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Markov Processes with Infinitely Divisible Limit Distributions: Some Examples

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ABSTRACT

A set of examples is described which suggests that members of a certain class of Markov processes have infinitely divisible limit distributions. A counter example rules out such a possibility and raises the question of what further restrictions are required to guarantee infinitely divisible limits. Some related examples illustrate the same occurrence of infinitely divisible limit distributions. For both settings, an easily checked necessary and sufficient condition is obtained for the existence of a limit distribution.

Key words and phrases: Markov processes, Markov chains, infinite divisibility, limit distributions, stationary distributions.

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1. **INTRODUCTION.** This paper is motivated by the consideration of the stochastic process $Z = \{Z_n, n \geq 1\}$, where $Z_n = X_1 + X_1 \cdot X_2 + \cdots + X_1 X_2 \cdots X_n$, $n \geq 1$, and the random variables $X_1, X_2, \cdots$ are iid. This is a simple random walk when $X_1 = \pm 1$ with equal probabilities, and has been called a "shooting gallery process" by one of the authors when the probabilities are unequal. Such processes are studied elsewhere.

Here, we are concerned with examples for which $E \log |X_1| < 0$. This condition guarantees that Z_∞ exists as the almost sure limit of Z_n. (See Section 3 below for a proof.) Clearly, when Z_∞ exists, it has the same distribution as $X \cdot (Z_\infty + 1)$, where X is independent of Z_∞ and distributed as X_1. It immediately follows that the characteristic function of Z_∞ satisfies the identity

$$\varphi_{Z_\infty}(t) = \int_{-\infty}^{\infty} \varphi_{Z_\infty + 1}(tx)\, F(dx), \tag{1}$$

where F is the distribution function of X. What we find intriguing is that the distribution of Z_∞ is infinitely divisible in a surprising number of instances.

Now consider the Markov process $Y = \{Y_n, n \geq 0\}$, "induced by F", defined by the recursive relationship

$$Y_n = X_n \cdot (Y_{n-1} + 1),\ n \geq 1, \tag{2}$$

where the X's are jointly independent of Y_0. When Z_∞ exists, Y has a limit distribution, namely that of Z_∞. To see this, observe that

$$Y_n = X_n \cdot (Y_{n-1} + 1) = X_n + X_n \cdot X_{n-1}(Y_{n-2} + 1) = \cdots$$

$$= X_n + X_n \cdot X_{n-1} + \cdots + X_n \cdot X_{n-1} \cdots X_1 \cdot Y_0.$$

So Y_n has the same distribution as $Z_{n-1} + (Z_n - Z_{n-1}) \cdot Y_0$ (since the X's are exchangeable), which converges to Z_∞ as $n \to \infty$.

Murray Rosenblatt has pointed out to one of the authors that the present mathematical setting can be viewed as arising from products of independent random matrices:

$$\begin{bmatrix} 1 & 0 \\ Z_n & P_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ X_1 & X_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ X_2 & X_2 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ X_n & X_n \end{bmatrix} \tag{3a}$$

and

$$\begin{bmatrix} 1 & 0 \\ Y_n & P_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ X_n & X_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_{n-1} & P_{n-1} \end{bmatrix}, \tag{3b}$$

where $P_n = X_1 X_2 \cdots X_n$, $n \geq 1$, and $P_0 = 1$. While limit theorems, in the spirit of the strong law of large numbers and the central limit theorem, have been formulated for *normalized* products of random matrices (For a review, see Joseph Watkins (1984), Y. Guivarc'h and A. Raugi (1984).) theories for *unnormalized* products are much less developed. (See, for instance, Murray Rosenblatt (1984) and Tze–Chien Sun (1984).) In particular, we are aware of no theory applicable to equations (3a,b). We will not attempt to pursue this plausible approach.

Section 2 discusses examples, including some interesting variants of (2). Example 5 shows that the class of possible distributions of Z_∞ does have members which are *not* infinitely divisible. However, we do not know how extensive this class is, nor a sufficient condition on F for infinite divisibility. Section 3 discusses the existence of Z_∞ and justifies some of the assertions made in Section 2.

2. EXAMPLES. The notation and discussion of Section 1 are assumed in this section.

Example 1. If X is uniformly distributed on the interval [0,1], then Z_∞ must exist, and equation (1) becomes

$$\varphi_{Z_\infty}(t) = \int_0^1 \varphi_{Z_\infty+1}(tx)\,dx = t^{-1}\int_0^t \varphi_{Z_\infty+1}(u)\,du.$$

Thus

$$t\cdot\varphi'_{Z_\infty}(t) + \varphi_{Z_\infty}(t) = \varphi_{Z_\infty+1}(t) = e^{it}\cdot\varphi_{Z_\infty}(t),$$

which has the unique solution

$$\varphi_{Z_\infty}(t) = \exp\{\int_0^1 \frac{e^{itv}-1}{v}\,dv\}. \tag{4}$$

The form of this characteristic function guarantees that Z_∞ is infinitely divisible. See, for instance, Feller (1966, pp. 533, 539, equation (3.10)). As noted in Section 1, the Markov process $Y = \{Y_n, n\geq 0\}$, defined by (2), converges in law to Z_∞. We cannot identify the distribution of Z_∞ more explicitly, but it easily follows from (4) that all of its moments are finite, and, in particular, that its k–th cumulant is k^{-1}, $k \geq 1$. □

Example 2. The following is a discrete variant of the Markov process described in Example 1. Consider a nonnegative integer–valued Markov chain $Y = \{Y_n, n\geq 0\}$ with transition probabilities which make Y_{n+1} uniformly distributed on the integers $0, 1, \cdots, Y_n+1$, $n\geq 0$. To show that there is a limiting distribution, one only needs to exhibit a stationary distribution, since the chain obviously is irreducible and aperiodic. The stationary distribution is that of a Poisson random variable Y_0 with mean one: For

$$P(Y_1 = k) = \sum_{m=k-1}^{\infty}(m+2)^{-1}\cdot e^{-1}/m! = e^{-1}\sum_{m=k-1}^{\infty}\{1/(m+1)! - 1/(m+2)!\} = e^{-1}/k! = P(Y_0 = k).$$

We note that, again, the limiting distribution is infinitely divisible. □

Example 3. If X is Bernoulli distributed with mean p, then Z_n is the minimum of n and the number of ones in the sequence $X_1, X_2, \cdots$ before the first occurrence of zero. Clearly, a limiting random variable Z_∞ exists which is geometrically distributed. It is well-known that the geometric distributions are infinitely divisible. The corresponding Markov process (induced by the distribution of X) has the same limiting distribution. □

Example 4. Suppose $X = -1, 0, 1$ with probabilities p, r, and p, respectively, with $2p + r = 1$, so that Z is a simple random walk when $r = 0$. When $r > 0$, Z is a stopped simple random walk with a geometrically distributed stopping time, and Z_∞ exists. Assume hereafter that $r > 0$. Again, Z_∞ is infinitely divisible; its characteristic function is given by

$$\varphi(t) = \exp\{\sum_{m\neq 0} \frac{e^{itm}-1}{m^2}\mu(m)\}, \tag{5}$$

where the canonical measure μ is concentrated on the nonzero integers and is defined by

$$\mu(m) = |m|\cdot s^{|m|},\ m \neq 0;\ s = \frac{1-\sqrt{1-4p^2}}{2p}. \tag{6}$$

See Section 3 below for a proof. □

Some extensions. Example 1 can easily be extended to include any beta distributed random variable with parameters α and 1: if X has a probability density function of the form $\alpha x^{\alpha-1}$ on the interval [0,1], then the limiting characteristic function is as in (4) but with the integral multiplied by α. This is still infinitely divisible. If, instead, X is uniformly distributed on the interval [–1,1], then there is a limiting infinitely divisible distribution with characteristic function

$$\varphi_{Z_\infty}(t) = \exp\{\int_{-1}^{1} \frac{e^{itv}-1}{2v} \mathrm{sign}(v)\, dv\} = \exp\{\int_{-1}^{1} \frac{e^{itv}-1}{v^2} \cdot \tfrac{1}{2}|v|\, dv\}.$$

Likewise, Example 2 can be extended. Suppose $M, M_0, M_1, \cdots$ are iid and independent of Y_0 with M taking values on the nonnegative integers. If Y_{n+1} is uniformly distributed on the integers $0, 1, \cdots, Y_n+M_n$, $n\geq 0$, there is a limiting distribution for the Markov chain if and only if $E\{\log(M+1)\} < \infty$. Again the limit is infinitely divisible and its characteristic function assumes the form

$$\varphi(t) = \exp\{\sum_{m=1}^{\infty} \frac{e^{itm}-1}{m} P(M \geq m)\}. \tag{7}$$

Example 5. The following is an example with a limiting distribution that is *not* infinitely divisible. Let X be 0 or 2 with equal probability, and let $p(r) = P(Z_\infty = r)$, $r=0,1,\cdots$. Since the random variables Z_∞ and $X\cdot(Z_\infty + 1)$ must have the same distribution, it is easy to check that Z_∞ is geometrically–distributed with parameter $\frac{1}{2}$ on the integers 0, 2, 6, 14, 30, $\cdots$, i.e., $p(r) = (\frac{1}{2})^n$ when $r = 2^n - 2$ $(n\geq 1)$, and $p(r) = 0$ elsewhere. If such a distribution were infinitely divisible, there would be another probability function $q(\cdot)$ defined on the nonnegative integers whose convolution with itself is $p(\cdot)$. Necessarily,

$$q(0)^2 = p(0) = \tfrac{1}{2}, \quad 2\cdot q(0)\cdot q(1) = p(1) = 0,$$
$$2\cdot q(0)\cdot q(2) + q(1)^2 = p(2) = \tfrac{1}{4},$$
$$2\cdot q(0)\cdot q(4) + 2\cdot q(1)\cdot q(3) + q(2)^2 = p(4) = 0.$$

These require $q(2)$ to be strictly positive and zero at the same time. Since this is impossible, Z_∞ is not infinitely divisible. □

Stopped shooting gallery processes. There is an interesting extension of Examples 3 and 4. (Again, proofs of assertions made here are in Section 3 below.) Suppose $X = -1, 0, 1$ with probabilities q, r, and p, respectively, with $p + q + r = 1$, so that Z is a shooting gallery process when $r = 0$. When $r > 0$, Z_∞ is a stopped shooting gallery process, with a geometrically distributed stopping time, and Z_∞ exists. Assume hereafter that $r > 0$. Then the characteristic function of Z_∞ has the form shown in (5), where the "canonical measure" μ, possibly a signed measure, is concentrated on the nonzero integers and defined by

$$\mu(m) = |m|\{(s)^{|m|} - (p-q)^{|m|}\},\ m \neq 0, \tag{8}$$

where

$$s = (1 - \sqrt{1-4p_1^2})/2p_1;\ p_1 = p(1+p^2-q^2)^{-1}. \tag{9}$$

This is a measure and, consequently, Z_∞ is infinitely divisible, if and only if

$$4p^3 + r(2-r)(2p-q) \geq 0. \tag{10}$$

In particular, Z_∞ is infinitely divisible whenever $2p \geq q$; thus, whenever $p \geq 1/3$. □

3. THEORY AND PROOFS. This section assumes the mathematical framework of Section 1 and, as indicated by subsection headings, refers to various parts of Section 2.

Existence of Z_∞. Assume the expectation $E \log |X_1|$ is defined, possibly infinite, where $\log |X_1|$ is interpreted as "$-\infty$" when $X_1 = 0$.

THEOREM 1. *The limit* Z_∞ *exists if and only if* $E \log |X_1| < 0$.

PROOF. This is a consequence of the strong law of large numbers when $E \log |X_1| < 0$, since $|Z_n - Z_m| \leq \sum_{k=m}^{n} |X_1 \cdots X_k|$, and the products must go to zero exponentially fast (m<n). Conversely, if $E \log |X_1| > 0$, then $|Z_n - Z_{n-1}| = |X_1 \cdots X_n|$ *grows* exponentially fast, and Z_∞ can not exist. If $E \log |X_1| = 0$, the *fluctuations* of the product $|X_1 \cdots X_n|$ get arbitrarily large infinitely often and, again, Z_∞ can not exist. (It is unclear whether anything can be said, in general, when the expectation is undefined.) □□

Justifications for Example 4. Suppose $X = -1, 0, 1$ with probabilities p, r, p, respectively, with $2p + r = 1$. Further, suppose $r > 0$, so that Z_∞ is well–defined. The main task is to show that Z_∞ is infinitely divisible by deriving the characteristic function shown in (5) with the canonical measure shown in (6).

Derivation. Given the event $[X_i = \pm 1, i=1,\cdots,n; X_{n+1}=0]$, it is clear that $Z_\infty = Z_n$ and the conditional characteristic function of the latter is $(\cos t)^n$. The event in question has probability $r(2p)^n$. So the characteristic function of Z_∞ can be written as

$$\varphi(t) = \frac{1-2p}{1-2p\cdot\cos t}. \tag{11}$$

To complete the derivation of (5), one needs to expand the logarithms of the numerator and denominator of (11). By setting $t = 0$ in the expansion for the denominator, one obtains the expansion for the numerator; so only the expansion for the denominator needs to be shown:

$$\begin{aligned} -\log(1-2p\cdot\cos t) &= (pe^{it} + pe^{-it}) + (pe^{it} + pe^{-it})^2/2 + \cdots \\ &= \sum_{n=1}^{\infty} \frac{p^n}{n} \sum_{k=0}^{n} \binom{n}{k} e^{it(2k-n)} \\ &= \sum_{n=1}^{\infty} \frac{p^n}{n} \sum_{m=-n}^{n} \left[\binom{n}{(m+n)/2} \right] e^{itm}, \end{aligned}$$

where the binomial coefficient is to be interpreted as zero when $(m+n)/2$ is not an integer. For $m \geq 1$, the coefficient of e^{itm} is

$$\sum_{n=m}^{\infty} \frac{p^n}{n} \left[{n \atop (m+n)/2} \right] = \frac{p^m}{m} \sum_{s=0}^{\infty} \frac{mp^{2s}}{m+2s} \binom{m+2s}{s} = m^{-1} \left(\frac{1 - \sqrt{1-4p^2}}{2p} \right)^m. \tag{12}$$

The latter equality depends on a combinatorial identity appearing in Riordan (1968, p. 147). The last expression in (12) explains the values of the canonical measure μ, appearing in (6), when $m \geq 1$. The values for negative m are derived in a similar way. The value of the canonical measure at $m = 0$ is immaterial since $e^{itm} - 1 = 1 - 1 = 0$; it has been set equal to zero. □

The expression $\left(\frac{1 - \sqrt{1-4p^2}}{2p}\right)^m$, appearing in (12), has a probabilistic interpretation. For no obvious reason, it is equal to $P(\max_n Z_n \geq m)$, $m \geq 1$. The proof of this, which will not be given, is based on some ideas described by Feller (1957, pp. 318–321; this is not fully covered in the third edition). The curious thing is that when $P(\max_n Z_n \geq m)$ is inserted into the characteristic function shown in (5), one obtains an expression which is strikingly similar to the characteristic function shown in (7). We have no explanation for this; the similarity could be completely spurious. □□

Justification for the stopped shooting gallery process. Suppose $X = -1, 0, 1$ with probabilities q, r, p, respectively, with $p + q + r = 1$. Further, suppose $r > 0$, so that Z_∞ is well–defined.

THEOREM 2. *The characteristic function of* Z_∞ *is as shown in equation* (5). *The signed measure* μ *is defined on the nonzero integers, as shown in equations* (8) *and* (9). *Consequently,* Z_∞ *is infinitely divisible if and only if inequality* (10) *is satisfied.*

PROOF. In the same way (11) is derived, one can show, in this more general setting, that the characteristic function of Z_∞ takes the form

$$\varphi(t) = \frac{(1 - p - q)(1 + [q-p]e^{-it})}{1 - 2p\cdot\cos t + p^2 - q^2}.$$

Conveniently, this can be written as $\varphi_1(t)\varphi_2(t)$, where $\varphi_1(t)$ is as in (11), with p and r replaced by

$$p_1 = p(1+p^2-q^2)^{-1},\ r_1 = r(r+2q)(1+p^2-q^2)^{-1}, \qquad (2p_1+r_1=1);$$

and where $\varphi_2(t) = r_2 + q_2e^{-it}$, with

$$q_2 = (q-p)(r+2q)^{-1},\ r_2 = (r+2q)^{-1} \qquad (q_2+r_2=1).$$

So $\varphi_1(t)$ is the infinitely divisible characteristic function for a geometrically–stopped simple random walk in the new parameters indexed by 1; and $\varphi_2(t)$ is the characteristic function of an independent Bernoulli random variable, with mean q_2, if $p \leq q$. If $p > q$, $\varphi_2(t)$ is not a characteristic function, but its reciprocal is the infinitely divisible characteristic function for a geometrically–distributed random variable with success probability $r+2q$. In any case, one can carry out separate logarithmic expansions of $\varphi_1(t)$ and $\varphi_2(t)$, combine them, obtain the representation shown in (5), and identify the signed measure μ shown in (8), where s is as shown in (9). Clearly, μ is a measure, so that Z_∞ is infinitely divisible if and only if $s \geq |p-q|$, which simplifies to (10). □□

Justification of equation (7).

Theorem 3. *Suppose the random variables* $M, M_0, M_1, \cdots$ *are iid and independent of* Y_0, *with* Y_0 *and* M *taking values on the nonnegative integers. If* Y_{n+1} *is uniformly distributed on the integers* $0, 1, \cdots, Y_n+M_n$, $n \geq 0$, *then the Markov chain* $Y = \{Y_n, n \geq 0\}$ *is positive recurrent, and it has an infinitely divisible limiting distribution with a characteristic function taking the form shown in* (7) *if and only if* $E\{\log(M+1)\} < \infty$.

Proof. According to Feller (1966, page 310), the characteristic function $\varphi(t)$ shown in (7) is an infinitely divisible characteristic function if and only if $\sum_{m=1}^{\infty} m^{-1}P(M \geq m) < \infty$. This is equivalent to the condition $E\{\log(M+1)\} < \infty$ appearing in the statement of the theorem.

Now suppose this condition holds. Since Y is obviously irreducible and aperiodic, the task of showing that it is positive recurrent, with the limiting distribution described in (7), reduces to showing that the distribution of Y_0 is stationary when its characteristic function is $\varphi(t)$, i.e., to showing that the resulting characteristic function for Y_1 is $\varphi(t)$ as well. Now

$$E\{\exp(itY_1)\} = E\{\frac{e^{it(Y_0+M_0+1)} - 1}{(Y_0+M_0+1)(e^{it}-1)}\} = i(e^{it}-1)^{-1}E\int_0^t e^{iu(Y_0+M_0+1)}du.$$

So one needs

$$i\int_0^t \varphi(u)\xi(u)e^{iu}\, du = (e^{it}-1)\varphi(t), \; -\infty < t < \infty, \tag{13}$$

where $\xi(u)$ is the characteristic function of M. In turn, it easily can be checked, by using a straightforward truncation argument, that it is enough to validate (13) when M is bounded. When M is bounded, $\varphi(t)$ has a continuous derivative everywhere, and it follows that (13) is equivalent to

$$\frac{\varphi'(t)}{\varphi(t)} = i(1-e^{-it})^{-1}(\xi(t)-1), \; -\infty < t < \infty.$$

(At $t = 0$, the right side takes the limiting form: $\xi'(0) = E(M)$. These are well–defined and finite when M is bounded.) So, in view of (7), the task is to validate the identity

$$\sum_{m=1}^{\infty} e^{itm}\, P(M \geq m) = (1-e^{-it})^{-1}(\xi(t)-1), \; -\infty < t < \infty.$$

(Differentiating (7) poses no problem since M is bounded and the sum has only a finite number of non–zero terms.) This identity is easily validated.

Now consider the converse; suppose $E\{\log (M+1)\} = \infty$. Then, as noted, (7) is not an infinitely divisible characteristic function. So the infinitely divisible characteristic functions of the form $\varphi_k(t) = \exp\{\sum_{m=1}^{k} \frac{e^{itm}-1}{m} P(M \geq m)\}$, $k \geq 1$, do not converge as $k \to \infty$; the corresponding distributions are not tight. And this implies Y is not positive recurrent. To make this precise, let $Y(k) = \{Y_n(k), n\geq 0\}$, with the same initial state $Y_0(k) = Y_0$, be a Markov chain defined in the same way as Y, but with each M_n replaced by $\min(M_n,k)$ $(n \geq 0)$, $k \geq 1$. Then $Y(k)$ has a limiting distribution with characteristic function $\varphi_k(t)$, $k \geq 1$, and it easily follows that the doubly indexed set $\{Y_n(k), n\geq 0, k\geq 1\}$ is not tight. Since, obviously, each Y_n is stochastically larger than $Y_n(k)$ for every k, the distributions of $\{Y_n, n\geq 0\}$ are not tight. This implies that Y is not positive recurrent. □

So, when $E\{\log (M+1)\} = \infty$, $Y_n \to \infty$ *in distribution*, and *in probability*, as $n \to \infty$. We remark that the same kind of reasoning can *not* be used to show $Y_n \to \infty$ *with probability one* as $n \to \infty$; the condition $E\{\log (M+1)\} = \infty$ may not be strong enough to guarantee a transient Markov chain.

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