CONTIGUITY IN NONPARAMETRIC ESTIMATION OF A CONDITIONAL FUNCTIONAL

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Induced order statistics play a vital role in the estimation of a functional of a conditional distribution. The popular notion of contiguity of probability measures is extended to such a conditional setup and incorporated in the study of the asymptotic theory of the estimators.

1. **Introduction.** Let $\{(X_i, Z_i),\ i \geq 1\}$ be a sequence of independent and identically distributed random vectors (i.i.d.r.v.) with a distribution function (d.f.) $\Pi(x,z)$, $(x,z) \in \mathbb{R}^2$, the real plane. Let $F(x) = \Pi(x,\infty)$, $x \in \mathbb{R}$ be the marginal d.f. of X and let $G(z|x)$ be the conditional d.f. of Z given X=x, for $z \in \mathbb{R}$, $x \in \mathbb{R}$. At a given point x_0, consider a functional of the conditional d.f. $G(\cdot|x_0)$:

$$(1.1)\quad \theta(x_0) = T(G(\cdot|x_0)),$$

and consider the problem of estimating $\theta(x_0)$ from the sample observations $(X_1,Z_1),\ldots,(X_n,Z_n)$. In this setup, the functional $T(\cdot)$ is of known form,

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while $G(\cdot|x_0)$ is not known. Our task is to construct suitable nonparametric estimators of $G(\cdot|x_0)$ and incorporate them in the formulation of suitable estimators of $\theta(x_0)$. In this context, based on a suitable norm $\|\cdot\|$ (viz., the Euclidean norm), we may set $Y_i = \|X_i - x_0\|$, $i=1,\dots,n$, so that we have the data transformation: $(X_i, Z_i) \to (Y_i, Z_i)$, $i=1,\dots,n$. Let $0 \le Y_{n1} \le \dots \le Y_{nn}$ be the order statistics corresponding to $Y_1,\dots,Y_n$, and let $Z_{n1},\dots,Z_{nn}$ be the *induced order statistics* (i.e., $Z_{ni} = Z_j$ if $Y_{ni} = Y_j$, for $i,j=1,\dots,n$). For every positive integer k ($\le n$), the *k-nearest neighbor* (k-NN) *empirical* d.f. of Z (with respect to x_0) is defined as

(1.2) $\hat{G}_{nk}(z) = k^{-1} \Sigma_{i=1}^{k} I(Z_{ni} \le z)$, $z \in \mathbb{R}$,

where I(A) stands for the indicator function of the set A. Then, for a suitably chosen k (relative to n), we consider

(1.3) $T_{nk} = T(\hat{G}_{nk})$ $(k \le n)$

as an estimator of $\theta(x_0) = T(G(\cdot|x_0))$. Although typically $T(\cdot)$ may not be a linear functional, under suitable differentiability conditions, T_{nk} may be approximated adequately by a linear one. However, the empirical d.f. $\hat{G}_{nk}$ in (1.2) is not based on i.i.d.r.v. (but on conditionally independent and non-i.d.r.v.'s), and hence, the usual treatment of i.i.d.r.v.'s may not hold here. Although, for $k(=k_n)$ increasing with n (as is usually the case), the asymptotic properties of $\hat{G}_{nk}$ may be studied by some direct analysis [viz., Bhattacharya and Gangopadhyay (1988)], we shall find it convenient to extend the notion of *contiguity* of probability measures to such a conditional setup, and to incorporate the same in a novel and considerably simpler approach to the desired asymptotic theory. Along with the preliminary notions, this contiguity is established in Section 2. Asymptotic distribution of T_{nk} in

(1.3) is considered in Section 3, and, in this context, the results of Section 2 are incorporated in a unified manner. Although the optimal rate of k_n is $O(n^{4/5})$, there is a bias term of comparable magnitude. Hence, we advocate the use of $k_n = O(n^{4/5-\eta})$, for some $\eta > 0$, where the bias is asymptotically negligible.

2. **A contiguity theorem for induced order statistics.** The following regularity conditions are assumed:

[A1] The d.f. F admits an absolutely continuous density function f, such that

(2.1) $f(x_0) > 0$,

and $f'(x) = (d/dx)f(x)$ exists in a neighborhood of x_0, and there exist positive numbers ϵ, k_0 and α, such that $|x-x_0| < \epsilon$ implies that

(2.2) $|f'(x) - f'(x_0)| \leq k_0|x-x_0|$.

[A2] The d.f. $G(z|x_0)$ has (a.a. $z \in R$) a continuous density $g(z|x_0)$ such that the partial derivatives $g_x(z|x) = (\partial/\partial x)g(z|x)$ and $g_{xx}(z|x) = (\partial^2/\partial x^2)g(z|x)$ exist in a neighborhood of x_0(a.a. z).

As in Section 1, let $Y_i = |X_i-x_0|$, $1 \leq i \leq n$, and let $F_Y(y)$, $y \geq 0$, be the marginal d.f. of Y_i. Then F_Y admits a density function

(2.3) $f_Y(y) = f(x_0+y) + f(x_0-y)$, $y \geq 0$,

so that by an appeal to (2.1) and (2.2), we conclude that [A1] holds for f_Y too. $G^*(z|y)$ (and $g^*(z|y)$) the conditional d.f. (and density) of Z, given Y=y, are given by

(2.4) $G^*(z|y) = \{f(x_0+y)\ G(z|x_0+y) + f(x_0-y)\ G(z|x_0-y)\}/f_Y(y)$,

(2.5) $g^*(z|y) = \{f(x_0+y)\ g(z|x_0+y) + f(x_0-y)\ g(z|x_0-y)\}/f_Y(y)$.

Note that

(2.6) $g^*(z|0) = g(z|x_0)$ and $G^*(z|0) = G(z|x_0)$, $z \in R$.

For notational simplicity, in the sequel, $g(z|x_0)$ and $G(z|x_0)$ will be denoted by $g(z)$ and $G(z)$, respectively.

[A3] There exist $\epsilon > 0$, $\alpha > 0$ and some Lebesgue measurable functions $u_1(z)$ and $u_2(z)$, such that for all $y : y \leq \epsilon$,

(2.7) $|(\partial/\partial y^2)\ g^*(z|y)| \leq u_1(z)$ a.a. z.,

and

(2.8) $|\{(\partial/\partial y^2)\ g^*(z|y)\} - \{(\partial/\partial y^2)\ g^*(z|y)|_{y=0}\}\ | \leq u_2(z)y^\alpha$.

[A4] There exists $\epsilon > 0$, such that

(2.9) $I(z|y) = E[\{(\partial/\partial y^2) \log g^*(z|y)\}^2 \mid Y=y] < \infty$,

uniformly in $y : 0 \leq y \leq \epsilon$.

In passing, we may remark that for [A3] to hold the following conditions suffice: There exist $\epsilon > 0$, $\alpha > 0$ and Lebesgue measurable functions (on $\mathbb{R}$) $u_1^*(z)$, $u_2^*(z)$ and $u_3^*(z)$, such that for $|x-x_0| \leq \epsilon$,

(2.10) $|g_x(z|x)| \leq u_1^*(z)$, $|g_{xx}(z|x)| \leq u_2^*(z)$;

(2.11) $|g_{xx}(z|x) - g_{xx}(z|x_0)| \leq u_3^*(z)\,|x-x_0|^\alpha$.

However, expressing in terms of $(\partial/\partial y^2)g^*(z|y)$, we are able to bring the analogy with the classical (location/regression/scale) case treated in Hájek and Sidák (1967).

Let us now denote by

(2.12) $g^*(z|Y_{ni}) = g_{ni}(z)$ and $G^*(z|Y_{ni}) = G_{ni}(z)$, $1 \leq i \leq n$.

Also, for every k ($\leq$n), let q_{nk} and Q_{nk} denote the joint conditional density and distribution of $Z_{n1},\dots,Z_{nk}$, given $Y_{n1},\dots,Y_{nk}$. Note that [viz., Bhattacharya (1974)] given $(Y_{n1},\dots,Y_{nn})$, $Z_{n1},\dots,Z_{nn}$ are conditionally independent, so that

$$(2.13)\quad q_{nk}(\cdot) = \prod_{i=1}^{k} g_{ni}(\cdot) \quad\text{and}\quad Q_{nk}(\cdot) = \prod_{i=1}^{k} G_{ni}(\cdot).$$

Let $Z^0_{n1},\dots,Z^0_{nk}$ be i.i.d.r.v.'s with a density (and d.f.) $g(z)$ and $G(z)$, and let

$$(2.14)\quad p_{nk}(\cdot) = \prod_{j=1}^{k} g(\cdot) \quad\text{and}\quad P_{nk}(\cdot) = \prod_{j=1}^{k} G(\cdot).$$

Thus, in (2.13) we conceive of the actual conditional density of the Z_{nj} (given the Y_{nj}), while in (2.14), we have an i.i.d. model with the (conditional) density $g(z)$ $(= g(z|x_0))$. In this conditional setup, we consider the following.

<u>Theorem 2.1.</u> *Let* $k = [tn^{\lambda}]$ *for some* λ $(0 < \lambda \le 4/5)$ *and for some* $t(0<a<t<b<\infty)$. *Then under the regularity conditions* [A1]-[A4], *the densities* q_{nk} *are contiguous to the densities* p_{nk}.

A detailed proof of the theorem is given in section 4.

3. Asymptotic distribution of $T(\hat{G}_{nk})$. By the use of the contiguity established in Theorem 2.1, we will be able to study the asymptotic distribution of $T(\hat{G}_{nk})$ under the usual regularity conditions on $T(\cdot)$ and some other conditions on $g^*(z|y)$; these are therefore introduced first.

Let $\mathcal{L}(A,B)$ be a set of continuous linear transformations from a topological vector space A to another B, and let C be a class of compact subsets of A, such that every subset consisting of a single point belongs to C. For $G \in A$ and for every $H \in K \in C$, we assume that

$$(3.1)\quad T(H) = T(G + (H-G)) = T(G) + \int T_1(G;z,x_0)\, d[H(z)-G(z)] + R_1(G;H-G),$$

where $G(\cdot) = G(\cdot|x_0)$ and

$$(3.2)\quad |R_1(G;\, H-G)| = o(\|H-G\|), \text{ uniformly in } H \in K,$$

and $\|\cdot\|$ stands for the "sup-norm". This relates to the so called first order

Hadamard-differentiability of $T(\cdot)$ at G, and $T_1(G;z,x_0)$ is called the first order compact (or Hadamard) derivative of $T(\cdot)$ at G; it is so normalized that

(3.3) $\int T_1(G;z,x_0)\, dG(z|x_0) = 0 \quad (\text{a.a. } x_0)$

Similarly, if we assume that

(3.4) $$T(H) = T(G) + \int T_1(G;z,x_0)\, d[H(z)-G(z)] + \tfrac{1}{2} \iint T_2(G;z,z',x_0)\, d[H(z)-G(z)]d[H(z')-G(z')] + R_2(G;H-G), \ \forall\, H \in K \in C,$$

where

(3.5) $\|R_2(G;H-G)\| = o(\|H-G\|^2)$, uniformly in $H \in K$,

then $T(\cdot)$ is second order Hadamard differentiable at G and $T_2(G;z,z',x_0)$ is the second order compact (or Hadamard) derivative of $T(\cdot)$ at G. We may set

(3.6) $T_2(G;z,z',x_0) \equiv T_2(G;z',z,x_0)$,

(3.7) $\int T_2(G;z,z',x_0)\, dG(z') = 0 = \int T_2(G;z,z',x_0)\, dG(z) \quad (\text{a.e.})$

Further, bearing in mind the contiguity in Theorem 2.1, we make the following assumption.

[B1] There exist $\epsilon > 0$, $\alpha > 0$ and $A < \infty$ such that uniformly in $y (0 \le y \le \epsilon)$,

(3.8) $\left|\int T_1(G;z,x_0)\, \{(\partial/\partial y^2) g^*(z|y)\}\, dz\right| \le A;$

(3.9) $$\left|\int T_1(G;z,x_0)\, \{(\partial/\partial y^2) g^*(z|y)\}\, dz - \int T_1(G;z,x_0) \{(\partial/\partial y^2) g^*(z|y)|_{y=0}\}\, dz\right| \le Ay^{\alpha}.$$

As in (2.10)-(2.11), the terms involving $(\partial/\partial y^2)$ may be replaced by $(\partial/\partial x)$ and $(\partial^2/\partial x^2)$. Then, we have the following.

Theorem 3.1. *Suppose that* (i) $T(\cdot)$ *is first order Hadamard-differentiable at* $G(\cdot|x_0)$, (ii) $k=k_n=[tn^{4/5}]$ *for some* $0 < a < t < b < \infty$, (iii) *the assumptions*

[A1]-[A4] and [B1] hold, and

(3.10) $0 < \sigma^2(x_0) = \int T_1^2\,(G;z,x_0)\, g(z|x_0)\, dz < \infty$.

Then, as $n \to \infty$,

(3.11) $n^{2/5}[T(\hat{G}_{nk}) - T(G)] \overset{\mathcal{D}}{\to} \mathcal{N}(\mu(x_0),\ \sigma^2(x_0)/t)$,

where

(3.12) $\mu(x_0) = t^2\{12f^2(x_0)\}^{-1} \int_{\mathbb{R}} T_1(G;z,x_0)q(z,x_0)\, dz$

and

(3.13) $q(z,x_0) = \frac{1}{2}\{g_{xx}(z|x_0) + 2f'(x_0)g_x(z|x_0)/f(x_0)\};\ z \in \mathbb{R}$.

In passing, we may remark that if in (3.11), we replace $n^{2/5}$ by $k^{1/2}$, then we would have

(3.14) $k^{1/2}[T(\hat{G}_{nk}) - T(G)] \overset{\mathcal{D}}{\to} \mathcal{N}(t^{1/2}\,\mu(x_0),\ \sigma^2(x_0))$.

In this form, the result also extends to the following.

<u>Corollary 3.1.1.</u> If $k = o(n^{4/5})$ and (i), (iii) and (3.10) hold, then,

(3.15) $k^{1/2}[T(\hat{G}_{nk}) - T(G)] \overset{\mathcal{D}}{\to} \mathcal{N}(0,\sigma^2(x_0))$, as $n \to \infty$.

The main difference between (3.14) and (3.15) is that for $k = O(n^{4/5})$, the asymptotic normal distribution has a non-vanishing centering constant, while for $k = o(n^{4/5})$, this bias term is asymptotically insignificant. This feature plays an important role in the choice of $\{k_n\}$. We consider the following lemmas and incorporate them in the proof of the theorem and corollary.

<u>Lemma 3.1.</u> Let $k = [tn^\lambda]$ for some λ $(0< \lambda<1)$ and $t(0<a<t<b<\infty)$. If the assumption [A1] holds, then

(a) for every B such that $Bf(x_0) > b > a > 0$, there exists an n_0 $(< \infty)$ such that

(3.16) $P[Y_{n[bn^{\lambda}]} > B\, n^{-1+\lambda}] \le \exp[-2\, n^{-1+2\lambda}\, (Bf(x_0)-b)^2], \quad \forall\, n \ge n_0.$

(b) If $U_{n1} < U_{n2} < \ldots < U_{nn}$ denote the order statistics in a random sample $U_1, \ldots, U_n$ of size n from uniform distribution (0,1), then for $B > b$, there exists an n_0 ($<\infty$) such that

(3.17) $P[U_{n[bn^{\lambda}]} > B\, n^{-1+\lambda}] \le \exp[-2\, n^{-1+2\lambda}(B-b)^2], \quad \forall\, n \ge n_0.$

Hence

(3.18) $U_{n[bn^{\lambda}]}$ and $Y_{n[bn^{\lambda}]}$ are $O(n^{-1+\lambda})$.

For a proof of lemma 3.1, we may refer to Bhattacharya and Mack (1987).

Lemma 3.2. Let $k = [tn^{4/5}]$ for $0<a<t<b<\infty$. Suppose that the assumption [A1] holds, then

(3.19) $k^{-1} \sum_{i=1}^{k} Y_{ni}^2 = \{12\, f^2(x_0)\}^{-1}\, (k/n)^2 + R_{nk},$

where

(3.20) $\max_{k \le [n^{4/5}b]} |R_{nk}| = o(n^{-2/5})$ a.s.

A similar result was proved by Bhattacharya and Mack (1987) assuming $f''(x)$ exists and is continuous at x_0. However, that proof can be easily modified so that the result will remain true under [A1].

Lemma 3.3. (a) under [A1], [A2] and [B1], we have

(3.21) $g^*(z|y) = g(z|x_0) + y^2\, q(z,x_0) + y^2\, r(z,y,x_0),$

where

(3.22) $q(z,x_0) = (\partial/\partial y^2)\, g^*(z|y)\big|_{y=0}$

$= \tfrac{1}{2}[g_{xx}(z|x_0) + 2\, f'(x_0) g_x(z|x_0)/f(x_0)].$

and

(3.23) $r(z,y,x_0) = (\partial/\partial y^2)\, g^*(z|y)\Big|_{y^*} - (\partial/\partial y^2)\, g^*(z|y)\Big|_{y=0}$,

for some y^* between 0 and y. Moreover, there exists $\epsilon > 0$ and M $(0 < M < \infty)$ such that uniformly for all $y(0 < y < \epsilon)$, and for some $\alpha > 0$

(3.24) $|\int T_1(G;z,x_0)\, q(z,x_0)\, dz| \le M$

(3.25) $|\int T_1(G;z,x_0)\, r(z,y,x_0)\, dz| \le M\, y^{\alpha}$.

(b) If in addition, we assume [A3], then for $0 < y < \epsilon$

(3.26) $\int_{-\infty}^{\infty} q(z,x_0)\, dz = 0$

and

(3.27) $\int_{-\infty}^{\infty} r(z,y,x_0)\, dz = 0$.

<u>Proof</u>: The expansion in (3.21) follows from a Taylor's series expansion of $g^*(z|y)$ about y=0. (3.22) can be established by expanding $f(x_0 \pm y)$ and $g(z|x_0 \pm y)$ about x_0, substituting the expansions in (2.5), taking derivative with respect to y^2 and evaluating the derivative at y=0. (3.24) and (3.25) are easy consequences of [B1]. (3.26) and (3.27) follow from the fact that [A3] allows us to interchange the integral and the derivative. ■

<u>Proof of Theorem 3.1</u> Considering the expansion in (3.1), we have

(3.28) $T(\hat{G}_{nk}) - T(G) = \int T_1(G;z,x_0)\, d[\hat{G}_{nk}(z)-G(z)] + o(\|\hat{G}_{nk}-G\|)$.

Since by Theorem 2.1, the densities q_{nk} are contiguous to the densities p_{nk}, and under p_{nk}

(3.29) $k^{1/2}\|\hat{G}_{nk}-G\| = 0_p(1)$,

it follows that under q_{nk}

(3.30) $k^{1/2}\|\hat{G}_{nk}-G\| = 0_p(1)$.

So, we can rewrite (3.28) as

(3.31) $T(\hat{G}_{nk}) - T(G) = k^{-1} \sum_{i=1}^{k} T_1(G;Z_{ni},x_0) + o_p(n^{-2/5})$.

or,

$$(3.32)\quad k^{1/2}\,[T(\hat{G}_{nk}) - T(G)] = \sigma(x_0)\, S_{nk} + o_p(1),$$

where

$$(3.33)\quad S_{nk} = k^{-1/2} \sum_{i=1}^{k} \phi(Z_{ni})$$

and

$$(3.34)\quad \phi(Z_{ni}) = \{\sigma(x_0)\}^{-1}\, T_1(G;Z_{ni},x_0).$$

Note that for all sequences $\{M_n\}$ of positive numbers tending to infinity

$$(3.35)\quad \int_{|z|>M_n} \phi^2(z)\, dG(z) \to 0$$

which in turn ensures that under p_{nk}

$$(3.36)\quad S_{nk} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Our objective is to show that there exists a sequence $\{a_{nk}\}$ of real numbers such that under q_{nk}

$$(3.37)\quad S_{nk} - a_{nk} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

Define

$$(3.38)\quad \psi'_{nk}(z) = \begin{cases} \phi(z) & \text{if } |\phi(z)| < k^{1/6}, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$(3.39)\quad \psi_{nk}(z) = \psi'_{nk}(z) - \int \psi'_{nk}(z) g(z)\, dz.$$

It can be shown easily that

$$(3.40)\quad \int \psi_{nk}(z) g(z)\, dz = 0,$$

$$(3.41)\quad \int (\psi_{nk}(z) - \phi(z))^2\, g(z)\, dz \to 0,$$

$$(3.42)\quad k^{-1/2} \sup_z \psi^2_{nk}(z) \to 0.$$

Now, if we define

(3.43) $h_{ni}(Z_{ni}) = \psi_{nk}(Z_{ni})$ and $\sigma_{ni} = k^{-½}$ for all i=1,2,...,k such that

(3.44) $S_{nk} = \sum_{i=1}^{k} \sigma_{ni}\, h_{ni}(Z_{ni})$,

then the proof of (3.37) will follow from the proof of the central limit theorem under contiguous alternatives in Behnen and Neuhaus (1975). Hence, we omit the details. However, note that the proof of their theorem shows that the sequence a_{nk} in (3.37) may have the form

(3.45) $a_{nk} = \sum_{i=1}^{k} \sigma_{ni} \int h_{ni}(z) g_{ni}(z)\, dz = k^{-½} \sum_{i=1}^{k} \int \psi_{nk}(z) g_{ni}(z)\, dz.$

Finally, we show that as $n \to \infty$

(3.46) $a_{nk} \to \{12\, f^2(x_0)\}^{-1}\, t^{5/2} \int_{-\infty}^{\infty} \phi(z)\, q(z,x_0)\, dz.$

Using the expansion in lemma 3.3., along with (3.45) and lemma 3.2, we have

$$
\begin{aligned}
(3.47)\quad a_{nk} &= k^{-½} \sum_{i=1}^{k} Y_{ni}^2 \int \psi_{nk}(z) q(z,x_0)\, dz \\
&\quad + k^{-½} \sum_{i=1}^{k} Y_{ni}^2 \int \psi_{nk}(z) r(z,Y_{ni},x_0)\, dz \\
&= k^{-½}[\{12\, f^2(x_0)\}^{-1}\, (k/n)^2 + o(n^{-2/5})] \int \psi_{nk}(z) q(z,x_0)\, dz \\
&\quad + k^{-½} \sum_{i=1}^{k} Y_{ni}^2 \int \psi_{nk}(z)\; r(z,Y_{ni},x_0)\, dz.
\end{aligned}
$$

By (3.39) and lemma 3.3; we have

$$
\begin{aligned}
(3.48)\quad \int \psi_{nk}\, q(z,x_0) dz &= \int \psi'_{nk}(z)\, q(z,x_0)\, dz - \left(\int \psi'_{nk}(z) g(z) dz\right)\left(\int q(z,x_0) dz\right) \\
&= \int_{|\phi(z)| \le k^{1/6}} \phi(z)\, q(z,x_0)\, dz \\
&\to \int \phi(z)\, q(z,x_0) dz < \infty,
\end{aligned}
$$

and

(3.49) $\int \psi_{nk}(z)\; r(z,y_{ni},x_0) dz$

$$= \int \psi'_{nk}(z)\ r(z,Y_{ni},x_0)\ dz - (\int \psi'_{nk}(z)dz)(\int r(z,Y_{ni},x_0)\ dz)$$
$$= o(1).$$

Hence by lemma 3.1, we have

(3.50) $$k^{-1/2} \sum_{i=1}^{k} Y_{ni}^2 \int \psi_{nk}(z)\ r(z,Y_{ni},x_0)\ dz = o(1) \quad \text{a.s.}$$

Combining (3.47), (3.48) and (3.50), we have

(3.51) $$a_{nk} \to \{12\ f^2(x_0)\}^{-1}\ t^{5/2} \int_{-\infty}^{\infty} \Phi(z)q(z,x_0)\ dz, \text{ as } n \to \infty,$$

where $k = [tn^{4/5}]$ for some t $(0 < a < t < b < \infty)$.

Now (3.32), (3.37) and (3.51) together imply

(3.52) $$n^{2/5}[T(\hat{G}_{nk}) - T(G)]$$
$$\xrightarrow{\mathcal{D}} \mathcal{N}[t^2\{12\ f^2(x_0)\}^{-1} \int_{-\infty}^{\infty} T_1(G;z,x_0)q(z,x_0)\ dz,\ \sigma^2(x_0)/t].$$

This completes the proof of theorem 3.1. ■

The proof of Corollary 3.1.1. follows on the same line and hence is omitted.

Next we consider the limiting distribution of $T(\hat{G}_{nk})$ under degenerate case; i.e., when the functional T is second order compact differentiable, and satisfies the following conditions

(3.53) $$\sigma^2(x_0) = \int T_1^2(G;z,x_0)\ g(z)\ dz = 0$$

and

(3.54) $$0 < \iint T_2^2(G;z_1,z_2,x_0)\ g(z_1)\ g(z_2)\ dz_1\ dz_2 < \infty.$$

Let $\{\lambda_\ell(x_0),\ \ell = 0,1,2,\dots\}$ denote the finite or infinite collection of eigenvalues of $T_2(G;\cdot,\cdot,x_0)$ corresponding to orthonormal eigenfunctions $\{\tau_\ell(\cdot,x_0);\ \ell = 0,1,2,\dots\}$, such that

(3.55) $$\int T_2(G;z_1,z_2,x_0)\ \tau_\ell(z_1,x_0)\ dG(z_1) = \lambda_\ell(x_0)\ \tau_\ell(z_2,x_0) \quad \text{a.e. (G)}$$

for all $\ell \geq 0$.

(3.56) $\int \tau_\ell(z,x_0)\ \tau_m(z,x_0)\ dG(z) = \delta_{\ell m}$,

where

(3.57) $\delta_{\ell m} = 1$ if $\ell = m$; $= 0$ if $\ell \neq m$.

In addition, let [B2] relate to the following conditions:

(3.58) $\sum_{\ell=0}^{\infty} \lambda_\ell(x_0) < \infty$;

and there exist $\epsilon > 0$, $\alpha > 0$ and $A < \infty$ such that for $0<y<\epsilon$ and for all $\ell \geq 0$,

(3.59) $\left|\int \tau_\ell(z,x_0)\ \{(\partial/\partial y^2)\ g^*(z|y)|_{y=0}\}\ dz\right| \leq A$,

(3.60) $\left|\int \tau_\ell(z,x_0)\ \{(\partial/\partial y^2)\ g^*(z|y)\}\ dz - \int \tau_\ell(z,x_0)\ \{(\partial/\partial y^2)\ g^*(z|y)|_{y=0}\}\ dz\right| \leq Ay^\alpha$.

Remark: A sufficient condition for (3.59) and (3.60) to hold is given by the following:

(3.61) $\left|\int \tau_\ell(z,x_0)\ g_x(z|x_0)\ dz\right| \leq A$,

(3.62) $\left|\int \tau_\ell(z,x_0)\ g_{xx}(z|x_0)\ dz\right| \leq A$,

and

(3.63) $|\tau_\ell(z,x_0)\ g_{xx}(z|x)\ dz - \int \tau_\ell(z,x_0)\ g_{xx}(z|x_0)\ dz| \leq A|x-x_0|^\alpha$,

for $|x-x_0| < \epsilon$.

The following lemma can be proved using an argument similar to the one used in lemma 3.3.

Lemma 3.4. Under the assumptions [A1], [A2] and [B2], the expansion (3.21) for $g^*(z|y)$ holds, and there exists $\epsilon > 0$ and M $(0 < M < \infty)$ such that for $|x-x_0| < \epsilon$,

(3.64) $|\tau_\ell(z,x_0)\ q(z,x_0)\ dz| \leq M$

and

(3.65) $|\tau_\ell(z,x_0)\ r(z,y,x_0)\ dz| \leq My^\alpha$.

If, in addition, we assume [A3], then for $|x-x_0| < \epsilon$

(3.60) $\int_{-\infty}^{\infty} q(z,x)\,dz = 0$ and $\int_{-\infty}^{\infty} r(z,y,x)\,dz = 0.$

<u>Theorem 3.2.</u> *Suppose that* (i) $T(\cdot)$ *is second order Hardamard-differentiable a* $G(\cdot|x_0)$, (ii) $k=k_n=[tn^{4/5}]$ *for some* $0<a<t<b<\infty$, (iii) *the assumptions* [A1]-[A4] and [B2] *hold, and* (iv) *the functional* $T(\cdot)$ *satisfies* (3.53)-(3.56). *Then as* $n \to \infty$,

$$(3.67)\quad 2n^{4/5}[T(\hat{G}_{nk}) - T(G) - k^{-1}\sum_{i=1}^{k} T_1(G;Z_{ni},x_0)] \overset{\mathcal{D}}{\to} \sum_{\ell=0}^{\infty} \lambda_\ell(x_0)\,(Z_\ell^0+a_\ell^0)^2,$$

where $Z_0^0, Z_1^0, \ldots$ *are i.i.d. normal random variables with mean* 0 *and variance* $(1/t)$ *and*

$$(3.68)\quad a_\ell^0 = t^2\{12\, f^2(x_0)\}^{-1} \int_{\mathbb{R}} \tau_\ell(z,x_0)\, q(z,x_0)\,dz.$$

Note that we can express (3.67) also in the following form

$$(3.69)\quad 2k[T(\hat{G}_{nk}) - T(G) - k^{-1}\sum_{i=1}^{k} T_1(G;Z_{ni},x_0)]$$

$$\overset{\mathcal{D}}{\to} \sum_{\ell=0}^{\infty} \lambda_\ell(x_0)\,(Z_\ell^{0*} + t^{1/2}\, a_\ell^0)^2,$$

where $Z_0^{0*}, Z_1^{0*}, \ldots$ are i.i.d. standard normal random variables.

In this form, the result also extends to the following.

<u>Corollary 3.2.1.</u> If $k = o(n^{4/5})$ and (i), (iii) and (iv) hold, then as $n \to \infty$,

$$(3.70)\quad 2k[T(\hat{G}_{nk}) - T(G) - k^{-1}\sum_{i=1}^{k} T_1(G;Z_{ni},x_0)] \overset{\mathcal{D}}{\to} \sum_{\ell=0}^{\infty} \lambda_\ell(x_0)\,(Z_\ell^{0*})^2.$$

As in the Theorem 3.1, we note that the main difference between (3.69) and (3.70) is that for $k = O(n^{4/5})$, the asymptotic distribution has a non vanishing centering constant, while for $k = o(n^{4/5})$, this bias term is insignificant.

Proof. Using the expansion given in (3.4), we have

$$(3.71)\quad T(\hat{G}_{nk}) - T(G) - k^{-1} \sum_{i=1}^{k} T_1(G;Z_{ni},x_0)$$

$$= (1/2k^2) \sum_{i=1}^{k} \sum_{j=1}^{k} T_2(G;Z_{ni},Z_{nj},x_0) + o(\|\hat{G}_{nk}-G\|^2).$$

As we have seen in the proof of Theorem 3.1, under q_{nk},

$$(3.72)\quad o(\|\hat{G}_{nk}-G\|^2) = o_p(n^{-4/5})$$

so that

$$(3.73)\quad 2k[T(\hat{G}_{nk}) - T(G) - k^{-1} \sum_{i=1}^{k} T_1(G;Z_{ni},x_0)]$$

$$= k^{-1} \sum_{i\neq j}^{k}\sum T_2(G;Z_{ni},Z_{nj},x_0) + k^{-1} \sum_{i=1}^{k} T_2(G;Z_{ni},Z_{ni},x_0) + o_p(1).$$

Define

$$(3.74)\quad T_2^*(G) = \int T_2(G;z,z,x_0)\, g(z)\, dz,$$

then by Dunford and Schwartz (1963) page 1087,

$$(3.75)\quad T_2^*(G) = \sum_{\ell\geq 0} \lambda_\ell(x_0) < \infty.$$

Under p_{nk}, by SLLN

$$(3.76)\quad k^{-1} \sum_{i=1}^{k} T_2(G;Z_{ni},Z_{ni},x_0) \to T_2^*(G) = \sum_{\ell\geq 0} \lambda_\ell(x_0), \qquad \text{a.s.}$$

Since the densities q_{nk} are contiguous to the densities p_{nk}, it follows that under q_{nk}

$$(3.77)\quad k^{-1} \sum_{i=1}^{k} T_2(G;Z_{ni},Z_{ni},x_0) \to \sum_{\ell\geq 0} \lambda_\ell(x_0), \text{ in probability.}$$

Now define

$$(3.78)\quad T_2^R(G;s,t,x_0) = \sum_{\ell=0}^{R} \lambda_\ell(x_0)\, \tau_\ell(s,x_0)\, \tau_\ell(t,x_0),$$

and

(3.79) $T_{nkR} = k^{-1} \sum_{i \neq j}^{k} \sum T_2^R(G;s,t,x_0) = \sum_{\ell=0}^{R} \lambda_\ell(x_0)[\xi_{nk\ell}^2 - C_{nk\ell}]$

where

(3.80) $\xi_{nk\ell} = k^{-1/2} \sum_{i=1}^{k} \tau_\ell(Z_{ni})$

(3.81) $C_{nk\ell} = k^{-1} \sum_{i=1}^{k} \tau_\ell^2(Z_{ni})$.

Using (3.56), the fact that the densities q_{nk} are contiguous to the densities p_{nk} and SLLN, we have under q_{nk}

(3.82) $C_{nk\ell} \to 1$ in probability.

By lemma 3.1, lemma 3.2 and lemma 3.4, we have under q_{nk}

(3.83) $$E(\xi_{nk\ell}) = k^{1/2}[k^{-1} \sum_{i=1}^{k} \int \tau_\ell(z,x_0)\, g_{ni}(z)\, dz]$$

$$= k^{1/2}[k^{-1} \sum_{i=1}^{k} Y_{ni}^2 \int \tau_\ell(z,x_0)q(z,x_0)\, dz$$

$$+ k^{-1} \sum_{i=1}^{k} Y_{ni}^2 \int \tau_\ell(z,x_0)r(z,Y_{ni},x_0)\, dz]$$

$$\to \{12\, f^2(x_0)\}^{-1}\, t^{5/2} \int \tau_\ell(z,x_0)q(z,x_0)dz$$

$$= a_\ell^* \text{ (say), a.s., } \forall\, \ell = 0,1,\ldots,R.$$

Now using an argument similar to the one used in proving Theorem 3.1, we have (under q_{nk})

(3.84) $(\xi_{nk1},\ldots,\xi_{nkR}) \xrightarrow{\mathfrak{D}} (Z_1^* + a_1^*,\ldots,Z_R^* + a_R^*)$,

where $Z_1^*,\ldots,Z_R^*$ are standard normal random variables.

So, it follows from (3.79), that for fixed R, under q_{nk}

(3.85) $T_{nkR} \xrightarrow{\mathfrak{D}} \sum_{\ell=0}^{R} \lambda_\ell(x_0)[(Z_\ell^* + a_\ell^*)^2 - 1]$

From Gregory (1977), we have that (3.85) implies that under q_{nk}

$$(3.86)\quad k^{-1} \sum_{i\neq j}^{k}\sum T_2(G;Z_{ni},Z_{nj},x_0) \overset{\mathcal{D}}{\to} \sum_{\ell=0}^{\infty} \lambda_\ell(x_0)[(Z_\ell^*+a_\ell^*)^2 - 1].$$

Finally, (3.67) follows from (3.73), (3.77), (3.86) and the fact that $k = [tn^{4/5}]$ for fixed $t(0<a<t<b<\infty)$.

This completes the proof of theorem 3.2. ■

The proof of Corollary 3.2.1. follows the same line of argument and hence is omitted.

We end this section with the following observation. We define a kernel estimator (with uniform kernel) of the conditional distribution $G(z|x_0)$ as follows:

$$(3.87)\quad \hat{G}_{nh}(z) = \{K_n(h)\}^{-1} \sum_{i=1}^{K_n(h)} 1(Z_{ni} \le z),$$

where h is the bandwidth and

$$(3.88)\quad K_n(h) = \sum_{i=1}^{n} 1(Y_{ni} \le h/2);$$

and, we may consider a kernel estimator of T(G) as

$$(3.89)\quad T_{nh} = T(\hat{G}_{nh}).$$

This situation is same as the k-nearest neighbor case, except that here $k=K_n(h)$ is a random variable. However, as Bhattacharya and Gangopadhyay (1988) have noted, if we choose $h = O(n^{-1/5})$, then the difference $n^{-4/5}[K_n(h)-nhf(x_0)]$ is asymptotically negligible. So, using a technique similar to theirs, the asymptotic distribution results of $T(\hat{G}_{nh})$ will follow from Theorem 2.1, Theorem 3.1 and Theorem 3.2.

4. Proof of the Theorem 2.1. First, we establish the following lemma:

Lemma 4.1 Under the assumptions [A1], [A2] and [A3], the expansion (3.21) for $g^*(z|y)$ hold. Also there exist $\epsilon > 0$ and Lebesgue integrable functions (on $\mathbb{R}$) $u_1(z)$ and $u_2(z)$ such that for $0<y<\epsilon$, $|q(z,x_0)|$ and $|r(z,y,x_0)|$ are bounded by $u_1(z)$ and $u_2(z)y^\alpha$ respectively, and

(4.1) $\int q(z,x_0)\,dz = 0, \quad \int r(z,y,x_0)\,dz = 0.$

The proof of the lemma is exactly the same as the proof of lemma 3.3.

Now, define

$$L_{nk} = q_{nk}/p_{nk} \quad \text{if } p_{nk} > 0$$
$$= 1 \quad \text{if } p_{nk} = q_{nk} = 0$$
$$= \infty \quad \text{if } p_{nk} = 0 < q_{nk}.$$

Then along the fashion of LeCam's first lemma (see, e.g., Hájek and Sidák (1967), page 203-204), we shall prove the following:

Lemma 4.2 $\log L_{nk}$ is asymptotically normal $(-\frac{1}{2}d^2, d^2)$, so that the densities q_{nk} are contiguous to the densities p_{nk}.

For this purpose, first note that

(4.4) $$\log L_{nk} = \sum_{i=1}^{k} \log[g_{ni}(Z_{ni})/g(Z_{ni})].$$

In the next lemma we will show that the summands in (4.4) are uniformly asymptotically negligible (UAN).

Lemma 4.3 Assume [A1]-[A4]. Then under p_{nk},

(4.5) $$\lim_{n\to\infty} \max_{1\le i\le k} P[|\{g_{ni}(Z_{ni})/g(Z_{ni})\} - 1| > \epsilon) = 0.$$

Proof: By Chebyehev's inequality

(4.6) $$P[|\{g_{ni}(Z_{ni})/g(Z_{ni})\} - 1| > \epsilon] \le E[\{g_{ni}(Z_{ni})/g(Z_{ni})\}-1]^2/\epsilon^2.$$

But by lemma 4.1, lemma 3.1 and [A4], under p_{nk}

(4.7) $E[\{g_{ni}(Z_{ni})/g(Z_{ni})\}-1]^2$

$$= Y_{ni}^4[\int \{q^2(z,x_0)/g(z)\}\, dz + \int \{r^2(z,Y_{ni},x_0)/g(z)\}\, dz + 2\int \{q(z,x_0)r(z,Y_{ni},x_0)/g(z)\}\, dz]$$

$$= o(1).$$ ■

From this point on, we restrict ourselves to the case $\lambda=4/5$ (i.e., $k = [tn^{4/5}]$ for $0<a<t<b<\infty$). At the end of this section we will indicate how the proof can be modified when $\lambda < 4/5$.

Lemma 4.4 Let

(4.8) $$W_{nk} = \sum_{i=1}^{k} \{[g_{ni}(Z_{ni})/g(Z_{ni})]^{1/2} - 1\}.$$

Assume [A1]-[A4] hold and the statistics W_{nk} are asymptotically normal $(-\frac{1}{4} d^2, d^2)$ under p_{nk}, then under p_{nk}

(4.9) $$\lim_{n\to\infty} P(|\log L_{nk} - W_{nk} + \tfrac{1}{4} d^2| > \epsilon) = 0, \quad \text{for every } \epsilon > 0,$$

and $\log L_{nk}$ is asymptotically normal $(-\frac{1}{2} d^2, d^2)$.

Proof: Since the UAN condition is satisfied (by lemma 4.3), the result follows from LeCam's second lemma (see, e.g., Hájek and Sidák (1967) page 205). ■

Thus, it is enough to show that under p_{nk}, W_{nk} is asymptotically normal $(-\frac{1}{4} d^2, d^2)$. For this, first we prove a series of results.

Lemma 4.5 Assume [A1]-[A4] hold. Then under p_{nk}

(4.10) $$E(W_{nk}) \to -(1/4)d^2,$$

where

(4.11) $$d^2 = \{80\, f^4(x_0)\}^{-1}\, t^5 \int_{-\infty}^{\infty} \{q^2(z,x_0)/g(z)\}dz.$$

Proof: Denote

(4.12) $$s(z|y) = [g(z|y)]^{1/2}$$

and

(4.13) $s(z) \equiv s(z|0) = [g(z)]^{½}$.

So, under p_{nk}

(4.14) $$E(W_{nk}) = 2 \sum_{i=1}^{k} \int [\{s(z|Y_{ni})/s(z)\} - 1]s^2(z)\, dz$$

$$= - \sum_{i=1}^{k} Y_{ni}^4 \int \{(s(z|Y_{ni})-s(z))/Y_{ni}^2\}^2\, dz$$

$$= - \sum_{i=1}^{k} Y_{ni}^4 \int \{(\partial/\partial y^2)\ s(z|y)\Big|_{y_{ni}^*}\}^2\, dz,$$

where $0 < y_{ni}^* \leq Y_{ni};\ i \geq 1,2,\ldots,k$.

Note that for some $\epsilon > 0$, by assumption [A4]

(4.15) $$\int\{(\partial/\partial y^2)\ s(z|y)\}^2 dz = \frac{1}{2}\int\{(\partial/\partial y^2)\ \log g(z|y)\}^2\ g(z|y)dz$$
$$= \frac{1}{2}\ I(z|y) < \infty,$$

uniformly for all y $(0<y<\epsilon)$.

Also, from lemma 4.1 it is seen easily that

(4.16) $(\partial/\partial y^2)\ s(z|y)|_{y=0} = (1/2)\ \{q(z,x_0)/(g(z))^{½}\}$.

So, (4.15) (4.16), along with lemma 3.1 give

(4.17) $$\lim_{n\to\infty} \int\{(\partial/\partial y^2)\ s(z|y)\Big|_{y_{ni}^*}\}^2 dz = \int\{(\partial/\partial y^2)\ s(z|y)\big|_{y=0}\}^2 dz$$
$$= (1/4)\ \int\{q^2(z,x_0)/g(z)\}\ dz,$$

uniformly i (=1,2,...,k).

Finally let $U_{n1} < \ldots < U_{nn}$ denote the order statistics of a random sample $(U_1,\ldots,U_n)$ of size n from the uniform distribution on (0,1). Then $Y_{ni} = h(U_{ni})$, where $h = F_Y^{-1}$. It can be seen easily that under assumption [A1] the following is true:

(i) $h(u) = F_Y^{-1}(u)$ is defined for $0 < u < \epsilon$ for some $\epsilon > 0$ as a unique solution of $F_Y(u) = u$,

(ii) $h'(u)$ is continuous at 0,

(iii) $h(0) = h''(0) = 0$, $h'(0) = \{2 f(x_0)\}^{-1}$.

Let F_n denote the empirical cdf of $U_1, \dots, U_n$; then

$$(4.18)\quad \max_{1\le j\le n} |U_{nj}-(j/n)| = \max_{1\le j\le n} |U_{nj}-F_n(U_{nj})| \le \sup_{0<u<1} |F_n(u)-u| = O_p(n^{-1/2}).$$

So,

$$(4.19)\quad U_{nj}^4 = (j/n)^4 + o_p(n^{-4/5}), \quad \text{for all } j; \ \sum_{i=1}^{k} U_{ni}^4 = O(k^5 n^{-4}).$$

By lemma 3.1

$$(4.20)\quad \sum_{i=1}^{k} Y_{ni}^4 = \sum_{i=1}^{k} \{h(U_{ni})\}^4$$

$$= \sum_{i=1}^{k} \{h'(0)U_{ni} + \{h'(\gamma U_{ni}) - h'(0)\}U_{ni}\}^4$$

$$= \sum_{i=1}^{k} U_{ni}^4 \{h'(0) + o_p(1)\}^4$$

$$= (h'(0))^4 \sum_{i=1}^{k} U_{ni}^4 + o_p(1).$$

Substituting (4.19) in (4.20), we have

$$(4.21)\quad \sum_{i=1}^{k} Y_{ni}^4 = (2 f(x_0))^{-4} \sum_{j=1}^{k} (j/n)^4 + o_p(1)$$

$$= (2 f(x_0))^{-4} [n^{-4/5} \sum_{j=1}^{[t n^{4/5}]} (j/n^{4/5})^4] + o_p(1)$$

$$= (2 f(x_0))^{-4} \int_0^t s^4 \, ds + o_p(1)$$

$$\to (2\ f(x_0))^{-4}\ (t^5/5) = \{80\ f^4(x_0)\}^{-1}\ t^5.$$

From (4.14), (4.17) and (4.21), we have

(4.22) $E(W_{nk}) \to -(1/4)d^2,$

where

(4.23) $d^2 = \{80\ f^4(x_0)\}^{-1}\ t^5 \int_{-\infty}^{\infty} \{q^2(z,x_0)/g(z)\}dz.$ ■

Lemma 4.6 Define

(4.24) $V_{nk} = \sum_{i=1}^{k} Y_{ni}^2\{q(Z_{ni},x_0)/g(Z_{ni})\}.$

Assume [A1]-[A4]. Then under p_{nk}

(4.25) $\mathrm{Var}(W_{nk} - V_{nk}) \to 0.$

Proof: Under p_{nk}

(4.26) $$\mathrm{Var}(W_{nk}-V_{nk}) = 4 \sum_{i=1}^{k} \mathrm{Var}[\frac{s(Z_{ni}|Y_{ni})}{s(Z_{ni})} - 1 - (1/2)Y_{ni}^2 \frac{q(Z_{ni},x_0)}{g(Z_{ni})}]$$

$$\le 4 \sum_{i=1}^{k} E[\frac{s(Z_{ni}|Y_{ni})-s(Z_{ni})}{s(Z_{ni})} - ½\ Y_{ni}^2 \frac{q(Z_{ni},x_0)}{g(Z_{ni})}]^2$$

$$\le 4 \sum_{i=1}^{k} Y_{ni}^4 \int[\{(s(z|Y_{ni}) - s(z))/Y_{ni}^2\} - ½\ \{q(z,x_0)/(g(z))^{½}\}]^2dz$$

$$\le 4 \sum_{i=1}^{k} Y_{ni}^4 \int[(\partial/\partial y^2)\ s(z|y)\Big|_{y_{ni}^*} - ½\ \{q(z,x_0)/(g(z))^{½}\}]^2dz$$

where $0 \le y_{ni}^* \le Y_{ni}$, $i=1,2,\ldots,k$.

So, in view of (4.15), (4.16), (4.21) and Theorem V.1.3. of Hájek and Sidák (1967), we can conclude that the last integral converges to 0 uniformly for all $i=1,2,\ldots,k$. ■

Proof of the theorem: Note that by lemma 4.1, under p_{nk}

$$(4.27)\quad E(V_{nk}) = \sum_{i=1}^{k} Y_{ni}^2 \int\{q(z,x_0)/g(z)\}\, g(z)\, dz$$

$$= \sum_{i=1}^{k} Y_{ni}^2 \int q(z,x_0)\, dz = 0,$$

and similarly under p_{nk}

$$(4.28)\quad \mathrm{Var}(V_{nk}) = \sum_{i=1}^{k} Y_{ni}^4 \int\{q^2(z,x_0)/g(z)\}\, dz$$
$$= \{[80\, f^4(x_0)]^{-1} t^5 + o_p(1)\} \int\{q^2(z,x_0)/g(z)\}\, dz$$

$$\to d^2.$$

So, it follows from Theorem V.1.2 of Hájek and Sidák (1967) that under p_{nk}

$$(4.29)\quad V_{nk} \xrightarrow{\mathscr{D}} \mathcal{N}(0,d^2).$$

Now, lemma 4.5 and lemma 4.6 together imply that under p_{nk}

$$(4.30)\quad E[W_{nk} - V_{nk} + (1/4)d^2] \to 0.$$

We combine (4.30) with lemma 4.4 to obtain that under p_{nk}

$$(4.31)\quad E[\log L_{nk} - V_{nk} + (1/2)d^2] \to 0.$$

Since (4.29) and (4.31) together imply that under p_{nk}

$$(4.32)\quad \log L_{nk} \xrightarrow{\mathscr{D}} \mathcal{N}(-\tfrac{1}{2}\, d^2, d^2),$$

from lemma 4.2, we conclude that the densities q_{nk} are contiguous to the densities p_{nk}, when $k = [tn^{4/5}]$ for some $0<a<t<b<\infty$.

Now, we consider the case when $k = [tn^{\lambda}]$ for $0 < \lambda < 4/5$. By lemma 3.1, we have

$$(4.33)\quad \sum_{i=1}^{k} Y_{ni}^4 = o(1).$$

So, in lemma 4.4, $E(W_{nk}) \to 0$ (i.e., $d^2 = 0$). So, it follows easily that under p_{nk}

(4.34) $\log L_{nk} \to 0$ in probability.

Hence, the LeCam first lemma holds through the degenerate normal law.

This completes the proof of Theorem 2.1. ∎

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