xrightarrow{\mathcal{D}} t_i(d) \quad \text{for } i \leq d,$$

and

$$(iv) \quad n^{-1/2} t_{i,n}(p) \xrightarrow{P} f \beta_i \quad \text{for } i > d,$$

where $F_i(d)$ and $t_i(d)$ are given in Lemma A.1 and f is a finite positive constant.

Proof: Let

$$\Phi_t = (Y_{0,t-1}, \dots, Y_{p-1,t-1})'$$

$$= (\Psi_t', \nu_t')',$$

$$\Psi_t = (Y_{0,t-1}, \dots, Y_{d-1,t-1})',$$

and

$$v_t = (Y_{d,t-1}, \dots, Y_{p-1,t-1})'$$

Define, $M_n = \text{diagonal } \{n^d, n^{d-1}, \dots, n, n^{1/2}, \dots, n^{1/2}\}$.

Then, from the arguments used in Theorem 5.1 of Hasza and Fuller (1979), and (A.10) we get

$$M_n^{-1} \sum_{t=1}^n \Phi_t \Phi_t' M_n^{-1} \xrightarrow{\beta} \begin{matrix} \beta_{d+1}^2 A \Sigma_d^{-1} A & 0 \\ 0 & T^{-1} \Gamma_Z^{-1} T^{-1} \end{matrix} \quad (\text{A.12})$$

where T is defined in (2.2) with $(p-d)$ in place of p ; Σ_d defined in Lemma 2;

Γ_Z is a $(p-d) \times (p-d)$ matrix with $\gamma_Z(i-j)$ as the (i,j) th element,

$\gamma_Z(h) = \lim_{t \rightarrow \infty} E[Z_t Z_{t-h}']$; and

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & & & & \\ i & 0 & \dots & 0 & 0 \end{bmatrix} : d \times d.$$

Also, from Lemma 2 and (A.11) it follows that

$$M_n^{-1} \sum_{t=1}^n \Phi_t Y_{p,t} \xrightarrow{\beta} \begin{matrix} -\beta_{d+1}^{-1} A \xi_d \\ T'b - T'\Gamma_Z c \end{matrix} \quad (\text{A.13})$$

where ξ_d is defined in Lemma 2, $b = (\gamma_Z(1), \dots, \gamma_Z(p-d))'$ and c is as defined in (2.2) with $p-d$ in place of p .

Now consider $F_{i,n}(p)$, the regression F-statistic for testing $\beta_1 = \dots = \beta_i = 0$ in (A.7). Note that,

$$F_{i,n}(p) = [i \hat{\sigma}_n^2(p)]^{-1} [R_n(\beta_1, \dots, \beta_p) - R_n(\beta_{i+1}, \dots, \beta_p)]$$

where $\hat{\sigma}_n^2(p)$ is the residual mean square of the regression (A.7), $R_n(\beta_{i+1}, \dots, \beta_p)$ denote the regression sums of squares of the regression (A.7) with $\beta_1, \beta_2, \dots, \beta_i$ set equal to zero. Since the expression (A.12) is block diagonal, it follows from (A.13) and Lemma A.1, for $i \leq d$,

$$F_{i,n}(p) \xrightarrow{D} F_i(d) .$$

and

$$t_{i,n}(p) \xrightarrow{D} t_i(d) .$$

Now,

$$\begin{aligned} n^{-1} F_{d+1,n}(p) &= [n(d+1)\hat{\sigma}_n^2(p)]^{-1} [R_n(\beta_1, \dots, \beta_p) - R_n(\beta_{d+2}, \dots, \beta_p)] \\ &\geq [n(d+1)\hat{\sigma}_n^2(p)]^{-1} [R_n(\beta_1, \dots, \beta_p) - R_n(\beta_1, \dots, \beta_d, \beta_{d+2}, \dots, \beta_p)] \\ &= [(d+1)\hat{\sigma}_n^2(p) n \hat{\sigma}_{p,n}^{d+1,d+1}]^{-1} \hat{\beta}_{d+1}^2 \end{aligned}$$

where $\hat{\sigma}_{p,n}^{d+1,d+1}$ is the $(d+1, d+1)$ th element of $[\sum_{t=1}^n \phi_t \phi_t']^{-1}$. Note that,

$$n \hat{\sigma}_{p,n}^{d+1,d+1} = \epsilon_{d+1}' M_n [\sum_{t=1}^n \phi_t \phi_t']^{-1} M_n \epsilon_{d+1}$$

$$P \rightarrow \text{the first element of } \Gamma^{-1} \Gamma_Z^{-1} \Gamma^{-1},$$

and

$$\hat{\beta}_{d+1}^2 \xrightarrow{P} \beta_{d+1}^2 ,$$

where ϵ_{d+1} is the $(d+1)$ th column of a $p \times p$ identity matrix. Therefore,

$F_{d+1,n}(p)$ diverges to infinity as n tends to infinity.

Also, note that for $i > d$,

$$t_{i,n}(p) = \left[\hat{\sigma}_n^2(p) \hat{\sigma}_{p,n}^{ii} \right]^{-1/2} \left[\hat{\beta}_i - \beta_i \right] + \left[\hat{\sigma}_n^2(p) \hat{\sigma}_{p,n}^{ii} \right]^{-1/2} \beta_i \quad (\text{A.14})$$

and hence from (A.12) and (A.13)

$$n^{-1/2} t_{i,n}(p) \xrightarrow{P} f \beta_i,$$

where f is a finite positive constant. []

We now complete the proof of Theorem 1.

Proof of Theorem 1: From Lemma A.1 and Lemma A.2, we know that for $i \leq d$,

$$F_{i,n}(p) \xrightarrow{D} F_i(d)$$

under the hypothesis H_d . From Lemma A.2, it follows that $F_{d+1,n}(p)$ diverges to infinity under the hypothesis H_d . Now, since

$$F_{d+j,n}(p) \geq (d+j)^{-1} (d+1) F_{d+1,n}(p)$$

for $j \geq 2$, we have that $F_{i,n}$ diverges to infinity for $i > d$, under the hypothesis H_d . []

We now prove Theorem 2.

Proof of Theorem 2: Note that $t_{i,n}^*(p)$ is the regression t -statistic for testing the hypothesis that the coefficient of V_{t-1} is zero in the regression of $(1-B)^{p-i+1} V_t$ on $V_{t-1}, (1-B)V_{t-1}, \dots, (1-B)^{p-i} V_{t-1}$, where $V_t = (1-B)^{i-1} Y_t$. Under the hypothesis H_d , for $i \leq d$, the process V_t has $(d-i+1)$ unit roots and $(p-d)$ stationary roots. Therefore, from the results of Dickey and Fuller (1979) and the arguments of Lemma A.2 it follows that, under the hypothesis H_d ,

$$\begin{aligned} \hat{\tau} &= t_1(1) \quad \text{for } i = d \\ t_{i,n}^*(p) &\xrightarrow{D} t_i^*(d) \quad \text{for } i < d, \end{aligned}$$

where $t_i^*(d)$ and $\hat{\tau}$ are as defined in Theorem 2. Also, as in (A.14) it

follows that under the hypothesis H_d , for $i > d$

$$n^{-1/2} t_{i,n}^*(p) \xrightarrow{P} f^* \beta_{i,p}^*$$

where f^* is a finite positive constant, $\beta_{i,p}^*$ is the probability limit of $\hat{\beta}_{i,p}^*$ and $\hat{\beta}_{i,p}^*$ is the regression coefficient of $Y_{i-1,t-1} = (1-B)^{i-1} Y_{t-1}$ in the regression of $(1-B)^p Y_t$ on $(1-B)^{i-1} Y_{t-1}, \dots, (1-B)^{p-1} Y_{t-1}$. We will show that under the hypothesis H_d , $\beta_{i,p}^*$ is a finite negative constant.

Note that for $i > d$, under the hypothesis H_d , the process $V_t = (1-B)^{i-1} Y_t$ is essentially a stationary process. It is easy to see that $\hat{\beta}_{i,p}^*$ is also the regression coefficient of V_{t-1} in the regression of $(1-B)V_t$ on $V_{t-1}, (1-B)V_{t-1}, \dots, (1-B)V_{t-\ell+1}$ where $\ell = p-i+1$. Since the d th difference of Y_t is essentially a stationary $(p-d)$ th autoregressive process and that V_t is an $(i-1)$ th difference of Y_t , it follows that

$$\gamma_V(h) = \lim_{t \rightarrow \infty} E[V_t V_{t-h}]$$

is a positive definite function. Define

$$\Gamma_V = \text{toeplitz}(\gamma_V(0), \dots, \gamma_V(\ell-1)),$$

$$\gamma_0 = (\gamma_V(1), \dots, \gamma_V(\ell))$$

and

$$\delta = \Gamma_V^{-1} \gamma_0. \tag{A.15}$$

Now, using the standard arguments it can be shown that, under the hypothesis H_d , the regression coefficients $\hat{\delta}$ of the regression V_t on $V_{t-1}, \dots, V_{t-\ell}$ converge in probability to δ . Also, since

$$\hat{\beta}_{i,p}^* = \sum_{i=1}^{\ell} \hat{\delta}_i - 1$$

we get

$$\hat{\beta}_{i,p}^* \rightarrow \beta_{i,p}^* = \sum_{i=1}^{\rho} \delta_i - 1 .$$

(See Fuller (1976), p. 374). Now, since δ is a solution to the Yule-Walker equations in (A.15), it follows from Theorem 2.2 of Pagano (1973) that the roots of the characteristic equation

$$m^{\rho} - \sum_{i=1}^{\rho} \delta_i m^{\rho-i} = 0 \quad (\text{A.16})$$

lie inside the unit circle. Therefore,

$$\begin{aligned} \beta_{i,p}^* &= -(1 - \sum_{i=1}^{\rho} \delta_i) \\ &= -\pi_{i=1}^{\rho} (1 - m_i) \end{aligned}$$

is negative, where m_1, \dots, m_{ρ} are the roots of the characteristic equation

(A.16). []

Appendix B

In this appendix we extend the testing procedures to autoregressive moving average processes. Consider the model

$$Y_t = \sum_{i=1}^p \alpha_i Y_{t-i} + \sum_{j=1}^q \kappa_j e_{t-j} + e_t \quad (\text{B.1})$$

where we assume that

- (i) $\{e_t\}$ is a sequence of iid $(0, \sigma^2)$ random variables with finite $(4+\delta)$ th moment for some $\delta > 0$,

(ii) the characteristic equations

$$m^p - \sum_{i=1}^p \alpha_i m^{p-i} = 0, \quad (B.2)$$

and

$$m^q + \sum_{j=1}^q \kappa_j m^{q-j} = 0, \quad (B.3)$$

have no common roots,

and

(iii) the roots of (B.3) lie inside the unit circle.

Let $1 \geq |m_1| \geq \dots \geq |m_p|$ denote the roots of (B.2). We wish to test the hypothesis $H_d: m_1 = \dots = m_d = 1$ and $|m_{d+1}| < 1$. Consider the reparameterization of (B.1)

$$(1-B)^p Y_t = \sum_{i=1}^p \beta_i (1-B)^{i-1} Y_{t-i} + \sum_{j=1}^q \kappa_j e_{t-1} + e_t. \quad (B.4)$$

From Lemma 1 we know that, under the hypothesis H_d , $\beta_1 = \dots = \beta_d = 0$ (and $\beta_{d+1} < 0$). Consider the regression models

$$(1-B)^p Y_t = \sum_{\lambda=1}^p a_\lambda (1-B)^{\lambda-1} Y_{t-1} + \sum_{j=1}^{k-p} b_j (1-B)^p Y_{t-j} + \text{error}, \quad (B.5)$$

and, for $i = 1, \dots, p$,

$$(1-B)^p Y_t = \sum_{\lambda=i}^p a_\lambda^* (1-B)^{\lambda-1} Y_{t-1} + \sum_{j=1}^{k-p} b_j^* (1-B)^p Y_{t-j} + \text{error}, \quad (B.6)$$

where k ($> p$) will be chosen to be a function of the sample size n . Let $F_{i,n}(p,k)$ denote the usual regression F-statistic for testing $a_1 = \dots = a_i = 0$ in (B.5) and let $t_{i,n}^*(p,k)$ denote the regression t-statistic for testing $a_i^* = 0$ in (B.6). We now present the asymptotic distributions of $F_{i,n}(p,k)$ and $t_{i,n}^*(p,k)$ under the hypothesis H_d for $i \geq d$. For the remainder of the paper, we assume that the choice of k is such that k^{-1} and $n^{-1}k^3$ tend to zero as n tends to infinity.

Result B.1: Suppose the hypothesis H_d is true. Then,

$$(i) F_{d,n}(p,k) \xrightarrow{D} F_d(d) ,$$

$$(ii) t_{d,n}^*(p,k) \xrightarrow{D} \hat{\tau} ,$$

(iii) for $i > d$, $t_{i,n}^*(p,k)$ diverges to negative infinity, and

(iv) $F_{i,n}(p,k)$ diverges to infinity for $i > d$,

where $F_d(d)$ and $\hat{\tau}$ are as defined in Theorems 1 and 2.

Based on Result B.1, we suggest two sequential procedures for testing the hypothesis H_d . Let $s \leq p$ denote our prior belief of the maximum number of unit roots present in the process.

Procedure 1: Consider the regression (B.5) and compute $F_{i,n}(p,k)$ where k is such that k^{-1} and $n^{-1}k^3$ tend to zero as n tends to infinity. Reject the hypothesis H_d (in favor of H_{d-1}) if $F_{i,n}(p,k) > F_i(i,\alpha)$, for $i=d, d+1, \dots, s$, where $F_i(i,\alpha)$ is the 100(1- α)% percentile of the distribution of $F_i(i)$. (See Theorem 1.)

Procedure 2: Consider the regressions (B.6) to compute $t_{i,n}^*(p,k)$, $i=1, \dots, s$, where k is such that k^{-1} and $n^{-1}k^3$ tend to zero as n tends to infinity. Reject the hypothesis H_d (in favor of H_{d-1}) if $t_{i,n}^*(p,k) < \hat{\tau}_\alpha$, for $i=d, d+1, \dots, s$, where $\hat{\tau}_\alpha$ are the percentiles of $\hat{\tau}$ given in Table 8.5.2 of Fuller (1976).

For either procedure, we get,

$$\lim_{n \rightarrow \infty} P_{H_j} [\text{Rejecting } H_d] = \begin{cases} \alpha & , j=d \\ \leq \alpha & , j>d \\ 1 & , j<d \end{cases} .$$

We will now outline a proof of Result B.1. Assume that the hypothesis H_d of exactly d unit roots is true.

1. Note that $Z_t = (1-B)^d Y_t$ is essentially a stationary and invertible ARMA (p-d,q) process.

2. Consider the regressions

$$Z_t = \sum_{i=1}^d a_i (1-B)^{i-1} Y_{t-1} + \sum_{j=1}^{k-d} \eta_j Z_{t-j} + \text{error} , \quad (\text{B.7})$$

$$(1-B)Z_t = \sum_{i=1}^{d+1} a_i (1-B)^{i-1} Y_{t-1} + \sum_{j=1}^{k-d-1} \eta_j^* (1-B)Z_{t-j} + \text{error} , \quad (\text{B.8})$$

$$Z_t = a_d^* (1-B)^{d-1} Y_{t-1} + \sum_{j=1}^{k-d} \eta_j^{**} Z_{t-j} + \text{error} , \quad (\text{B.9})$$

and

$$(1-B)^{i-d} Z_t = a_i^* (1-B)^{i-d-1} Z_{t-1} + \sum_{j=1}^{k-i} \nu_j (1-B)^{i-d} Z_{t-j} + \text{error} . \quad (\text{B.10})$$

Note that (B.7) and (B.8) are reparameterizations of (B.5) and (B.9) and (B.10) are reparameterizations of (B.6). Therefore, $F_{d,n}(p,k)$ is the same as the regression F-statistic for testing $a_1 = \dots = a_d = 0$ in (B.7);

$F_{d+1,n}(p,k)$ is the regression F-statistic for testing $a_1 = \dots = a_{d+1} = 0$ in (B.8); $t_{d,n}(p,k)$ is the same as the t-statistic for testing $a_d^* = 0$ in (B.9); and $t_{i,n}^*(p,k)$ is the same as the t-statistic for testing $a_i^* = 0$ in (B.10). Also, $t_{i,n}^*$ is the regression on t-statistic for testing $\sum_{j=1}^{k-i+1} \delta_j = 1$ in the regression

$$(1-B)^{i-d-1} Z_t = \sum_{j=1}^{k-i+1} \delta_j (1-B)^{i-d-1} Z_{t-j} + \text{error} . \quad (\text{B.11})$$

3. Said and Dickey (1984) showed that $t_{d,n}^*$ converges in distribution to $\hat{\tau}$ as n tends to infinity. (See Theorem 6.1 and Section 7 of Said and Dickey (1984).) Also, using the arguments of Said and Dickey (1984) and the arguments used in the proof of Lemma A.2, it can be shown that $F_{d,n}(p,k)$ converges in distribution to $F_d(d)$ as n tends to infinity. The main difference for the general ARMA case is that in the expressions (A.10)

and (A.11),

$$\beta_{d+1}^* = - [\pi_{j=1}^q (1-m_j^*)]^{-1} \pi_{i=d+1}^p (1-m_i) \quad (\text{B.12})$$

is used in place of β_{d+1} , where $1 > |m_1^*| \geq \dots \geq |m_q^*|$ are the roots of the characteristic equation (B.3). (See also Yajima (1985).)

4. Now, to show that $t_{d+1,n}^*(p,k)$ diverges to negative infinity, consider the regression (B.11) with $i=d+1$, i.e.,

$$Z_t = \sum_{j=1}^{k-d+1} \delta_j Z_{t-j} + \text{error} . \quad (\text{B.13})$$

Note that $t_{d+1,n}^*(p,k)$ is the regression t-statistic for testing the hypothesis $\sum_{j=1}^{k-d+1} \delta_j = 1$ in (B.13). Therefore,

$$k^{1/2} (n-k)^{-1/2} t_{d+1,n}^*(p,k) = [\hat{\sigma}_n^2(p,k) k^{-1} \mathbf{1}' \hat{R}_Z^{-1} \mathbf{1}]^{-1/2} (\mathbf{1}' \hat{\delta}_Z - 1), \quad (\text{B.14})$$

where $\hat{\delta}_Z = \hat{R}_Z^{-1} \hat{r}_Z$, $\hat{R}_Z = (n-k)^{-1} \sum_t [X_t(k) X_t'(k)]$, $\mathbf{1} = (1, \dots, 1)'$, $\hat{r}_Z = (n-k)^{-1} \sum_t [X_t(k) Z_t]$, $\hat{\sigma}_n^2(p,k)$ is the residual mean square error and $X_t(k) = (Z_{t-1}, \dots, Z_{t-k+d-1})'$. Since $|m_1^*| < 1$ and $|m_{d+1}| < 1$, from Theorem 1 of Berk (1974) it follows that, as n tends to infinity,

$$\hat{\sigma}_n^2(p,k) \xrightarrow{P} \sigma^2$$

$$\mathbf{1}' \hat{\delta}_Z - 1 = \mathbf{1}' (\hat{\delta}_Z - \delta) + \mathbf{1}' \delta - 1$$

$$\xrightarrow{P} \beta_{d+1}^*$$

and $k^{-1} \mathbf{1}' \hat{R}_Z^{-1} \mathbf{1}$ is bounded and bounded away from zero, where β_{d+1}^* is defined in (B.12). (In Berk's notation, $\delta = -a(k)$ and $1 + \sum_{i=1}^{\infty} a_i = -\beta_{d+1}^*$ is positive.) Therefore, $t_{d+1,n}^*(p,k)$ diverges to negative infinity.

5. We will now show that $F_{d+1,n}(p,k)$ diverges to infinity and hence $F_{d+j,n}(p,k) (\geq (d+j)^{-1}(d+1) F_{d+1,n}(p,k))$ diverges to infinity as n tends to infinity. Consider the regressions (B.8) and (B.10) with $i = d+1$. Then, in the notation of regression sums of squares introduced in (A.4),

$$F_{d+1,n}(p,k) = [(d+1)(n-2k-d)^{-1}(M_1 - M_2)]^{-1}(M_2 - M_4)$$

and

$$t_{d+1,n}^{*2}(p,k) = [(n-2k)^{-1}(M_1 - M_3)]^{-1}(M_3 - M_4)$$

where $M_1 > M_2 \geq M_3 \geq M_4$, $M_1 = \sum_t (Z_t - Z_{t-1})^2$,

$$M_2 = R_n(a_1, \dots, a_{d+1}, \eta_1^*, \dots, \eta_{k-d-1}^*),$$

$$M_3 = R_n(a_{d+1}^*, v_1, \dots, v_{k-d-1}) = R_n(a_{d+1}, \eta_1^*, \dots, \eta_{k-d-1}^*), \text{ and}$$

$$M_4 = R_n(v_1, \dots, v_{k-d-1}) = R_n(\eta_1^*, \dots, \eta_{k-d-1}^*).$$

Note that, since $(M_1 - M_4)(M_2 - M_3) \geq 0$,

$$(M_1 - M_2)^{-1}(M_2 - M_4) \geq (M_1 - M_3)^{-1}(M_3 - M_4).$$

Therefore,

$$F_{d+1,n}(p,k) \geq [(d+1)(n-2k)]^{-1}(n-2k-d)t_{d+1,n}^{*2}(p,k)$$

and hence $F_{d+1,n}(p,k)$ diverges to infinity.

6. We will now show that $t_{i,n}^*(p,k)$ diverges to negative infinity for $i > d+1$. Recall that $t_{i,n}^*(p,k)$ is the regression t -statistic for testing $\sum_{j=1}^{k-i+1} \delta_j = 1$ in the regression

$$V_t = \sum_{j=1}^{k-i+1} \delta_j V_{t-j} + \text{error}, \quad (\text{B.15})$$

where $V_t = (1-B)^{i-d-1} Z_t$, i.e.,

$$(n-k)^{-1/2} t_{i,n}^*(p,k) = [\hat{\sigma}_n^2(p,k) \mathbf{1}' \hat{R}_V^{-1} \mathbf{1}]^{-1/2} (\mathbf{1}' \hat{\delta}_V - 1),$$

where $\hat{\delta}_V = \hat{R}_V^{-1} \hat{r}_V$, $\hat{R}_V = (n-k)^{-1} \sum_t [X_t(k) X_t'(k)]$,
 $\hat{r}_V = (n-k)^{-1} \sum_t [X_t(k) V_t]$, $\hat{\sigma}_n^2(p, k)$ is the residual mean square error of
the regression (B.15) and $X_t(k) = (V_{t-1}, \dots, V_{t-k+i-1})'$. Since V_t is the
 $(i-d-1)$ th difference of Z_t , we can write

$$\hat{R}_V = C_{i-d-1} C_{i-d} \dots C_1 \hat{R}_Z C_1' \dots C_{i-d-1}' \quad (B.16)$$

where $\hat{R}_Z = (n-k)^{-1} \sum_t [X_t^*(k) X_t^{*'}(k)]$, $X_t^*(k) = (Z_{t-1}, \dots, Z_{t-k+d})'$, and

$$C_j = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & & & & & \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix} \quad : (k-d-j) \times (k-d-j+1) .$$

Therefore, from Theorem 6.5.5 of Anderson (1971), we get

$$\lambda_{\min}(\hat{R}_V) \geq \lambda_{\min}(\hat{R}_Z) 2^{i-d-1} \prod_{j=1}^{i-d-1} [1 - \cos \pi(k-d-j+1)^{-1}]$$

and

$$\lambda_{\max}(\hat{R}_V) \leq \lambda_{\max}(\hat{R}_Z) 2^{i-d-1} \prod_{j=1}^{i-d-1} [1 + \cos \pi(k-d-j+1)^{-1}] .$$

Since $[1 - \cos \pi(k+1)^{-1}]^{-1} = O(k^2)$, we get

$$\lambda_{\max}(\hat{R}_V^{-1}) \leq M_1 \lambda_{\max}(\hat{R}_Z^{-1}) k^{2(i-d-1)}$$

and

$$\lambda_{\min}(\hat{R}_V^{-1}) \geq M_2 \lambda_{\min}(\hat{R}_Z^{-1}) ,$$

where M_1 and M_2 are finite constants and $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the
minimum and maximum eigenvalues of A , respectively. From Berk's results,
we know that λ_{\min} and λ_{\max} of \hat{R}_Z are bounded and bounded away from zero.

Note that,

$$k^{1/2} (n-k)^{-1/2} |t_{i,n}^*(p,k)| \geq [M_1^{-1} (n-k)^{-1} \Sigma_t V_t^2 \lambda_{\min}(\hat{R}_Z)]^{-1/2} \\ k^{-(i-d-1)} |1' \hat{\delta}_V - 1| .$$

We will show that $k^{-(i-d-1)} (1' \hat{\delta}_V - 1)$ converges to a negative constant and hence that $t_{i,n}^*(p,k)$ diverges to negative infinity. We will present the proof for the case $i = d+2$ and indicate how to extend for $i > d+2$. Consider the case $i = d+2$. The regression (B.15) then is

$$Z_t - Z_{t-1} = \Sigma_{j=1}^{k-d-1} \delta_j (Z_{t-j} - Z_{t-j-1}) + \text{error},$$

or, equivalently,

$$Z_t = \Sigma_{j=1}^{k-d} \theta_j Z_{t-j} + \text{error}, \quad (\text{B.17})$$

where $\Sigma_{j=1}^{k-d} \theta_j$ is restricted to be equal to one. Note that

$$\delta_j = \Sigma_{\ell=1}^j \theta_\ell - 1, \text{ for } j=1, \dots, k-d-1 \text{ and hence}$$

$$1' \delta = \Sigma_{j=1}^{k-d-1} (k-d-j) \theta_j - (k-d-1). \text{ From equation (97) of Searle}$$

(1971, p. 206), the restricted least squares estimate of Θ in (B.17) is,

$$\hat{\Theta}^* = \hat{\Theta} - \hat{R}_Z^{-1} 1 (1' \hat{R}_Z^{-1} 1)^{-1} (1' \hat{\Theta} - 1), \quad (\text{B.18})$$

$$\text{where } \hat{\Theta} = \hat{R}_Z^{-1} r_Z.$$

Therefore,

$$(k-d)^{-1} (1' \hat{\delta}_V - 1) = (k-d)^{-1} \Sigma_{j=1}^{k-d-1} (k-d-j) \hat{\Theta}_j^* - 1 \\ = -b_k' \hat{\Theta} + b_k' \hat{R}_Z^{-1} 1 (1' \hat{R}_Z^{-1} 1)^{-1} (1' \hat{\Theta} - 1) \quad (\text{B.19})$$

where $b_k = (k-d)^{-1} (1, 2, \dots, k-d)'$. From Theorem 1 of Berk (1974) it follows

that

$$b'_k(\hat{\theta} - \theta) \xrightarrow{P} 0,$$

$$(k-d)^{-1} b'_k(\hat{R}_Z^{-1} - R_Z^{-1}) \mathbf{1} \xrightarrow{P} 0$$

$$(k-d)^{-1} \mathbf{1}'(\hat{R}_Z^{-1} - R_Z^{-1}) \mathbf{1} \xrightarrow{P} 0$$

and

$$\mathbf{1}'(\hat{\theta} - \theta) \xrightarrow{P} 0,$$

where $R_Z = \lim_{t \rightarrow \infty} \text{Cov}(X_t^*(k))$.

Since Z_t has stationary and invertible roots, we get $|\theta_j| < M\lambda^j$ for some $0 < \lambda < 1$ and $0 < M < \infty$. Therefore, as k tends to infinity,

$$\mathbf{1}'\theta - 1 \rightarrow \sum_{j=1}^{\infty} \theta_j - 1 = \beta_{d+1}^* < 0,$$

and

$$b'_k \theta = (k-d)^{-1} \sum_{j=1}^{k-d} j \theta_j \rightarrow 0.$$

Also, from the arguments of Theorem 10.2.7 of Anderson (1971) we get, as k tends to infinity,

$$(k-d)^{-1} b'_k R_Z^{-1} \mathbf{1} \rightarrow \frac{1}{2} [2\pi f_Z(0)]^{-1}$$

and

$$(k-d)^{-1} \mathbf{1}' R_Z^{-1} \mathbf{1} \rightarrow [2\pi f_Z(0)]^{-1},$$

where $f_Z(\cdot)$ is the spectral density of an ARMA $(p-d, q)$ process with m_{d+1}, \dots, m_p as the roots of the AR polynomial and m_1^*, \dots, m_q^* as the roots of the MA polynomial.