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BOOTSTRAP CONFIDENCE INTERVALS FOR CONDITIONAL QUANTILE FUNCTIONS

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SUMMARY. Based on the (k_n-) nearest neighbor as well as the (h_n-) kernel methods of estimation, bootstrap confidence intervals for a conditional quantile function are considered. Along with a Bahadur-type representation for bootstrap sample quantiles, the crucial choices of k_n and h_n are examined, and the related asymptotic theory is presented in a systematic manner.

1. INTRODUCTION

Let $\{(X_i, Z_i), i \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random vectors (r.v.) with a distribution function (d.f.) $\Pi(x,z)$, $(x,z) \in R^2$ $(=(-\infty,\infty)^2)$. Let $F(x) = \Pi(x,\infty)$, $x \in R$, and let $G(z|x)$ be the conditional d.f. of Z given X=x, for $z \in R$, $x \in R$. A *conditional quantile function* (of Z given X=x) is defined by

$$\xi_p(x) = \inf\{z : G(z|x) \geq p\}, \; x \in R \qquad (0 < p < 1). \tag{1.1}$$

Asymptotic normality of *kernel estimators* of $\xi_p(x)$ (in the fixed design case) was studied by Cheng (1983), while, in the random design case, Stute

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(1986) considered *nearest neighbor* (NN-) type estimators of $\xi_p(x)$. In this context, k_n, the cardinality of NN, is taken as $O(n^{2/3})$, and h_n, the bandwidth in the (uniform) kernel method, is taken as $O(n^{-1/3})$. Recently, Bhattacharya and Gangopadhyay (1988) have studied both the NN and kernel type estimators of $\xi_p(x)$, and incorporating the celebrated *Bahadur* (1966) *representation for sample quantiles* in a conditional framework, they were able to consider $k_n = O(n^{4/5})$ and $h_n = O(n^{-1/5})$. However, in this optimal choice of k_n and h_n, *bias* terms crop up, raising the question of attainability of such optimal rates.

The object of the present study is to construct *bootstrap confidence intervals* for $\xi_p(x)$ employing both the NN and kernel methods. It is shown that if, for some $\eta > 0$, $k = k_n = O(n^{4/5-\eta})$ and $h = h_n = O(n^{-1/5-\eta})$, then the proposed bootstrap methods work out neatly and the leading bias terms may be eliminated readily. Thus, without any essential reduction in the rate of convergence, the bias terms are eliminated while the other asymptotic properties are retained. In this context too, a Bahadur type representation for the bootstrap sample (conditional) distributions plays a vital role. Based on *local bootstrap samples*, estimators of $\xi_p(x)$ are considered, and incorporating a Bahadur type representation, their asymptotic properties are studied in a systematic manner. This shows that *bootstrap distributions* can be validly used to approximate closely the sampling distributions of estimators of $\xi_p(x)$, and this, in turn, provides asymptotic confidence intervals for the conditional quantile function $\xi_p(x)$.

Along with the preliminary notions, the main theorems are presented in Section 2. Section 3 deals with a Bahadur-type representation for local bootstrap samples, and this is then incorporated in Section 4 in the derivation of the main theorems. In this context, certain other asymptotic results on bootstrapping, having interests of their own, are also considered along with. The concluding Section deals with some general remarks.

2. THE MAIN RESULTS

Let $\xi_p(x)$ be defined by (1.1), and consider a fixed $p \in (0,1)$ and x_0 : $-\infty < x_0 < \infty$. Since x_0 and p are held fixed, for notational simplicity, we may write $\xi_p(x_0) = \xi_0$. The following regularity conditions are assumed to be true:

[A1] The d.f. F admits an absolutely continuous density function f, such that

$$\text{(a)} \quad f(x_0) = f_0 > 0, \tag{2.1}$$

and (b) $f''(x) = (d^2/dx^2) f(x)$ exists in a neighborhood of x_0, and there exist positive numbers ϵ and k_0, such that $|x-x_0| \le \epsilon$ implies that $|f''(x)-f''(x_0)| \le k_0|x-x_0|$.

[A2] The d.f. $G(z|x_0)$ has a continuous density function $g(z|x_0)$ for all z close to ξ_0, such that

$$\text{(a)} \quad g(\xi_0|x_0) > 0 \quad \text{where} \quad G(\xi_0|x_0) = p, \tag{2.2}$$

and, (b) the partial derivatives $g_z(z|x)$ and $g_{xx}(z|x)$ of $g(z|x)$ and $G_{xx}(z|x)$ of $G(z|x)$ exist in a neighborhood of (x_0,ξ_0), and there exist positive constants ϵ and k_0, such that $|x-x_0| \le \epsilon$ and $|z-\xi_0| \le \epsilon$ together imply that

$$|g_z(z|x)| \le k_0, \quad |g_x(z|x_0)| \le k_0, \quad |g_{xx}(z|x_0)| \le k_0,$$

$$|g_{xx}(z|x_0) - g_{xx}(z|x)| \le k_0|x-x_0|, \tag{2.3}$$

$$|G_{xx}(z|x) - G_{xx}(z|x_0)| \le k_0|x-x_0|.$$

Incidentially, (a) in [A2] insures that ξ_0 is uniquely defined by $G(\xi_0|x_0) = p$.

Consider next the transformation $(X_i,Z_i) \to (Y_i,Z_i)$ where $Y_i = |X_i-x_0|$, for $i \geq 1$. The marginal d.f. of Y_i is denoted by $F_Y(y)$, so that by [A1], $F_Y(y)$ admits a density function $f_Y(y) = f(x_0+y) + f(x_0-y)$, and for y "close to" 0, the condition (b) in [A1] holds for $f''_Y(y)$ as well. The conditional density $g^*(z|y)$ of Z_i, given $Y_i=y$, and the corresponding d.f. $G^*(z|y)$ are given by

$$g^*(z|y) = [f(x_0+y)g(z|x_0+y) + f(x_0-y)g(z|x_0-y)]/f_Y(y), \tag{2.4}$$

$$G^*(z|y) = [f(x_0-y)G(z|x_0+y) + f(x_0-y)G(z|x_0-y)]/f_Y(y).$$

Note that

$$g^*(z|0) = g(z|x_0) \quad \text{and} \quad G^*(z|0) = G(z|x_0). \tag{2.6}$$

In the sequel, we shall write $G(z|x_0) = G(z)$ and $g(z|x_0) = g(z)$.

For the collection $\{(Y_1,Z_1),\ldots,(Y_n,Z_n)\}$ of r.v.'s, let $Y_{n1} < \ldots < Y_{nn}$ be the *order statistics* corresponding to $Y_1,\ldots,Y_n$, and let $Z_{n1},\ldots,Z_{nn}$ be the *induced order statistics* (i.e., $Z_{ni} = Z_j$ if $Y_{ni} = Y_j$, for $i,j=1,\ldots,n$). For every positive integer k ($\leq n$), the *k-NN empirical d.f.* of Z (with respect to x_0) is given by

$$\hat{G}_{nk}(z) = k^{-1} \sum_{i=1}^{k} 1(Z_{ni} \leq z), \quad z \in R, \tag{2.7}$$

where 1(A) stands for the indicator function of the set A. The following estimators of ξ_0 are due to Bhattacharya and Gangopadhyay (1988):

(i) The k-NN estimator of ξ_0 is

$$\hat{\xi}_{nk} = \text{the [kp]th order statistic of } Z_{n1},\ldots,Z_{nk}$$
$$= \inf\{z : \hat{G}_{nk}(z) \geq k^{-1}[kp]\}. \tag{2.8}$$

(ii) The kernel estimator (with uniform kernel and bandwidth h) is

$$\tilde{\xi}_{nh} = \inf\{z : \hat{G}_{nK_n(h)} \geq [K_n(h)]^{-1}[pK_n(h)]\} \tag{2.9}$$

where

$$K_n(h) = \sum_{i=1}^{n} 1(Y_i \leq \tfrac{1}{2}h) \tag{2.10}$$

is a positive integer valued r.v. Note that by (2.8) and (2.9),

$$\tilde{\xi}_{nh} = \hat{\xi}_{nK_n(h)} \; , \tag{2.11}$$

where h is usually non-stochastic and $K_n(h)$ is stochastic in nature. In order that $\hat{\xi}_{nk}$ (and $\tilde{\xi}_{nh}$) are consistent estimators of ξ_0, we need that as $n \to \infty$, $k = k_n \to \infty$ (and $h = h_n \downarrow 0$). For the asymptotic normality results, usually, one needs that $k_n = 0(n^a) = nh_n$, for some $a \in (0,1)$: Stute (1986) considered the case of $a = 2/3$, while Bhattacharya and Gangopadhyay (1988) studied the case of $a = 4/5$. However, in the later case, there is generally a *bias* term of the order $n^{-2/5}$, and it may be necessary to eliminate the bias term in drawing statistical conclusions on ξ_0. We shall see that this bias term can be eliminated if we choose $a < 4/5$. We shall discuss the case of $a \geq 4/5$ in the concluding section.

To formulate the main results, we define, for every n,

$$I_n(a,b) = \{k : k_{n0} = [n^{4/5-2\delta}a] \leq k \leq [n^{4/5-2\delta}\, b] = k_{n1}\}, \tag{2.12}$$

where $0 < a < b < \infty$ and $0 < \delta < 1/10$. Also, let

$$J_n(c,d) = [cn^{-1/5-2\delta},\ dn^{-1/5-2\delta}],\quad 0 < c < d < \infty,\quad 0 < \delta < \frac{1}{10}. \tag{2.13}$$

Then the following asymptotic results follow along the lines of Bhattacharya and Gangopadhyay (1988) [and hence, the derivations are omitted]:

THEOREM 2.1. *Under assumptions* [A1] *and* [A2], *as* $n \to \infty$,

$$\hat{\xi}_{nk} - \xi_0 = (k/n)^2\, \beta(\xi_0) + \{kg(\xi_0)\}^{-1} \sum_{i=1}^{k} [1(Z^0_{ni} > \xi_0)-(1-p)] + R_{nk}, \tag{2.14}$$

where

$$\beta(\xi_0) = -[f(x_0)G_{xx}(\xi_0|x_0) + 2f'(x_0)G_x(\xi_0|x_0)]/\{24\ f^3(x_0)g(\xi_0)\},$$

$$Z^0_{ni} = G^{-1} \circ G^*(Z_{ni}|Y_{ni}),\ i=1,\dots,n,$$

and

$$\max_{k\in I_n(a,b)} |R_{nk}| = O(n^{-3/5+3\delta/2} \log n) \quad \text{a.s.}$$

THEOREM 2.2. *Under the hypothesis of Theorem 2.1, as* $n \to \infty$,

$$\{n^{2/5-\delta}[\hat{\xi}_{n[tn^{4/5-2\delta}]}-\xi_0],\quad t \in [a,b]\}$$

$$\overset{\mathfrak{D}}{\to} \{p(1-p)\}^{1/2}\ \{[g(\xi_0)]^{-1}\ t^{-1}B(t),\quad t \in [a,b]\}\ , \tag{2.15}$$

where $B = \{B(t),\ t \in R^+\}$ *is a standard Wiener process on* R^+, *and* $\overset{\mathfrak{D}}{\to}$ *stands for the convergence in law. Hence, for every* $k : k = [tn^{4/5-2\delta}]$, $t \in [a,b]$, *as* $n \to \infty$,

$$n^{2/5-\delta}[\hat{\xi}_{nk}-\xi_0] \overset{\mathfrak{D}}{\to} N(0,t^{-1}\ p(1-p)/g^2(\xi_0)). \tag{2.16}$$

THEOREM 2.3. *Under Assumptions* [A1] *and* [A2], *as* $n \to \infty$,

$$\tilde{\xi}_{nh} - \xi_0 = \beta(\xi_0)\ h^2\ f^2(x_0) +$$

$$\{[nhf(x_0)]g(\xi_0)\}^{-1} \sum_{i=1}^{[nhf(x_0)]} \{1(Z^0_{ni} > \xi_0) - (1-p)\} + R^*_{nh} , \qquad (2.17)$$

where $\beta(\xi_0)$ *and* Z^0_{ni} *are defined as in Theorem* 2.1 *and*

$$\sup_{h\in J_n(c,d)} |R^*_{nh}| = O(n^{-3/5+3\delta/2} \log n) \quad a.s. \qquad (2.18)$$

THEOREM 2.4. *Under the hypothesis of Theorem* 2.3, *as* $n \to \infty$,

$$\{n^{2/5-\delta}[\tilde{\xi}_{n(tn^{-1/5-2\delta})} - \xi_0], \quad t \in [c,d]\}$$

$$\stackrel{\mathcal{D}}{\to} \{p(1-p)\}^{1/2} \{f(x_0)g(\xi_0)\}^{-1} \{t^{-1}B(t), \quad t \in [c,d]\}, \qquad (2.19)$$

where $\{B(t),\ t \in R^+\}$ *is defined as in Theorem* 2.1. *Hence, for every* $t \in [c,d]$, *as* $n \to \infty$,

$$n^{2/5-\delta}[\tilde{\xi}_{n(tn^{-1/5-2\delta})} - \xi_0] \stackrel{\mathcal{D}}{\to} \mathcal{N}(0, \frac{p(1-p)t^{-1}}{f^2(x_0)g^2(\xi_0)}). \qquad (2.20)$$

Our main interest lies in the construction of (*local*) *bootstrap confidence intervals* for ξ_0 based on both the k-NN and kernel methods, and to study their asymptotic properties. For this purpose, in the k-NN procedure, a bootstrap sample $(Z^*_{n1},\dots,Z^*_{nk})$ is obtained from the empirical d.f. $\hat{G}_{nk}$, defined by (2.7). Thus, given the induced order statistics $(Z_{n1},\dots,Z_{nk})$, the Z^*_{ni} are conditionally i.i.d.r.v. with the d.f. $\hat{G}_{nk}$. Similarly, in the kernel method, given the Z_{ni} and $h(> 0)$, we define $k_n(h)$ by (2.10) and then a bootstrap sample of size $K_n(h)$ is obtained by resampling from all the Z_{ni} for which $Y_i = |X_i-x_0| \le h/2$. First, let us introduce the bootstrap sample estimates of ξ_0.

For the k-NN procedue, let $\hat{G}^*_{nk}$ be the bootstrap sample d.f. (based on $Z^*_{n1},\ldots,Z^*_{nk}$). Then, the bootstrap estimator is

$$\begin{aligned}\hat{\xi}^*_{nk} &= [kp]\text{th order statistic of } Z^*_{n1},\ldots,Z^*_{nk} \\ &= \inf\{z : \hat{G}^*_{nk}(z) \ge k^{-1}[kp]\}. \end{aligned} \tag{2.21}$$

Similarly, in the kernel method, the bootstrap estimator is

$$\tilde{\xi}^*_{nh} = \inf\{z : \hat{G}^*_{nK_n(h)}(z) \ge \{K_n(h)\}^{-1}[pK_n(h)]\}, \tag{2.22}$$

where $K_n(h) = \sum_{i=1}^{n} 1(Y_i \le h/2)$ is conditionally held fixed. Now, parallel to Theorems 2.1 through 2.4, we have the following asymptotic representation for the bootstrap estimates.

THEOREM 2.5. *Under* [A1] *and* [A2], *for every* $k \in I_n(a,b)$, *defined by* (2.13), $0 < a < b < \infty$, *as* $n \to \infty$,

$$\hat{\xi}^*_{nk} - \hat{\xi}_{nk} = \{kg(\xi_0)\}^{-1} \sum_{i=1}^{k} \{1(Z^{0*}_{ni} > \xi_0) - (1-p)\} + R^{*0}_{nk}, \tag{2.23}$$

where

$$Z^{0*}_{ni} = G^{-1} \circ \hat{G}_{nk}(Z^*_{ni}), \quad i=1,\ldots,k, \tag{2.24}$$

are i.i.d.r.v. with the d.f. $G(\cdot)$, *and as* $n \to \infty$,

$$\max_{k\in I_n(a,b)} |R^{*0}_{nk}| = O(n^{-3/5+3\delta/2} \log n) \quad a.s. \tag{2.25}$$

THEOREM 2.6. *Under the hypothesis of Theorem* 2.5, as $n \to \infty$

$$\{n^{2/5-\delta}(\hat{\xi}^*_{n[tn^{4/5-2\delta}]} - \hat{\xi}_{n[tn^{4/5-2\delta}]}),\ t \in [a,b]\}$$

$$\overset{\mathfrak{D}}{\to} \{p(1-p)\}^{1/2} \{g(\xi_0)\}^{-1} \{t^{-1}B(t),\ t \in [a,b]\}, \tag{2.26}$$

where B(t), $t \in R^+$ *is defined as in Theorem* 2.2. *Hence, for each* t,

$$n^{2/5-\delta}(\hat{\xi}^*_{n[tn^{4/5-2\delta}]} - \hat{\xi}_{n[tn^{4/5-2\delta}]}) \overset{\mathfrak{D}}{\to} \mathcal{N}(0, \frac{p(1-p)t^{-1}}{g^2(\xi_0)}), \tag{2.27}$$

for every $\delta : 0 < \delta < 1/10$, *and every* $0 < a < b < \infty$.

THEOREM 2.7. *Under* [A1], [A2], *for every* $h \in J_n(c,d) : 0 < c < d < \infty$,

$$\tilde{\xi}^*_{nh} - \tilde{\xi}_{nh} = \{g(\xi_0)[nhf(x_0)]\}^{-1} \sum_{i=1}^{[nhf(x_0)]} \{1(Z^{0*}_{ni} > \xi_0) - (1-p)\}$$

$$+ R^{**}_{nh}, \tag{2.28}$$

where the Z^{0*}_{ni} *are defined by* (2.24), *and*

$$\sup_{h \in J_n(c,d)} |R^{**}_{nh}| = O(n^{-3/5+3\delta/2} \log n) \ \text{a.s.}, \ \text{as} \ n \to \infty. \tag{2.29}$$

THEOREM 2.8. *Under the hypothesis of Theorem* 2.7, *as* $n \to \infty$,

$$\{n^{2/5-\delta}(\tilde{\xi}^*_{n(tn^{-1/5-2\delta})} - \tilde{\xi}_{n(tn^{-1/5-2\delta})},\ t \in [c,d]\}$$

$$\overset{\mathfrak{D}}{\to} \{p(1-p)/f^2(x_0)g^2(\xi_0)\}^{1/2} \{t^{-1} B(t),\ t \in [c,d]\}, \tag{2.30}$$

where B(t), $t \in R^+$ *is defined as in* (2.26) *and* $0 < c < d < \infty$. *Hence,*

$$n^{2/5-\delta}(\tilde{\xi}^*_{nh} - \tilde{\xi}_{nh}) \overset{\mathfrak{D}}{\to} \mathcal{N}(0,\ p(1-p)/\{f(x_0)g(\xi_0)\}^2)\ . \tag{2.31}$$

It is interesting to note that for each r(=1,2,3,4), the asymptotic representation in Theorem 2.r for the original sample and Theorem 2•r+4 for the bootstrap sample are the same. As such, one can generate a set of M bootstrap samples, and from each one, compute the bootstrap estimates, and then use the empirical quantiles (for the $\hat{\xi}^*_{nk}-\hat{\xi}_{nk}$ or $\tilde{\xi}^*_{nh}-\tilde{\xi}_{nh}$) to set the desired bootstrap confidence intervals for ξ_0. The asymptotic properties of such bootstrap intervals would then follow from Theorems 2.5 through 2.8.

3. BAHADUR REPRESENTATION FOR BOOTSTRAP (CONDITIONAL) QUANTILES

For the proofs of Theorems 2.5 through 2.8, we need a few preliminary lemmas and a Bahadur (1966) type representation for local bootstrap conditional distributions. These are considered in this section.

Note that given the (X_i or the) Y_{ni}, $Z_{n1},\dots,Z_{nk}$ are conditionally independent r.v.'s with d.f.'s $G^*(\cdot|Y_{n1}),\dots,G^*(\cdot|Y_{nk})$ respectively [viz., Bhattacharya (1974)]. We define

$$g^*(z|Y_{ni}) = g^*_{ni}(z),\ G^*(z|Y_{ni}) = G^*_{ni}(z),\ i \geq 1,\quad z \in R, \tag{3.1}$$

$$\vec{g}^*_{nk}(z) = k^{-1}\sum_{i=1}^{k} g^*_{ni}(z),\ \vec{G}^*_{nk}(z) = k^{-1}\sum_{i=1}^{k} G^*_{ni}(z),\ z \in R, \tag{3.2}$$

$$\xi_{nk} : \vec{G}^*_{nk}(\xi_{nk}) = p = G(\xi_0). \tag{3.3}$$

Further, note that, by definition,

$$0 < Y_{n1} < \dots < Y_{nk}, \tag{3.4}$$

and by Lemma 3.1 (to follow), $Y_{nk} = O(n^{-1/5-2\delta})$ a.s., as $n \to \infty$ [when [A1] holds and $k \in I_n(a,b)$]. As such, we may be tempted in making local expansions for $\vec{g}^*_{nk}$ and $\vec{G}^*_{nk}$ in terms of powers of the Y_{ni}. Towards this, we have the following:

<u>Lemma 3.1</u> *Under* [A1] *and for* $k \in I_n(a,b)$, *for every* $B : Bf(x_0) > b > a > 0$, *there exists an* $n_0(< \infty)$, *such that*

$$P\{Y_{n[bn^{4/5-2\delta}]} > Bn^{-1/5-2\delta}\}$$

$$\leq \exp\{-2n^{3/5-4\delta}\,(Bf(x_0)-b)^2\}, \quad \forall\ n \geq n_0, \qquad (3.5)$$

so that $Y_{n[bn^{4/5-2\delta}]} = O(n^{-1/5-2\delta})$ a.s., *as* $n \to \infty$.

For a proof of the lemma, we may refer to Bhattacharya and Mack (1987), and hence, the details are omitted. Likewise, Lemmas 3.2 through 3.5 (on the base sample) are formulated along the same lines as in Bhattacharya and Gangopadhyay (1988) [with only a change in the order of k], and hence, these will be stated without proof. We adopt the same notations as in Section 2 [see (2.1) through (2.6), for example].

<u>Lemma 3.2</u> *Under* [A1] *and* [A2],

$$g^*(z|y) = g(z|x_0) + \tfrac{1}{2}\, y^2 q(z;x_0) + y^3\, r(z,y,x_0), \qquad (3.6)$$

$$G^*(z|y) = G(z|x_0) + \tfrac{1}{2}\, y^2 Q(z;x_0) + y^3\, R(z,y,x_0), \qquad (3.7)$$

where $q(z;x_0) = g_{xx}(z|x_0) + 2\, f'(x_0)\, g_x(z|x_0)/f(x_0)$, $Q(z;x_0) = G_{xx}(z|x_0) + 2\, f'(x_0)G_x(z|x_0)/f(x_0)$, *and there exist* $\epsilon > 0$ *and* $M(0 < M < \infty)$, *such that for every* $y : 0 \leq y \leq \epsilon$, and $|z-\xi_0| \leq \epsilon$, $|q(z;x_0)|$, $|Q(z;x_0)|$, $|r(z,y;x_0)|$ *and* $|R(z,y,x_0)|$ *are all bounded by* M.

<u>LEMMA 3.3</u> *Under the hypothesis of Lemma* 3.2, *for* $a_n \sim n^{-2/5+\delta} \log n$,

$$P\{\max_{k\in I_n(a,b)}\ \sup_{|z-\xi_{nk}|\leq a_n} |\hat{G}_{nk}(z) - \bar{G}_{nk}(z) - \hat{G}_{nk}(\xi_{nk}) + \bar{G}_{nk}(\xi_{nk})|$$

$$< M\, n^{-3/5+3\delta/2} \log n \quad \text{i.o.}\} = 0, \qquad (3.8)$$

where $M(< \infty)$ *is a generic constant.*

Lemma 3.4 *Under the hypothesis of Lemma* 3.3,

$$P\{\max_{k\in I_n(a,b)} |\hat{\xi}_{nk}-\xi_{nk}| > a_n \quad \text{i.o.}\} = 0. \tag{3.9}$$

Lemma 3.5 *For every* $B(> b/f(x_0))$, *there exist* n_0 *and* c (*both* $< \infty$), *such that in the sample space of infinite sequences* $\{(Y_i,Z_i),\ 1 \le i \le n;\ n \ge 1\}$, $Y_{n[bn^{4/5-2\delta}]} \le B\, n^{-1/5-2\delta}$ *implies that* $\max_{k\in I_n(a,b)} |\xi_{nk}-\xi_0| \le C\, n^{-2/5-4\delta}$, *for every* $n \ge n_0$. *Hence, for every* $0 < a < b < \infty$,

$$\max_{k\in I_n(a,b)} |\xi_{nk}-\xi_0| = O(n^{-2/5-4\delta}) \quad \text{a.s., as } n \to \infty. \tag{3.10}$$

With these results at hand, we now proceed on to consider parallel results for the bootstrap sample: $Z^*_{n1},\dots,Z^*_{nk}$. Let $a_n = C_1 n^{-2/5+\delta} \log n$ where C_1 ($< \infty$) is an arbitrarily (large) positive number. Let then

$$H_{nk}(z) = |\hat{G}^*_{nk}(z) - \hat{G}^*_{nk}(\xi_{nk}) - \hat{G}_{nk}(z) + \hat{G}_{nk}(\xi_{nk})|, \tag{3.11}$$

$$H^*_{nk} = \sup_{|z-\xi_{nk}|\le a_n} H_{nk}(z), \tag{3.12}$$

$$H^*_n = \max_{k\in I_n(a,b)} H^*_{nk} \tag{3.13}$$

Then the following Theorem is an extension of the classical Bahadur (1966) representation for the conditional empirical d.f.'s in a bootstrap model.

THEOREM 3.1. *Under* [A1] *and* [A2], *defining* $I_n(a,b)$ *as in* (2.12),

$$P\{H^*_n > M_1\, n^{-3/5+3\delta/2} \log n \quad \text{i.o.}\} = 0, \tag{3.14}$$

where $M_1(< \infty)$ *depends on* $B(>b/f(x_0))$.

<u>Proof:</u> Let $b_n = n^{1/5-\delta/2}$. $I_{nk} = [\xi_{nk}-a_n-\xi_{nk}+a_n]$

$$J_{nk,r} = [\xi_{nk} + r(a_n/b_n),\ \xi_{nk} + (r+1)(a_n/b_n)] \tag{3.15}$$

$$= [\eta_{nk,r},\ \eta_{nk,r+1}], \tag{3.16}$$

$$r = -b_n,\ -b_n+1,\dots,-1,0,1,\dots,b_n-1,$$

then

$$H_n^* = \max_{k\in I_n(a,b)}\ \max_{-b_n\le r\le b_n-1}\ \sup_{z\in J_{nk,r}} |H_{nk}(z)|. \tag{3.17}$$

If follows from the monotonicity of $\hat{G}_{nk}(\cdot)$ and $\overline{G}_{nk}(\cdot)$ that for $z \in J_{nk,r} = [\eta_{nk,r},\ \eta_{nk,r+1}]$,

$$H_{nk}(\eta_{nk,r}) - \alpha_{nk,r} \le H_{nk}(z) \le H_{nk}(\eta_{nk,r+1}) + \alpha_{nk,r} \tag{3.18}$$

where $H_{nk}(\cdot)$ is given by (3.11), and

$$\alpha_{nk,r} = \hat{G}_{nk}(\eta_{nk,r+1}) - \hat{G}_{nk}(\eta_{nk,r}) \tag{3.19}$$

Hence

$$H_{nk}^* = \sup_{|z-\xi_{nk}|\le a_n} |H_{nk}(z)|$$

$$\le \max_{-b_n\le r\le b_n} |H_{nk}(\eta_{nk,r})| + \max_{-b_n\le r\le b_n-1} \alpha_{nk,r}. \tag{3.20}$$

Now by lemma 3.3,

$$\alpha_{nk,r} = \hat{G}_{nk}(\eta_{nk,r+1}) - \hat{G}_{nk}(\eta_{nk,r})$$

$$= \overline{G}_{nk}^*(\eta_{nk,r+1}) - \overline{G}_{nk}^*(\eta_{nk,r}) + O(n^{-3/5+3\delta/2}\log n)$$

$$= \left(\frac{a_n}{b_n}\right) \vec{g}^{\,*}_{nk}(\eta^*_{nk}) + O(n^{-3/5+3\delta/2} \log n) \tag{3.21}$$

where

$$\eta_{nk,r} \le \eta^*_{nk} \le \eta_{nk,r+1} .$$

But by lemma 3.5, on the set

$S_n = \{Y_{n[bn^{4/5-2\delta}]} \le B\, n^{-1/5-2\delta}\}$, for any z lying between $\eta_{nk,r}$ and $\eta_{nk,r+1}$, we have

$$\begin{aligned} |z-\xi_0| &\le |\eta_{nk,r+1} - \eta_{nk,r}| + |\eta_{nk,r} - \xi_0| \\ &\le (a_n/b_n) + |\xi_{nk}-\xi_0| + r(a_n/b_n) \qquad (3.22) \\ &\le C_2\, n^{-2/5-4\delta} , \text{ for large constant } C_2. \end{aligned}$$

So, on the set S_n, by lemma 3.1 and lemma 3.2,

$$\begin{aligned} \max_{-b_n \le r \le b_n} \alpha_{nk,r} = (a_n/b_n) \Big[&\sup_{|z-\xi_0| \le C_2 n^{-2/5-4\delta}} g(z;x_0) \\ &+ B^2\, n^{-2/5-4\delta} \sup_{|z-\xi_0| \le C_2 n^{-2/5-4\delta}} |q(z;x_0)| \\ &+ 2\, B^3\, n^{-3/5-6\delta} \sup_{0 \le y \le B n^{-1/5-2\delta},\ |z-\xi_0| \le C_2 n^{-2/5-4\delta}} |r(z,y,x_0)| \Big] \\ &+ O(n^{-3/5+3\delta/2} \log n) \\ &\le C_3 g(\xi_0) n^{-3/5+3\delta/2} \log n, \text{ for large constant } C_3. \end{aligned} \tag{3.23}$$

Hence:

$$P[H_n^* > M_1 n^{-3/5+3\delta/2} \log n] \le P[\max_{k\in I_n(a,b)} \max_{-b_n\le r\le b_n} |H_{nk}(\eta_{nk,r})|$$

$$> (M_1 - C_3 g(\xi_0)) n^{-3/5+3\delta/2} \log n] + P(S_n^c) \tag{3.24}$$

Now let $M_2 = M_1 - C_3\, g(\xi_0)$. Then

$$|H_{nk}(\eta_{nk,r})| > M_2\, n^{-3/5+3\delta/2} \log n, \tag{3.25}$$

iff

$$|\{\hat{G}_{nk}^*(\eta_{nk,r}) - \hat{G}_{nk}^*(\xi_{nk})\} - \{\hat{G}_{nk}(\eta_{nk,r}) - \hat{G}_{nk}(\xi_{nk})\}|$$

$$> M_2 n^{-3/5+3\delta/2} \log n, \tag{3.26}$$

iff

$$|\sum_{i=1}^{k} \{1(\xi_{nk} \le Z_{ni}^* \le \xi_{nk} + r(a_n/b_n)) - (\hat{G}_{nk}(\eta_{nk,r}) - \hat{G}_{nk}(\xi_{nk}))\}|$$

$$> k(M_2\, n^{-3/5+3\delta/2} \log n), \tag{3.27}$$

iff

$$|\sum_{i=1}^{k} \{U_{nki} - \mu_{nk}\}| > k(M_2\, n^{-3/5+3\delta/2} \log n), \tag{3.28}$$

where

$$U_{nki} = 1(\xi_{nk} \le Z_{ni}^* \le \xi_{nk} + r(a_n/b_n)) \tag{3.29}$$

and

$$\mu_{nk} = E(U_{nki} \mid Z_{n1}, \ldots, Z_{nk}). \tag{3.30}$$

Also by lemma 3.3,

$$\mu_{nk} = \hat{G}_{nk}(\xi_{nk} + r(a_n/b_n)) - \hat{G}_{nk}(\xi_{nk})$$

$$= \overline{G}_{nk}^*(\xi_{nk} + r(a_n/b_n)) - \overline{G}_{nk}^*(\xi_{nk}) + O(n^{-3/5+3\delta/2} \log n)$$

$$\le a_n \, \vec{g}^{*}_{nk}(\xi^{*}_{nk}) + O(n^{-3/5+3\delta/2} \log n), \tag{3.31}$$

where ξ^{*}_{nk} is lying between ξ_{nk} and $\eta_{nk,r}$. But for any z lying between ξ_{nk} and $\eta_{nk,r}$, by lemma 3.5, on the set S_n,

$$|z-\xi_0| \le |z-\xi_{nk}| + |\xi_{nk}-\xi_0| \le C_4 \, a_n, \text{ for}$$

some large constant C_4. Using lemma 3.2, we can now conclude that for large n, on the set S_n,

$$\mu_{nk} \le a_n[\sup_{|z-\xi_0|\le C_4 a_n} g(z|x_0) + B^2 n^{-2/5-4\delta} \sup_{|z-\xi_0|\le C_4 a_n} |q(z;x_0)|$$

$$+ 2 B^3 n^{-3/5-6\delta} \sup_{0\le y\le Bn^{-1/5-2\delta}, |z-\xi_0|\le C_4 a_n} |r(z,y,x_0)|]$$

$$+ O(n^{-3/5+3\delta/2} \log n) \le M_3 \, g(\xi_0)a_n, \text{ for some large constant } M_3. \tag{3.32}$$

So, by Bernstein's inequality we have

$$P[H_{nk}(\eta_{nk,r}) > M_2 \, n^{-3/5+3\delta/2} \log n]$$

$$= P[|\sum_{i=1}^{k} \{U_{nki}-\mu_{nk}\}| > k(M_2 \, n^{-3/5+3\delta/2} \log n)]$$

$$= EP[|\sum_{i=1}^{k} \{U_{nki} - E(U_{nki}|Z_{n1},\dots,Z_{nk})\}|$$

$$> k(M_2 \, n^{-3/5+3\delta/2} \log n) \mid Z_{n1},\dots,Z_{nk}]$$

$$\le 2 \exp(- \frac{M_2 a}{4 \, M_3 C_1 g(\xi_0)} \log n), \tag{3.33}$$

for sufficiently large n.

Thus,

$$P(H_n^* > M_1\ n^{-3/5+3\delta/2}\ \log n)$$

$$\leq 4(b-a)n^{4/5}n^{1/5-\delta/2}\ n^{-M_2a[4M_3C_1g(\xi_0)]^{-1}} + P(S_n^c)$$

$$= 4(b-a)\ n^{1-\delta/2-M_2a[4M_3C_1g(\xi_0)]^{-1}} + \exp[-2n^{3/5-4\delta}(Bf(x_0)-b)^2]$$

(3.34)

Now choose the constants in such a way that

$$M_2a[4\ M_3\ C_1\ g(\xi_0)]^{-1} > 2 - \delta/2, \quad \text{then}$$

$$\sum_{n=1}^{\infty} P(H_n^* > M_1\ n^{-3/5+3\delta/2}\ \log n) < \infty. \tag{3.35}$$

This concludes the proof of Theorem 3.1. ∎

Lemma 3.6

$$P(\max_{k\in I_n(a,b)} |\hat{\xi}_{nk}^* - \hat{\xi}_{nk}| > a_n \quad \text{i.o.}) = 0.$$

Proof. First note that by lemma 3.4, it is enough to show

$$P(\max_{k\in I_n(a,b)} |\hat{\xi}_{nk}^* - \xi_{nk}| > C_5\ a_n) = 0, \tag{3.35}$$

for some constant C_5.

First note that $\hat{\xi}_{nk}^* \leq \xi_{nk}-a_n$ implies

$$k^{-1} \sum_{i=1}^{k} \{1(Z_{ni}^* \leq \xi_{nk} - C_5a_n) - \hat{G}_{nk}(\xi_{nk}-C_5a_n)\}$$

$$\geq \frac{[kp]}{k} - \hat{G}_{nk}(\xi_{nk}-C_5a_n). \tag{3.37}$$

By lemma 3.3 and lemma 3.4, we have

$$\frac{[kp]}{p} - \hat{G}_{nk}(\xi_{nk}-C_5 a_n)$$

$$= [\hat{G}_{nk}(\hat{\xi}_{nk}) - \hat{G}_{nk}(\xi_{nk})] + [\hat{G}_{nk}(\xi_{nk}) - \hat{G}_{nk}(\xi_{nk}-C_5 a_n)]$$

$$= [\vec{G}^*_{nk}(\hat{\xi}_{nk}) - \vec{G}^*_{nk}(\xi_{nk})] + [\vec{G}^*_{nk}(\xi_{nk}) - \vec{G}^*_{nk}(\xi_{nk}-C_5 a_n)]$$

$$+ O(n^{-3/5+3\delta/2} \log n)$$

$$= (\hat{\xi}_{nk}-\xi_{nk})\vec{g}^*_{nk}(\xi^*_{nk}) + C_5\, a_n\, \vec{g}^*_{nk}(\xi^{**}_{nk}) + O(n^{-3/5+3\delta/2} \log n). \qquad (3.38)$$

First note that both $|\xi^*_{nk}-\xi_0|$ and $|\xi^{**}_{nk}-\xi_0|$ are bounded by $C_6 a_n$ for some large constant C_6. Then by lemma 3.1, lemma 3.2 and lemma 3.4, on the set S_n, for large n

$$\min_{k\in I_n(a,b)} \{\frac{[kp]}{k} - \hat{G}_{nk}(\xi_{nk}-c_5 a_n)\} \geq ½\, g(\xi_0)a_n, \qquad (3.39)$$

choosing C_5 large enough. Hence, for large n,

$$P[\min_{k\in I_n(a,b)} (\hat{\xi}^*_{n,k}-\xi_{nk}) \leq -C_5 a_n]$$

$$\leq \sum_{k\in I_n(a,b)} EP[|k^{-1} \sum_{i=1}^{k} \{1(Z^*_{ni} \leq \xi_{nk}-C_5 a_n) - \hat{G}_{nk}(\xi_{nk}-C_5 a_n)\}|$$

$$\geq ½\, g(\xi_0)a_n \mid Z_{n1},\ldots,Z_{nk}] + P(S^c_n)$$

$$\leq 2(b-a)n^{4/5-2\delta} \exp(-½\, C_1 a\, g^2(\xi_0)(\log n)^2) + P(S^c_n), \qquad (3.40)$$

by Hoeffding (1963). Since $\sum_{n=1}^{\infty} P(S^c_n) < \infty$ by lemma 3.1 and

$$\sum_{n=1}^{\infty} n^{4/5-2\delta} \exp[-\tfrac{1}{2} C_1 a\, g^2(\xi_0)(\log n)^2] < \infty, \tag{3.41}$$

we have

$$P(\min_{k\in I_n(a,b)} (\hat{\xi}^{*}_{nk}-\xi_{nk}) \le -C_5 a_n \quad \text{i.o.}) = 0. \tag{3.42}$$

In the same way

$$P(\max_{k\in I_n(a,b)} (\hat{\xi}^{*}_{nk}-\xi_{nk}) \ge C_5 a_n \quad \text{i.o.}) = 0, \tag{3.43}$$

and the lemma is proved. ∎

Proof of Theorem 2.5

First note that Theorem 3.1, lemma 3.3 and lemma 3.4 together imply

$$P[\max_{k\in I_n(a,b)} \sup_{|z-\hat{\xi}_{nk}|\le a_n} |\{\hat{G}^{*}_{nk}(z) - \hat{G}^{*}_{nk}(\hat{\xi}_{nk})\} - \{\hat{G}_{nk}(z) - \hat{G}_{nk}(\hat{\xi}_{nk})\}| > M_4\, n^{-3/5+3\delta/2} \log n \quad \text{i.o.}] = 0, \tag{3.44}$$

for some large constant M_4.

So, by lemma 3.3 and lemma 3.4, we now have

$$\begin{aligned} p - \hat{G}^{*}_{nk}(\hat{\xi}_{nk}) &= \hat{G}_{nk}(\hat{\xi}^{*}_{nk}) - \hat{G}_{nk}(\hat{\xi}_{nk}) + R^{*0}_{nk}(1) \\ &= \vec{G}^{*}_{nk}(\hat{\xi}^{*}_{nk}) - \vec{G}^{*}_{nk}(\hat{\xi}_{nk}) + R^{*0}_{nk}(1) \\ &= (\hat{\xi}^{*}_{nk}-\hat{\xi}_{nk})\vec{g}^{*}_{nk}(\tilde{\xi}_{nk}) + R^{*0}_{nk}(1), \end{aligned} \tag{3.45}$$

where $\tilde{\xi}_{nk}$ lies between $\hat{\xi}^{*}_{nk}$ and $\hat{\xi}_{nk}$, and

$$\max_{k\in I_n(a,b)} |R^{*0}_{nk}(1)| = O(n^{-3/5+3\delta/2} \log n), \quad \text{a.s.} \tag{3.46}$$

By lemma 3.5, $\max_{k\in I_n(a,b)} |\tilde{\xi}_{nk}-\hat{\xi}_{nk}| = O(n^{-2/5+\delta} \log n)$, a.s.

Then by lemma 3.1 and lemma 3.2,

$$\max_{k\in I_n(a,b)} |\vec{g}^{*}_{nk}(\tilde{\xi}_{nk}) - g(\xi_0)| = O(n^{-2/5+\delta} \log n), \quad \text{a.s.} \tag{3.47}$$

Again letting $V_{nki} = 1(Z^{*}_{ni} \le \hat{\xi}_{nk})$, we have

$$P(\max_{k\in I_n(a,b)} |\frac{[kp]}{k} - \hat{G}^{*}_{nk}(\hat{\xi}_{nk})| > n^{-2/5+\delta} \log n)$$

$$\le \sum_{k\in I_n(a,b)} EP[|k^{-1} \sum_{i=1}^{k} \{V_{nki}-E(V_{nki}|Z_{n1},\dots,Z_{nk})\}| > n^{-2/5+\delta} \log n \mid Z_{n1},\dots,Z_{nk}]$$

$$\le 2(b-a)n^{4/5-2\delta} \exp(-2(b-a)(\log n)^2), \tag{3.48}$$

by Theorem 1 of Hoeffding (1963), and

$$\sum_{n=1}^{\infty} n^{4/5-2\delta} \exp[-2(b-a)(\log n)^2] < \infty. \tag{3.49}$$

Hence

$$p - \hat{G}^{*}_{nk}(\hat{\xi}_{nk}) = O(n^{-2/5+\delta} \log n), \quad \text{a.s.} \tag{3.50}$$

From (3.45), (3.47), (3.50), we have

$$\max_{k\in I_n(a,b)} |(\hat{\xi}^{*}_{nk}-\hat{\xi}_{nk}) - \{g(\xi_0)\}^{-1}[p-\hat{G}^{*}_{nk}(\hat{\xi}_{nk})]|$$

$$= O(n^{-3/5+3\delta/2} \log n), \quad \text{a.s.} \tag{3.51}$$

Since $\quad p - \hat{G}^{*}_{nk}(\hat{\xi}_{nk}) = k^{-1} \sum_{i=1}^{k} [1(Z^{*}_{ni} > \hat{\xi}_{nk}) - (1-p)],$

we now have the following representation

$$\hat{\xi}^{*}_{nk} - \hat{\xi}_{nk} = \{kg(\xi_0)\}^{-1} \sum_{i=1}^{k} [1(Z^{*}_{ni} > \hat{\xi}_{nk}) - (1-p)] + R^{*0}_{nk}, \tag{3.52}$$

where

$$\max_{k \in I_n(a,b)} |R^{*0}_{nk}| = O(n^{-3/5+3\delta/2} \log n), \quad \text{a.s.} \tag{3.53}$$

This representation can be modified to the following forms:

$$\hat{\xi}^{*}_{nk} - \hat{\xi}_{nk} = \{kg(\xi_0)\}^{-1} \sum_{i=1}^{k} [1(Z^{*}_{ni} > \xi_0) - (1-\hat{G}_{nk}(\xi_0))] + R^{*0}_{nk} \tag{3.54}$$

and

$$\hat{\xi}^{*}_{nk} - \hat{\xi}_{nk} = \{kg(\xi_0)\}^{-1} \sum_{i=1}^{k} [1(Z^{0*}_{ni} > \xi_0) - (1-p)] + R^{*0}_{nk}, \tag{3.55}$$

where

$$\max_{k \in I_n(a,b)} |R^{*0}_{nk}| = O(n^{-3/5+3\delta/2} \log n), \quad \text{a.s.}$$

in both (3.54) and (3.55), and $Z^{0*}_{ni} = G^{-1} \circ \hat{G}_{nk}(Z^{*}_{ni})$ and $G(\cdot) = G(\cdot|x_0)$ is the conditional cdf of Z given $X=x_0$. Note that since $Z^{*}_{n1}, \ldots, Z^{*}_{nk}$ are conditionally iid given $(Z_{n1}, \ldots, Z_{nk})$ with Z^{*}_{ni} having cdf $\hat{G}_{nk}$,

$$P(Z^{0*}_{ni} \le z_i, \quad i=1,2,\ldots,k)$$

$$= EP[\hat{G}_{nk}(Z^{*}_{ni}) \le G(z_i), \ i=1,2,\ldots k \mid Z_{n1}, \ldots, Z_{nk}]$$

$$= \prod_{i=1}^{k} G(z_i), \tag{3.56}$$

so that for each k, $Z^*_{n1},\ldots,Z^*_{nk}$ are iid with cdf $G(\cdot)$.

To obtain (3.54) from (3.53), we need to show

$$\max_{k\in I_n(a,b)} \left|k^{-1} \sum_{i=1}^{k} [\{(Z^*_{ni} \le \hat{\xi}_{nk}) - 1(Z^*_{ni} \le \xi_0)\} - \{\hat{G}_{nk}(\hat{\xi}_{nk}) - \hat{G}_{nk}(\xi_0)]\right|$$

$$= 0(n^{-3/5+3\delta/2} \log n), \quad \text{a.s.} \tag{3.57}$$

Since by lemma 3.4 and lemma 3.5,

$$\max_{k\in I_n(a,b)} |\hat{\xi}_{nk}-\xi_0| = 0(n^{-2/5+\delta} \log n), \tag{3.58}$$

the result follows immediately from (3.44).

To obtain (3.55) from (3.54), it is enough to show that

$$\max_{k\in I_n(a,b)} \left|k^{-1} \sum_{i=1}^{k} \{W_{ni}-\mu_{nk}\}\right| = 0(n^{-3/5+3\delta/2} \log n), \text{ a.s.} \tag{3.59}$$

where $W_{ni} = 1(Z^{0*}_{ni} > G^{-1} \circ \hat{G}_{nk}(\xi_0)) - 1(Z^{0*}_{ni} > \xi_0)$

and $\mu_{nk} = G(\xi_0) - \hat{G}_{nk}(\xi_0) = E(W_{ni} \mid Z_{n1},\ldots,Z_{nk})$.

Now,

$$\mu_{nk} = \{G(\xi_0) - \overline{G}^*_{nk}(\xi_0)\} + \{\overline{G}^*_{nk}(\xi_0) - \hat{G}_{nk}(\xi_0)\} \tag{3.60}$$

By lemma 2, for large n,

$|G(\xi_0) - \overline{G}^*_{nk}(\xi_0)| \le B^2|Q(\xi_0)|n^{-2/5-4\delta}$ on the set

$S_n = \{Y_{n,[n^{4/5}b]} \le Bn^{-1/5-2\delta}\}$, and hence

$$G^{*}_{ni}(\xi_0) \le G(\xi_0) + B^2|Q(\xi_0)|n^{-2/5-4\delta} \quad \text{on} \quad S_n . \tag{3.61}$$

Now use (3.61) and Bernstein's inequality to show:

$$P[\max_{k\in I_n(a,b)} |\hat{G}_{nk}(\xi_0) - \vec{G}^{*}_{nk}(\xi_0)| > n^{-2/5+\delta} \log n]$$

$$= EP[\max_{k\in I_n(a,b)} |k^{-1} \sum_{i=1}^{k} \{1(Z_{ni} > \xi_0) - \vec{G}^{*}_{ni}(\xi_0\}| > n^{-2/5+\delta} \log n \mid \mathcal{A}] + P(S_n^c)$$

$$\le 2(b-a)n^{4/5-2\delta} \exp[-2a(4G(\xi_0))^{-1} (\log n)^2] + P(S_n^c), \tag{3.62}$$

where $\mathcal{A} = \sigma(Y_1, Y_2, \ldots)$.

By lemma 3.1, this implies

$$\max_{k\in I_n(a,b)} |\hat{G}_{nk}(\xi_0) - \vec{G}^{*}_{nk}(\xi_0)| = O(n^{-2/5+\delta} \log n), \quad \text{a.s.} \tag{3.63}$$

Hence combining (3.61) and (3.63), we have

$$\max_{k\in I_n(a,b)} |\mu_{nk}| \le M_5(n^{-2/5+\delta} \log n), \quad \text{a.s.} \tag{3.64}$$

for some constant M_5.

Now, we use the Bernstein's inequality and lemma 3.1 to obtain

$$\sum_{n=1}^{\infty} P[\max_{k\in I_n(a,b)} |k^{-1} \sum_{i=1}^{k} (W_{ni}-\mu_{nk})| > n^{-3/5+3\delta/2} \log n]$$

$$\le 2(b-a) \sum_{n=1}^{\infty} n^{4/5-2\delta} \exp[-a(4\,M_5)^{-1} \log n] + \sum_{n=1}^{\infty} P(S_n^c) < \infty \tag{3.65}$$

This proves (3.59) and the representation (3.55) is established.

This completes the proof of Theorem 2.5. ∎

Proof of Theorem 2.7

First, we use (2.11) to rewrite the representation given in Theorem 2.5 as follows:

$$\tilde{\xi}^*_{nh} - \tilde{\xi}_{nh} = \{K_n(h)g(\xi_0)\}^{-1} \sum_{i=1}^{K_n(h)} [1(Z^{0*}_{ni} > \xi_0) - (1-p)] + R^{*0}_{nK_n(h)} . \qquad (3.66)$$

Our objective at this point is to show

(1) $$\sup_{h\in J_n(c,d)} |R^{*0}_{n\,K_n(h)}| = O(n^{-3/5+3\delta/2} \log n), \text{ a.s.} \qquad (3.67)$$

(2) we can replace $K_n(h)$ by $[nhf(x_0)]$ in the first term of the expansion without slowing down the rate of convergence of the remainder term.

Our approach is very similar to the one of Bhattacharya and Gangopadhyay (1988). First note:

Lemma 3.7. Let $\Delta_n(h) = K_n(h) - nhf(x_0)$. Then

$$\sup_{h\in J_n(c,d)} |\Delta_n(h)| = O(n^{2/5-\delta} \log n), \text{ a.s.}$$

Proof: Let $\mu(h) = P(Y \le h/2)$. Now write

$$\begin{aligned}\Delta_n(h) &= [\sum_{i=1}^{n} (1(Y_i \le h/2) - \mu(h))] + [n\,\mu(h) - nhf(x_0)] \\ &= \Delta_{n1}(h) + \Delta_{n2}(h), \text{ say.} \qquad (3.68)\end{aligned}$$

But, by condition 1,

$$|\mu(h) - hf(x_0)| \le (h^3/24) \{ \sup_{|x-x_0|\le h/2} |f''(x_0)|\}. \qquad (3.69)$$

Thus $$\sup_{h\in J_n(c,d)} |\Delta_{n2}(h)| = O(n^{2/5-6\delta}), \text{ a.s.} \qquad (3.70)$$

Now, since $\mu'(h) < 2f(x_0)$ on $J_n(c,d)$, we divide $J_n(c,d)$ into υ_n =

$2(d-c)\ f(x_0)\ n^{2/5-\delta}(\log n)^{-1}$ equal intervals to ensure that $n\mu(h)$ increases by at most $(n^{2/5-\delta}\log n)$ at the (υ_{n+1}) endpoints of these intervals, then $\sup_{h\in J_n(c,d)} |\Delta_{n1}(h)| \le 2n^{2/5-\delta}\log n]$. Thus

$$P[\sup_{h\in J_n(c,d)} |\Delta_{n1}(h)| > 2n^{2/5-\delta}\log n]$$

$$\le (\upsilon_n+1) \sup_{h\in J_n(c,d)} P[|\Delta_{n1}(h)| > n^{2/5-\delta}\log n] \quad (3.71)$$

Finally, since $\mu(h) = hf(x_0) + n^{-1}\ \Delta_{n2}(h) \le 2df(x_0)n^{-1/5-2\delta}$ for all $h \in J_n(c,d)$,

$$\sup_{h\in J_n(c,d)} P[|\Delta_{n1}(h)| > n^{2/5-\delta}\log n]$$

$$\le 2\ \exp[-(\log n)^2/(8\ df(x_0))], \quad (3.72)$$

by Bernstein's inequality, and $\sum_{n=1}^{\infty} n^{2/5-\delta}\exp[-a(\log n)^2] < \infty$ for all $a > 0$. Hence

$$\sup_{h\in J_n(c,d)} |\Delta_{n1}(h)| = O(n^{2/5-\delta}\log n),\ \text{a.s.} \quad (3.73)$$

and the lemma is proved. ∎

Now to prove (3.67), for $0 < c < d$, let $a = cf(x_0)/2 < 2df(x_0) = b$, and let

$$A_n = \{\sup_{h\in J_n(c,d)} |R^{*0}_{nk_n}(h)| > M\ n^{-3/5+3\delta/2}\log n\}$$

$$B_n = \{\max_{k\in I_n(a,b)} |R^{*0}_{nk}| > M\ n^{-3/5+3\delta/2}\log n\}$$

$$C_n = \{ \max_{h \in J_n(c,d)} |\Delta_n(h)| > M\, n^{2/5-\delta} \log n \}.$$

So there exists $N_0 = N_0(M)$ such that for $n > N_0$, C_n^c implies $K_n(h) \in I_n(a,b)$ for all $h \in J_n(c,d)$. Thus

$$C_n^c \cap A_n \subset C_n^c \cap B_n \quad \text{for } n > N_0.$$

It now follows that for sufficiently large M,

$$P[A_n \ \text{i.o.}] \le P[C_n \ \text{i.o.}] + P[\bigcup_{N \ge 1} \bigcap_{n \ge N} C_n^c \text{ and } A_n \ \text{i.o.}]$$

$$\le P[C_n \ \text{i.o.}] + P[\bigcup_{N > N_0} \bigcap_{n \ge N} C_n^c \text{ and } B_n \ \text{i.o.}]$$

$$= 0,$$

since for large M, $P(C_n \ \text{i.o.}) = 0$ by lemma 3.7 and $P[B_n \ \text{i.o.}] = 0$ by Theorem 2.5. This proves (3.67).

Finally, let

$$U_{ni} = 1(Z_{ni}^{*0} > \xi_0) - (1-p) \tag{3.74}$$

and

$$m_n(h) = [nhf(x_0)]. \tag{3.75}$$

Then

$$\{K_n(h)g(\xi_0)\}^{-1} \sum_{1}^{K_n(h)} U_{ni} = \{m_n(h)g(\xi_0)\}^{-1} \sum_{1}^{m_n(h)} U_{ni} + R''_{nh} + R'''_{nh}, \tag{3.76}$$

where

$$R''_{nh} = \{\Delta_n(h)/K_n(h)\}\{m_n(h)g(\xi_0)\}^{-1} \sum_{1}^{m_n(h)} U_{ni}. \tag{3.77}$$

$$R'''_{nh} = \{1-\Delta_n(h)/K_n(h)\}\{m_n(h)g(\xi_0)\}^{-1} \left[\sum_1^{K_n(h)} U_{ni} - \sum_1^{m_n(h)} U_{ni} \right], \tag{3.78}$$

where $U_{n1},\dots,U_{nn}$ are iid with mean 0. By lemma 8,

$$\sup_{h\in J_n(c,d)} |\Delta_n(h)/K_n(h)| = O(n^{-2/5+\delta} \log n), \text{ a.s.} \tag{3.79}$$

and

$$\sup_{h\in J_n(c,d)} |(m_n(h))^{-1} \sum_1^{m_n(h)} U_{ni}| = \max_{n^{4/5-2\delta}cf(x_0)\le k\le n^{4/5-2\delta}df(x_0)} |k^{-1} \sum_1^{k} U_{ni}|$$

$$= O(n^{-1/5+\delta/2}), \quad \text{a.s.} \tag{3.80}$$

by an application of Theorem 1 of Hoeffding (1963). Hence

$$\sup_{h\in J_n(c,d)} |R''_{nh}| = O(n^{-3/5+3\delta/2} \log n), \quad \text{a.s.} \tag{3.81}$$

Now consider the jump points of $m_n(h) = [nhf(x_0)]$ in $J_n(c,d)$ together with the end points $n^{-1/5-2\delta}$ and $n^{-1/5-2\delta}$, and call these points

$$n^{-1/5-2\delta} c = h_{n0} < h_{n1} < \dots < h_{n\nu_n} = n^{-1/5-2\delta} d, \tag{3.82}$$

then $\nu_n \le n^{4/5-2\delta}(d-c)f(x_0)$, $h_{n,j+1}-h_{n,j} \le (nf(x_0))^{-1}$, and $m_n(h)$ is constant on each of the ν_n intervals $[h_{nj}, h_{n,j+1})$. At the same time, $K_n(h)$ is also integer-valued and non-decreasing and $|U_{ni}| \le 1$. Hence for each j and for all $h_{nj} \le h < h_{n,j+1}$,

$$\left| \sum_1^{K_n(h)} U_{ni} - \sum_1^{m_n(h)} U_{ni} \right| = \left| \sum_1^{K_n(h_{nj})} U_{ni} - \sum_1^{m_n(h_{nj})} U_{ni} \right|$$

$$+ \left| \sum_{K_n(h_{nj})+1}^{K_n(h)} U_{ni} \right| \leq \left| \sum_{1}^{K_n(h_{nj})} U_{ni} - \sum_{1}^{m_n(h_{nj})} U_{ni} \right|$$

$$+ \{K_n(h_{n,j+1}) - K_n(h_{nj})\}. \tag{3.83}$$

Therefore, if we can show that

$$\max_{0 \leq j \leq \nu_n} \left| \sum_{1}^{K_n(h_{nj})} U_{ni} - \sum_{1}^{m_n(h_{nj})} U_{ni} \right| = O(n^{1/5-\delta/2} \log n), \text{ a.s.} \tag{3.84}$$

and

$$\max_{0 \leq j \leq \nu_n} |K_n(h_{n,j+1}) - K_n(h_{nj})| = O(n^{1/5-\delta/2} \log n), \text{ a.s.} \tag{3.85}$$

then by Lemma 3.7, we have in (3.78),

$$\sup_{h \in J_n(c,d)} |R_{nh}'''| = O(n^{-3/5+3\delta/2} \log n), \text{ a.s.}, \tag{3.86}$$

since

$$\sup_{h \in J_n(c,d)} |1-\Delta_n(h)/K_n(h)|(m_n(h))^{-1} = O(n^{-4/5+2\delta}), \text{ a.s.} \tag{3.87}$$

To prove (3.85), note that $K(h_{n,j+1}) - K(h_{nj})$ is Binomial (n, Π_{nj}), where $\Pi_{nj} = P(h_{n,j}/2 < Y_i < h_{n,j+1}/2) = n^{-1}[1+O(1)]$, since $h_{n,j+1} - h_{nj} \leq (nf(x_0))^{-1}$. Hence for large n and for all j,

$$P[K_n(h_{n,j+1}) - K_n(h_{nj}) \geq 2 M n^{1/5-\delta/2} \log n]$$

$$\leq P[n^{-1}(\{k_n(h_{n,j+1}) - K_n(h_{nj})\} - \Pi_{nj}) \geq M n^{-4/5-\delta/2} \log n]$$

$$\leq 2 \exp[-(M/4)n^{1/5-\delta/2} \log n], \tag{3.88}$$

by a variation of the Bernstein's inequality, and the fact that for

υ_n = (constant) $n^{4/5-2\delta}$,

$$\sum_{n=1}^{\infty} \upsilon_n \exp[-(M/4)n^{1/5-\delta/2} \log n] < \infty.$$

This proves (3.85).

Finally, note that by Hoeffding's inequality,

$$P[| \sum_{1}^{K_n(h_{nj})} U_{ni} - \sum_{1}^{m_n(h_{nj})} U_{ni}| > M\, n^{1/5-\delta/2} \log n]$$

$$\leq 2\, E \exp[-2\, M^2\, n^{2/5-\delta}(\log n)^2/|K_n(h_{nj})-m_n(h_{nj})|]$$

$$\leq 2\, P[|K_n(h_{nj}) - m_n(h_{nj})| > n^{2/5-\delta} \log n]$$

$$+ 2 \exp(-2\, M \log n). \tag{3.89}$$

Since

$$\sum_{n=1}^{\infty} \upsilon_n \exp[-2\, M \log n] = \text{Const.} \sum_{n=1}^{\infty} n^{4/5-2\delta-2M} < \infty; \tag{3.90}$$

for sufficiently large M, to establish (3.84), we only need to show that

$$\sum_{n=1}^{\infty} n^{4/5-2\delta}\, P[|K_n(h_{nj}) - m_n(h_{nj})| > M\, n^{2/5-\delta} \log n] < \infty \tag{3.91}$$

But $K_n(h) - m_n(h) = K_n(h) - [nhf(x_0)]$ differs from $\Delta_n(h) = K_n(h) - nhf(x_0)$ by at most 1, and it is shown in the proof of lemma 3.7 that

$$P[|\Delta_n(h)| > M\, n^{2/5-\delta} \log n] < \exp[a(\log n)^2], \tag{3.92}$$

for some $a > 0$, which implies (3.91), and thus (3.84) is established.

Now note that (3.66), (3.67), (3.76), (3.81), and (3.86) together imply the result given in Theorem 2.7. ∎

4. PROOFS OF THEOREM 2.6 AND 2.8.

Proof of Theorem 2.6.

In the representation of $\hat{\xi}^*_{nk}$ of theorem 2.5, take $k = [n^{4/5-2\delta}t] = n^{4/5-2\delta}t + \epsilon_n(t)$ with $0 \lesssim \epsilon_n(t) \lesssim 1$. After a little rearrangement of terms this leads to

$$n^{2/5-\delta}(\xi^*_{n[tn^{4/5-2\delta}]} - \hat{\xi}_{n[tn^{4/5-2\delta}]})$$

$$= \{\sqrt{p(1-p)}\, g(\xi_0)\}^{-1}\ t^{-1}\ [n^{-2/5+\delta} \sum_{i=1}^{[n^{4/5-2\delta}t]} W^*_{ni}]$$

$$+ R_{n1}(t) + R_{n2}(t), \tag{4.1}$$

where

$$W^*_{ni} = [1(Z^{0*}_{ni} > \xi_0) - (1-p)]/\sqrt{p(1-p)}, \quad i=1,2,\ldots,n, \tag{4.2}$$

are iid with mean 0 and variance 1 for each n in view of (3.56). Also, the remainder term $R_{n1}(t) = n^{2/5-\delta} R_{n,[n^{4/5-2\delta}t]}$. So,

$$\sup_{a\lesssim t\lesssim b} |R_{n1}(t)| = n^{2/5-\delta} \max_{k\in I_n(a,b)} |R^{*0}_{nk}| = 0(n^{-1/5+\delta/2} \log n), \quad \text{a.s.} \tag{4.3}$$

The other remainder term $R_{n2}(t)$ comes from the discrepancy $0 \lesssim \epsilon_n(t) \lesssim 1$ due to replacing $k = [n^{4/5-2\delta}t]$ by $n^{4/5-2\delta}t$ in the first term of the representation. But

$$\sup_{a\lesssim t\lesssim b} |n^{2/5-\delta} \sum_{1}^{[n^{4/5-2\delta}t]} W^*_{ni}| = 0_p(1), \tag{4.4}$$

hence

$$\sup_{a\le t\le b} |R_{n2}(t)| = 0_p(n^{-4/5+2\delta}) \tag{4.5}$$

Thus we have, with $\sigma = \sqrt{p(1-p)}/g(\xi_0)$, (4.6)

$$n^{2/5-\delta}(\hat{\xi}^*_{n[tn^{4/5-2\delta}]} - \hat{\xi}_{n[tn^{4/5-2\delta}]}) = \sigma\, t^{-1}[n^{-2/5+\delta} \sum_{1}^{[n^{4/5-2\delta}t]} W^*_{ni}] + o_p(1) \tag{4.7}$$

uniformly in $a \le t \le b$.

Now we use Theorem 1, page 452 of Gikhman and Skorokhod (1969) to see

$$\{n^{-2/5+\delta} \sum_{1}^{[n^{4/5-2\delta}t]} W^*_{ni},\ a \le t \le b\} \overset{\mathcal{D}}{\to} \{B(t);\ a \le t \le b\}. \tag{4.8}$$

This proves the lemma. ∎

Proof of Theorem 2.8 is exactly the same.

5. REMARKS

1. Similar results can be obtained by choosing $k = 0(n^{4/5}(\log n)^{-1})$ or $h = 0(n^{-1/5}(\log n)^{-1})$. Thus, $\delta > 0$ may be replaced by this less stringent condition.

2. If we choose $k = 0(n^{4/5})$, [such that $(n^{-4/5}\ k) \to t$, as $n\to\infty$] this will result in a non-zero bias term $(\beta\ t^2)$ in the asymptotic distribution of $\hat{\xi}_{nk}$. So under this set up, to obtain a bootstrap confidence interval of ξ_0, we need to obtain a consistent estimator of β. A consistent estimator of β may be found by replacing $f(x_0)$, $G_{xx}(\xi|x_0)$, $f'(x_0)$, $G_x(\xi|x_0)$ and $g(\xi)$ by their respective consistent estimators. However, note that if we define

$$\Delta_{nb_n} = \frac{\hat{G}_{nk}(\hat{\xi}_{nk}|x_0+b_n) - 2\,\hat{G}_{nk}(\hat{\xi}_{nk}|x_0) + \hat{G}_{nk}(\hat{\xi}_{nk}|x_0-b_n)}{2\,b_n^2} \;;$$

then it can be seen easily that for Δ_{nb_n} to be a consistent estimator of $G_{xx}(\xi|x_0)$, we need to choose b_n such that $b_n^{-2}\,O(n^{-2/5}\log n) \to 0$ as $n \to \infty$. But the corresponding rate of b_n can not be achieved with $k = O(n^{4/5})$. Similar problems are encountered in the kernel case as well if we choose the bandwidth $h = O(n^{-1/5})$.

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