UNIFORM SECOND ORDER ASYMPTOTIC LINEARITY OF M-STATISTICS IN LINEAR MODELS

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Abstract. For conventional linear models, uniform second order asymptotic linearity (in regression parameters) of M-statistics is studied under a variety of regularity conditions on the score functions. Parallel results are also derived for studentized versions of these M-statistics.

1. Introduction

Let $X_1,\dots,X_n$ be independent random variables (r.v.) with distribution functions (d.f.) $F_1,\dots,F_n$ respectively, all defined on the real line R^1, where

(1.1) $\quad F_i(x) = F(x - \beta' c_i),\ i = 1,\dots,n,$

$\beta' = (\beta_1,\dots,\beta_p)$ is an unknown vector of $p(\geq 1)$ parameters, and the $c_i = (c_{i1},\dots,c_{ip})'$ are known vectors of regression constants. Consider a **score function** $\psi : R^1 \to R^1$, and define

(1.2) $\quad M_n(b) = \Sigma_{i=1}^n c_i\ \psi(X_i - b' c_i)\ ,\ b \in R^p.$

Then the usual M-estimator ($\hat{\beta}$) of β is obtained by equating $M_n(b)$ to 0 in a suitable norm. For the study of general properties of such M-estimators, the following stochastic processes play a vital role. Let $Y_1,\dots,Y_n$ be independent and identically distributed (i.i.d.) r.v. (with a d.f. F), and let

(1.3) $\quad S_n(t) = \Sigma_{i=1}^n c_i [\ \psi(Y_i - n^{-\frac{1}{2}} t' c_i) - \psi(Y_i)\ ]\ ,\quad t \in R^p.$

Under suitable regularity conditions [viz., Jurečková(1983)] it is known that

(1.4) $\quad \sup\{\ ||n^{-\frac{1}{2}} S_n(t) + n^{-1} C_n t \gamma(\psi,F)||\ : ||t|| \leq M\ \} \to 0$, in probability,

as $n \to \infty$, where M is an arbitrary finite positive number, $\gamma(\psi,F)$ is a suitable constant (depending on ψ and F) and $C_n = \Sigma_{i=1}^n c_i c_i'$; this is known as the *uniform first order asymptotic linearity of M-statistics in regression parameter.* Results stronger than (1.4) have also been established under diverse regularity conditions [viz., Jurečková and Sen (1981a,b), Jurečková (1985), Jurečková and Sen (1987) and others]. Following the line of attack of the last two papers (dealing with the simple location model), we intend to study *uniform second order asymptotic linearity results* in the general case of linear models. Precise formulation of these second order results depends very much on the nature of the score function ψ and the density function f(.) corresponding to the d.f. F. A more general formulation of the problem is needed to handle the case of **studentized M-statistics** (as would be introduced in the sequel). From the applications point of view, we may remark that the M-estimators are not scale-equivariant , and hence, when the scale parameter for the density f(.) is not known, there is a profound need for studentizing the score function (viz., taking $\psi((X_i - b'c_i)/s_n)$ for suitable estimator (s_n) of the unknown scale factor); we intend to treat this studentized case also in detail. Concerning the score function, in practice, we may usually have either of the following three types :

(i) ψ is a step-function having finitely many jumps of finite magnitudes;

(ii) ψ is absolutely continuous with a derivative ψ' which is a step-function;

(iii) ψ is absolutely continuous with an absolutely continuous derivative ψ'.

Since somewhat different techniques are needed to handle these three cases, we shall treat them separately. However, for each type of ψ , both the classical and studentized versions of the M-statistics are considered. Section 2 deals with the second order (uniform) asymptotic linearity results for case (i), Section 3 with case (ii) and Section 4 with case (iii). Some general remarks are appended in the last section. It will be seen that the rates of convergence as well as the needed regularity conditions are possibly different in the different cases.

2. Second Order Results When ψ is a Step-Function.

Let $\psi : R^1 \to R^1$ be a step-function

(2.1) $\psi(x) = \alpha_j$, for $x \in (r_j, r_{j+1}]$, $j=0,\ldots,k$,

where $\alpha_0,\ldots, \alpha_k$ are real (distinct) numbers and $-\infty = r_0 < r_1 <\ldots < r_k < r_{k+1} = +\infty$, k being a positive integer. Further, we assume that there exists a positive definite (p.d.) matrix $\underset{\sim}{Q}$ such that

$$(2.2) \quad n^{-1}\underset{\sim}{C}_n = n^{-1}\Sigma_{i=1}^{n}\underset{\sim}{c}_i\underset{\sim}{c}_i' \rightarrow \underset{\sim}{Q}, \text{ as } n \rightarrow \infty,$$

and

$$(2.3) \quad n^{-1}\Sigma_{i=1}^{n} ||\underset{\sim}{c}_i||^3 = O(1), \text{ as } n \rightarrow \infty,$$

where $||.||$ stands for the Euclidean norm. Concerning the d.f. F , we assume that in a neighbourhood of r_j , F has bounded derivatives f and f' , for each j=1,...,k. Then, we have the following.

<u>THEOREM 2.1.</u> <u>Under the assumptions made above</u>, <u>for any</u> M > 0,

$$(2.4) \quad \underline{\text{Sup}}\{ ||n^{-\frac{1}{2}}\underset{\sim}{S}_n(\underset{\sim}{t}) + n^{-1}\underset{\sim}{C}_n\underset{\sim}{t}\gamma || : ||\underset{\sim}{t}|| \leq M \} = o_p(n^{-\frac{1}{4}}), \ \underline{\text{as}} \ n \rightarrow \infty,$$

<u>where</u>

$$(2.5) \quad \gamma = \Sigma_{j=1}^{k}(\alpha_j - \alpha_{j-1})f(r_j).$$

Side by side, we consider the case of a studentized M-statistics, so that the proofs for both the cases can be formulated in a common vein. Corresponding to (1.2), we consider a Studentized M-statistics

$$(2.6) \quad \underset{\sim}{M}_n^*(\underset{\sim}{b};s_n) = \Sigma_{i=1}^{n} \underset{\sim}{c}_i \, \psi((X_i-\underset{\sim}{b}'\underset{\sim}{c}_i)/s_n), \ \underset{\sim}{b} \in R^p,$$

where s_n is a suitable estimator of the scale factor σ , and we assume that

$$(2.7) \quad n^{\frac{1}{2}} | s_n - \sigma | = O_p(1).$$

In this case, we extend (1.3) to

$$(2.8) \quad \underset{\sim}{S}_n(\underset{\sim}{t},u) = \Sigma_{i=1}^{n}\underset{\sim}{c}_i[\psi((Y_i - n^{-\frac{1}{2}}\underset{\sim}{t}'\underset{\sim}{c}_i)/\sigma e^{n^{-\frac{1}{2}}u}) - \psi((Y_i/\sigma))], \ \underset{\sim}{t} \in R^p, \ u \in R^1.$$

Then, we have the following.

<u>THEOREM 2.2</u>. <u>Under the assumptions of Theorem</u> 2.1, <u>for any</u> M > 0, <u>as</u> n → ∞ ,

$$(2.9) \quad \sup\{||n^{-\frac{1}{2}}\underset{\sim}{S}_n(\underset{\sim}{t},u) + n^{-1}\underset{\sim}{C}_n\underset{\sim}{t}\gamma_1 + n^{-1}\Sigma_{i=1}^{n}\underset{\sim}{c}_i u\gamma_2||:||\underset{\sim}{t}||\leq M, \ |u| \leq M\} = o_p(n^{-\frac{1}{4}}),$$

<u>where</u>

$$(2.10) \quad \gamma_1 = \Sigma_{j=1}^{k}(\alpha_j - \alpha_{j-1})f(\sigma r_j) \ \underline{\text{and}} \ \gamma_2 = \Sigma_{j=1}^{k} r_j(\alpha_j - \alpha_{j-1})f(\sigma r_j).$$

<u>Proofs of the theorems</u>. Theorem 2.1 is a special case of Theorem 2.2 , so that it suffices only to consider the proof of Theorem 2.2. For this purpose, we may assume without any loss of generality that ψ has a single step , i.e., $\psi(x)$ is equal to 0 or 1 according as x is $\leq$ or > r , where r may even be taken as equal to 0. Also, we may take σ = 1. Identifying the coordinatewise structure in (2.9), it also suffices to consider only the case of $S_{n1}(\underset{\sim}{t},u)$. Let us denote by

$$(2.11)\quad S_n^o(\underset{\sim}{t},u) = S_{n1}(\underset{\sim}{t},u) - ES_{n1}(\underset{\sim}{t},u)$$
$$= -\Sigma_{i=1}^n c_{i1}\{I(Y_i \le r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t}) - I(Y_i \le r\sigma)$$
$$- F(r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t}) + F(r\sigma)\}.$$

Let $W = \{W(s),\ s \in R^1\}$ be a Wiener process, and define

$$(2.12)\quad \tau_i(t,u) = \text{time for } W(s) \text{ to exit the interval } (-a_i,\ b_i),\ i \ge 1,$$

where

$$(2.13)\quad (-a_i,b_i) = (-|c_{i1}||F(r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t}) - F(r\sigma)|,$$
$$|c_{i1}|[1-|F(r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}}c_i't) - F(r\sigma)|]),\ \text{if } c_{i1}[r\sigma(e^{n^{-\frac{1}{2}}u}-1) + n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t} \ge 0;$$
$$= (-|c_{i1}|[1-|F(r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t}) - F(r\sigma)|],$$
$$|c_{i1}||F(r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t}) - F(r\sigma)|),\ \text{if } c_{i1}(r\sigma(e^{n^{-\frac{1}{2}}u}-1)+n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t} \le 0,$$

for $i \ge 1$. Then by an appeal to the Skorokhod embedding of Wiener process, we have

$$(2.14)\quad n^{-\frac{1}{4}} S_n^o(\underset{\sim}{t},u) =_{\mathcal{D}} n^{-\frac{1}{4}} W(\Sigma_{i=1}^n \tau_i(\underset{\sim}{t},u)) =_{\mathcal{D}} W(n^{-\frac{1}{2}}\Sigma_{i=1}^n\tau_i(\underset{\sim}{t},u)),\quad \forall(\underset{\sim}{t},u),$$

where $=_{\mathcal{D}}$ stands for the equality in law or distribution. For $n \ge n_o$ and $||\underset{\sim}{t}||, |u| \le M$,

$$(2.15)\quad |F(r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t}) - F(r\sigma)|$$
$$\le |F(r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t}) - F(r\sigma e^{n^{-\frac{1}{2}}u})| + |F(r\sigma e^{n^{-\frac{1}{2}}u}) - F(r\sigma)|$$
$$\le M.K.n^{-\frac{1}{2}}(1 + ||\underset{\sim}{c}_i||),\ \text{where K is a constant.}$$

Hence, if $||\underset{\sim}{t}|| \le M$ and $|u| \le M$,

$$(2.16)\quad \tau_i(\underset{\sim}{t},u) \le V_i(M) = V_i^+(M) + V_i^-(M),\quad i \ge 1,$$

where

$$(2.17)\quad V_i^+(M) = \text{time for } W(s) \text{ to exit the interval } (-n^{-\frac{1}{2}}MK|c_{i1}|(1+||\underset{\sim}{c}_i||),\ |c_{i1}|);$$

$$(2.18)\quad V_i^-(M) = \text{time for } W(s) \text{ to exit the interval } (-|c_{i1}|,\ n^{-\frac{1}{2}}MK|c_{i1}|(1+||\underset{\sim}{c}_i||)).$$

Hence,

$$(2.19)\quad \sup\{n^{-\frac{1}{4}}|S_n^o(\underset{\sim}{t},u)| : ||\underset{\sim}{t}|| \le M,\ |u| \le M\}$$
$$=_{\mathcal{D}} \sup\{|W(n^{-\frac{1}{2}}\Sigma_{i=1}^n \tau_i(\underset{\sim}{t},u))| : ||\underset{\sim}{t}|| \le M,\ |u| \le M\}$$
$$\le \sup\{|W(s)| : 0 \le s \le n^{-\frac{1}{2}}\Sigma_{i=1}^n V_i(M)\}.$$

By (2.17) and (2.18),

$$(2.20)\quad E(n^{-\frac{1}{2}}\Sigma_{i=1}^n V_i(M)) = 2MKn^{-1}\Sigma_{i=1}^n c_{i1}^2(1 + ||\underset{\sim}{c}_i||) \le K_o,\quad \forall\ n \ge n_o,$$

where K_o is a finite, positive constant. Hence, given $\varepsilon > 0$, there exists a $C > 0$, such that

(2.21) $\quad P\{ n^{-\frac{1}{2}} \Sigma_{i=1}^{n} V_i(M) > C \} \leq \varepsilon/2 , \quad \forall\, n \geq n_o .$

Moreover, given $\varepsilon > 0$ and $C > 0$, there exists a $K^* > 0$, such that

(2.22) $\quad P\{ \sup\{|W(s)|: 0 \leq s \leq C\} > K^* \} \leq \varepsilon/2 .$

Combining (2.19), (2.21) and (2.22), we obtain that for $n \geq n_o$,

(2.23) $\quad P\{ \sup\{|n^{-\frac{1}{4}} S_n^o(\underset{\sim}{t},u)| : ||\underset{\sim}{t}||, |u| \leq M \} > K^* \} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon ,$

so that

(2.24) $\quad \sup\{ n^{-\frac{1}{4}} |S_n(\underset{\sim}{t},u) - ES_n^o(\underset{\sim}{t},u)| : ||\underset{\sim}{t}|| \leq M, |u| \leq M \} = o_p(1).$

It remains to show that uniformly in $||\underset{\sim}{t}||, |u| \leq M$, as $n \to \infty$,

(2.25) $\quad A_n = n^{-\frac{1}{4}} | ES_n(\underset{\sim}{t},u) + n^{-\frac{1}{2}} \gamma_1 \Sigma_{i=1}^{n} c_{i1} \underset{\sim}{c}_i' \underset{\sim}{t} + n^{-\frac{1}{2}} \gamma_2 \Sigma_{i=1}^{n} c_{i1} u | = O(1).$

Note that by (2.8)

$$(2.26) \quad A_n = n^{-\frac{1}{4}} | \Sigma_{i=1}^{n} c_{i1} [F(r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}} \underset{\sim}{c}_i' \underset{\sim}{t}) - F(r\sigma)] + n^{-\frac{1}{2}} \gamma_1 \Sigma_{i=1}^{n} c_{i1} \underset{\sim}{c}_i' \underset{\sim}{t}$$
$$n^{-\frac{1}{2}} \gamma_2 \Sigma_{i=1}^{n} c_{i1} u |$$
$$= n^{-\frac{1}{4}} \Big| \Sigma_{i=1}^{n} c_{i1} [F(r\sigma e^{n^{-\frac{1}{2}}u} + n^{-\frac{1}{2}} \underset{\sim}{c}_i' \underset{\sim}{t}) - F(r\sigma e^{n^{-\frac{1}{2}}u}) - n^{-\frac{1}{2}} \underset{\sim}{c}_i' \underset{\sim}{t} f(r\sigma e^{n^{-\frac{1}{2}}u})]$$
$$+ \Sigma_{i=1}^{n} c_{i1} [F(r\sigma e^{n^{-\frac{1}{2}}u}) - F(r\sigma) - r\sigma n^{-\frac{1}{2}} u f(r\sigma)]$$
$$+ n^{-\frac{1}{2}} \Sigma_{i=1}^{n} c_{i1} \underset{\sim}{c}_i' \underset{\sim}{t} [f(r\sigma e^{n^{-\frac{1}{2}}u}) - f(r\sigma)] \Big|$$
$$\leq A_{n1} + A_{n2} + A_{n3} \quad \text{(say)}.$$

Since $F(a_n + n^{-\frac{1}{2}} b_n) - F(a_n) - n^{-\frac{1}{2}} b_n f(a_n) = \int_0^{n^{-\frac{1}{2}} b_n} [f(a_n + u) - f(a_n)]du$, using (2.2), (2.3) and the boundedness of f and f' (in a neighbourhood of r), we obtain by some simple steps that each of A_{n1} , A_{n2} and A_{n3} is $O(n^{-\frac{1}{4}})$, uniformly in $||\underset{\sim}{t}|| \leq M$ and $|u| \leq M$. This shows that (2.24) holds, and the proof of the theorem is complete.

3. SOAR when ψ is Absolutely Continuous but ψ' is a Step-Function.

As in Section 1, let $Y_1, \ldots, Y_n$ be i.i.d.r.v.'s with a d.f. F , and define the stochastic processes $\underset{\sim}{S}_n(\underset{\sim}{t})$, $\underset{\sim}{t} \in R^p$ and $\underset{\sim}{S}_n(\underset{\sim}{t},u)$, $(\underset{\sim}{t},u) \in R^{p+1}$, as in (1.3) and (2.8) respectively. In either case, we assume now that ψ is an absolutely continuous function with a derivative ψ' which is a step-function, namely,

(3.1) $\quad \psi'(x) = \alpha_\nu$, for $r_\nu < x \leq r_{\nu+1}$, $\nu = 1, \ldots, k$

where $\alpha_o, \ldots, \alpha_k$ are real numbers($\alpha_o = \alpha_k = 0$) and $-\infty = r_o < r_1 < \ldots < r_k < r_{k+1} = \infty$. Thus, ψ is a continuous, piecewise linear function and it is a constant for $x \leq r_1$ or $x \geq r_k$(as is the case with the Huber or the Hampel score functions). Regarding the

c_i, we assume that (2.2) holds, and we strengthen (2.3) to

$$(3.2) \qquad n^{-1} \Sigma_{i=1}^{n} ||c_i||^4 = O(1) \text{ , as } n \to \infty \text{ .}$$

Further, we replace (2.5) by

$$(3.3) \qquad \gamma = \int \psi'(x) \, dF(x) \text{ .}$$

THEOREM 3.1. Suppose that F has a bounded defivative f in a neighbourhood of $r_1, \ldots, r_k$ and ψ and the c_i satisfy the conditions mentioned above. Then

$$(3.4) \qquad \sup\{ ||S_n(t) + n^{-\frac{1}{2}} C_n t\gamma || : ||t|| \leq M \} = O_p(1) \text{ , as } n \to \infty \text{ .}$$

We shall find it convenient to consider the studentized case side by side. We replace (2.10) by

$$(3.5) \quad \gamma_1 = \sigma^{-1} \int \psi'(x/\sigma) dF(x) \quad \text{and} \quad \gamma_2 = \sigma^{-1} \int x\psi'(x/\sigma) dF(x) \text{.}$$

THEOREM 3.2. Under the assumptions made above, as $n \to \infty$,

$$(3.6) \qquad \sup\{ ||S_n(t,u) + n^{-\frac{1}{2}}[C_n t\gamma_1 + \Sigma_{i=1}^{n} c_i u\gamma_2]|| : ||t|| \leq M, |u| \leq M \} = O_p(1).$$

Proof of the theorems. As (3.4) is a particular case of (3.6), it suffices to show that (3.6) holds. Further, as ψ has been assumed to be flat at the two tails, and ψ' has finitely many steps, it suffices to consider the particular case where

$$(3.7) \qquad \psi(x) = \begin{cases} \psi(r_1) \text{ , for } x < r_1 \\ x \text{ , } \quad \text{for } r_1 \leq x \leq r_2 \\ \psi(r_2) \text{ ,for } x > r_2 \text{ ,} \end{cases} \quad -\infty < r_1 < r_2 < \infty \text{ .}$$

Also, without any loss of generality we may put $\sigma = 1$ and consider only the case of $S_{n1}(t,u)$. Further, by virtue of (2.2) and (3.7), for n adequately large,

$$(3.8) \quad |\Sigma_{i=1}^{n} c_{i1}[\psi(e^{-n^{-\frac{1}{2}}u}(Y_i - n^{-\frac{1}{2}} c_i' t)) - \psi(Y_i - n^{-\frac{1}{2}}(uY_i + c_i' t))]| = O(1) \text{ ,}$$

uniformly in $|u| \leq M$ and $||t|| \leq M$, for every finite M. [Note that ψ in (3.7) is first order Lipschitz.] As such, we shall replace $\psi((Y_i - n^{-\frac{1}{2}} c_i' t)/e^{n^{-\frac{1}{2}}u})$ by $\psi(Y_i - n^{-\frac{1}{2}}(u + c_i' t))$, for every i(=1,...,n), in (2.8). With these adjustments, we may note that for any pair (t_1, u_1) and (t_2, u_2) of distinct points

$$(3.9) \quad \text{Var}(S_{n1}(t_1,u_1) - S_{n1}(t_2,u_2)) \leq \Sigma_{i=1}^{n} c_{i1}^2 E[\psi(Y_i - n^{-\frac{1}{2}}(u_1 Y_i + c_i' t_1)) - \psi(Y_i - n^{-\frac{1}{2}}(u_2 Y_i + c_i' t_2))]^2$$
$$\leq K^* \{(u_1 - u_2)^2 + ||t_1 - t_2||^2 \}, \quad K^* < \infty,$$

uniformly in $|u_j| \leq M$ and $||t_j|| \leq M$, for j =1,2; the last step in (3.9) is again a consequence of the Lipschitz character of ψ. In the same vein, note that

$$(3.10)\quad |E[S_{n1}(\underset{\sim}{t}_1,u_1)-S_{n1}(\underset{\sim}{t}_2,u_2)] + n^{-\frac{1}{2}}[\Sigma_{i=1}^n c_{i1}\underset{\sim}{c}_i'(\underset{\sim}{t}_1-\underset{\sim}{t}_2)\gamma_1 + \gamma_2\Sigma_{i=1}^n c_{i1}(u_1-u_2)]|$$
$$\leq K^{**}\{\ ||\underset{\sim}{t}_1 - \underset{\sim}{t}_2|| + |u_1 - u_2|\};\quad K^{**} < \infty,$$

uniformly in $|u_j| \leq M$ and $||\underset{\sim}{t}_j|| \leq M$, for j=1,2. By (3.9) and (3.10), we have

$$(3.11)\quad E[\ S_{n1}(\underset{\sim}{t}_1,u_1)-S_{n1}(\underset{\sim}{t}_2,u_2) + n^{-\frac{1}{2}}\ \Sigma_{i=1}^n c_{i1}[\gamma_1(\underset{\sim}{t}_1-\underset{\sim}{t}_2)'\underset{\sim}{c}_i + \gamma_2(u_1-u_2)]\ ]^2$$
$$\leq K_o\{\ (u_1-u_2)^2 + ||\underset{\sim}{t}_1 - \underset{\sim}{t}_2||^2\ \};\quad K_o < \infty,$$

uniformly in $|u_j| \leq M$ and $||\underset{\sim}{t}_j|| \leq M$, j=1,2.

The process in (3.6) may not vanish along the lower boundary (where one or more of the coordinates $(\underset{\sim}{t},u)$ is null), although at $(\underset{\sim}{t},u) = \underset{\sim}{0}$, it vanishes. In order to make use of some existing results on multi-parameter stochastic processes, we first consider the following reparametrization. Let

$$(3.12)\quad S_n^*(\underset{\sim}{t},u) = \Sigma_{\varepsilon_o=0,1}\Sigma_{\varepsilon_1=0,1}\cdots\Sigma_{\varepsilon_p=0,1}(-1)^{\Sigma_{j=0}^p \varepsilon_j}\ S_{n1}(t_1-\varepsilon_1 t_1,\ldots,t_p-\varepsilon_p t_p,u-\varepsilon_o u).$$

Note that for $S_n(\underset{\sim}{t},u)$ in (3.6), the centring part is linear, and hence, for $S_n^*(\underset{\sim}{t},u)$ the corresponding centering part is null. As such we may rewrite $S_{n1}(\underset{\sim}{t},u) - ES_{n1}(\underset{\sim}{t},u)$ as a linear combination of 2^{p+1} processes of the type (3.12) of dimension $\leq$ p+1,and $S_{n1}(\underset{\sim}{0},0)$ appears in the last term in this set (but $S_{n1}(\underset{\sim}{0},0)= 0$). Each of these processes vanishes along their lower boundary. Hence, it suffices to show that each of the processes of the type (3.12) is uniformly bounded in probability(over the domain T = $[-M,M]^{p+1}$). For this purpose, we denote by $\underset{\sim}{E} = \text{Diag}(\varepsilon_1,\ldots,\varepsilon_p)$ and let

$$(3.13)\quad \psi^*(Y_i;\underset{\sim}{t},u) = \Sigma_{\{\varepsilon_j=0,1;0\leq j\leq p\}}(-)^{\varepsilon_0+\cdots+\varepsilon_p}\ \psi((Y_i-n^{-\frac{1}{2}}\underset{\sim}{c}_i'(\underset{\sim}{I}-\underset{\sim}{E})\underset{\sim}{t})/e^{n^{-\frac{1}{2}}u(1-\varepsilon_0)}),$$

for i =1,...,n. Note that by (3.2),

$$(3.14)\quad |(Y_i - n^{-\frac{1}{2}}\underset{\sim}{c}_i'(\underset{\sim}{I}-\underset{\sim}{E})\underset{\sim}{t})/e^{n^{-\frac{1}{2}}u(1-\varepsilon_0)} - Y_i| \leq 2Mn^{-\frac{1}{2}}(|Y_i| + ||\underset{\sim}{c}_i||) = O(n^{-\frac{1}{4}}),$$

uniformly in i(=1,...,n) and $|u| \leq M$ and $||\underset{\sim}{t}|| \leq M$, whatever $\underset{\sim}{E}$ may be (over the 2^{p+1} realizations). Also, on each of the three open intervals $(-\infty,r_1)$, (r_1,r_2) and (r_2,∞), ψ is linear, so that the corresponding ψ^* is equal to 0. Thus, ψ^* can only be different from 0 in a small (i.e., $O(n^{-\frac{1}{4}})$) neighbourhood of r_1 and r_2. If we denote by $r = \max\{|r_1|, |r_2|\}$, and make use of the Lipschitz character of ψ, then we can bound ψ^* by $CMn^{-\frac{1}{2}}(r + ||\underset{\sim}{c}_i||)I_i$, where C (< ∞) is a finite constant ($\leq 2^p$) and I_i is 1 or 0 according as Y_i is in this neighbourhood of r_1 (or r_2) or not. Note that the indicator variables I_i are all independent. Using (3.12), (3.13) and the above bound, we obtain that

(3.15) $\sup\{|S_n^*(\underset{\sim}{t},u)|: ||\underset{\sim}{t}|| \le M, |u| \le M\} \le n^{-\frac{1}{2}}\Sigma_{i=1}^n |c_{i1}| CM(r + ||\underset{\sim}{c}_i||) I_i$,

where the I_i are (nonnegative) indicator variables with

(3.16) $EI_i \le C^*(r + ||\underset{\sim}{c}_i||)n^{-\frac{1}{2}}$, $i=1,\dots,n$; C^* ($< \infty$) depends on M and the pdf f(.) at r_1 and r_2.

Taking expectation on both sides of (3.15) and using (3.16) along with (3.2), we conclude that $E[\sup\{|S_n^*(\underset{\sim}{t},u)|: ||\underset{\sim}{t}|| \le M, |u| \le M\}] = O(1)$, while using (3.2) and the binomial character of the I_i, it follows that the right hand side of (3.15) has a variance of the order $n^{-\frac{1}{4}}$ (which converges to 0 as $n \to \infty$), so that it is bounded in probability. Hence, the left hand side of (3.15) is bounded in probability, and this completes the proof of the theorem.

In passing, we may remark that in (3.7), we have taken $-\infty < r_1 < r_2 < \infty$. It is possible to choose $r_1 = -\infty$ or $r_2 = +\infty$, provided we assume that for the density f(y) of Y_1 , yf(y) converges to 0 as $y \to -\infty$ (or $+\infty$), and this is insured by the finiteness of γ in (3.3) (when ψ' is different from 0 for all finite x), which in turn is guaranteed by the existence of the first moment of Y.

4. SOAR for Absolutely Continuous ψ and Absolutely Continuous ψ'

We shall extend the results of Section 3 to the case where ψ' is itself absolutely continuous, and we denote the derivative of ψ' by ψ''. Naturally, in this context, we may need some extra conditions on ψ' and ψ'' . Let us denote by

(4.1) $\psi''_\delta(y) = \sup\{ |\psi''(y+u)| : |u| \le \delta \}, \quad \delta > 0.$

THEOREM 4.1. Suppose that the $\underset{\sim}{c}_i$ satisfy (2.2) and (3.2), γ , defined by (3.3), is finite, and that for some $\nu > 1$, $\delta_o > 0$, $E[\{\psi''_\delta(Y)\}^\nu] < \infty$, for every δ : $0 < \delta \le \delta_o$. Then (3.4) holds.

Let us then define

(4.2) $\bar{\psi}'_\delta(y) = \sup\{ |\psi'(e^{-v}(y+u))|: |u| \le \delta, |v| \le \delta \}; \quad \delta > 0$,

and $\bar{\psi}''_\delta(.)$ is defined analogously.

THEOREM 4.2. Suppose that the $\underset{\sim}{c}_i$ satisfy (2.2) and (3.2), γ_1 and γ_2, defined by (3.5) are finite, and for some $\nu > 1$ and $\delta_o > 0$, (4.3) $E[|Y\bar{\psi}''_\delta(Y)|^\nu] < \infty$ and $E[|Y^2\psi''_\delta(Y)|^\nu] < \infty$, for all $\delta: 0 < \delta \leq \delta_o$.

Then (3.6) holds.

Proof of the theorems. Note that for the studentized case, we need (4.3) which is more stringent than the parallel condition in Theorem 4.1. If ψ' can be expressed as a difference of two monotone functions, then in (4.3), $\bar{\psi}'_\delta(.)$ may also be replaced by $\psi'_\delta(.)$. Further, in most of the cases, ψ'' is decreasing in the two tails, and hence, the second condition in (4.3) may be less restrictive than the usual moment condition on the Y needed in the studentized case. In this proof also, we consider specifically the case of $S_{n1}(\underset{\sim}{t},u)$ and take $\sigma = 1$. For simplicity of the proof, we treat the case of Theorem 4.1 in detail, and only mention the modifications needed for the other theorem.

As a first step, we adopt the representations in (3.12) and (3.13) (without the index u). Then, note that for each i(=1,...,n),

(4.4) $\psi(Y_i - n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t}) = \psi(Y_i) - n^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t}\psi'(Y_i) + (2n)^{-1}(\underset{\sim}{c}_i'\underset{\sim}{t})^2\psi''(Y_i - hn^{-\frac{1}{2}}\underset{\sim}{c}_i'\underset{\sim}{t})$; $h \in (0,1)$.

Further, note that letting $\underset{\sim}{E}$ as in before (3.13)(ε_o deleted),

(4.5) $\Sigma_{\{\varepsilon_j=0,1\ ,1\leq j\leq p\}}\{\underset{\sim}{c}_i'(\underset{\sim}{I}-\underset{\sim}{E})\underset{\sim}{t}\}(-1)^{Tr\underset{\sim}{E}} = 0$ and $\Sigma_{\{\varepsilon_j=0,1;1\leq j\geq p\}}(-1)^{Tr\underset{\sim}{E}} = 0$,

so that

(4.6) $\psi^*(Y_i;\underset{\sim}{t}) = 0 - 0 + (2n)^{-1}\Sigma_{\{\varepsilon_j=0,1;1\leq j\leq p\}}\{\underset{\sim}{c}_i'(\underset{\sim}{I}-\underset{\sim}{E})\underset{\sim}{t}\}^2\psi''(Y_i - hn^{-\frac{1}{2}}\underset{\sim}{c}_i'(\underset{\sim}{I}-\underset{\sim}{E})\underset{\sim}{t})(+1)^{Tr\underset{\sim}{E}}$.

Since by (3.2), $n^{-\frac{1}{2}}|\underset{\sim}{c}_i\underset{\sim}{t}|$ is $O(n^{-\frac{1}{4}})$ uniformly in i(=1,...,n) and $||\underset{\sim}{t}|| \leq M$, it follows from (4.6) that

(4.7) $\sup\{|\psi^*(Y_i;\underset{\sim}{t})|: ||\underset{\sim}{t}|| \leq M\} \leq (2n)^{-1}\underset{\sim}{c}_i'\underset{\sim}{c}_i M^2 \psi''_{\delta_n}(Y_i)2^p$; $\delta_n = O(n^{-\frac{1}{4}})$, as $n \to \infty$.

Thus, parallel to (3.15), here we have

(4.8) $\sup\{|S_n^*(\underset{\sim}{t})|: ||\underset{\sim}{t}|| \leq M\} \leq M^2 2^p (2n)^{-1}\Sigma_{i=1}^n \underset{\sim}{c}_i'\underset{\sim}{c}_i\psi''_{\delta_n}(Y_i)$; $\delta_n < \delta_o \ \forall n \geq n_o$.

Under the assumed condition that $E[\{\psi''_\delta(Y)\}^\nu] < \infty$, for some $\nu > 1$, and (3.2), we may use the Markov law of large numbers and verify that the right hand side of (4.8) converges in probability to a finite positive constant. This completes the proof of Theorem 4.1.

To prove Theorem 4.2, first, we show that (3.8) holds uniformly in $||\underset{\sim}{t}|| \leq M$ and $|u| \leq M$, in probability, when n is large. Towards this note that $e^{-a}(y - b) =$

$(y-a-by)+[y(e^{-a}-1+a)+b(1-e^{-a})]$. As such, we may write

$$(4.9)\qquad \psi((Y_i-n^{-\frac{1}{2}}\underset{\sim}{c}_i^{\prime}\underset{\sim}{t})/e^{n^{-\frac{1}{2}}u}) = \psi(Y_i-n^{-\frac{1}{2}}(uY_i+\underset{\sim}{c}_i^{\prime}\underset{\sim}{t})) + [Y_i(e^{-un^{-\frac{1}{2}}}-1+n^{-\frac{1}{2}}u)+n^{-\frac{1}{2}}\underset{\sim}{c}_i^{\prime}\underset{\sim}{t}(1-e^{-un^{-\frac{1}{2}}})]\,\psi'(Y_{ni}^{*}),\quad i=1,\ldots,n,$$

where Y_{ni}^{*} lies between $(Y_i-n^{-\frac{1}{2}}\underset{\sim}{c}_i^{\prime}\underset{\sim}{t})/e^{un^{-\frac{1}{2}}})$ and $(Y_i-n^{-\frac{1}{2}}(\underset{\sim}{c}_i^{\prime}\underset{\sim}{t}+uY_i))$. Thus, proceeding as in before (and making use of (3.2)), we obtain that

$$(4.10)\qquad \sup\{\,|\Sigma_{i=1}^{n}c_{i1}[\psi((Y_i-n^{-\frac{1}{2}}\underset{\sim}{c}_i^{\prime}\underset{\sim}{t})/e^{un^{-\frac{1}{2}}})-\psi(Y_i-n^{-\frac{1}{2}}(\underset{\sim}{c}_i^{\prime}\underset{\sim}{t}+uY_i))]|\ :||\underset{\sim}{t}||\le M,\ |u|\le M\}$$
$$\le C\{n^{-1}\Sigma_{i=1}^{n}|c_{i1}|\{|Y_i|+||\underset{\sim}{c}_i||\}\,\bar{\psi}_{\delta_n}^{\prime}(Y_i)\ ;\quad \delta_n\le\delta_o\ ,\ \forall\, n\ge n_o\ ,$$

where C is a finite positive constant depending on M. As such, using (3.2), the first condition in (4.3) and the Markov law of large numbers, it follows that the right hand side of (4.10) stochastically converges to a finite positive quantity, and hence, (3.8) holds, in probability. Having shown (4.10), we may replace in $S_{n1}(\underset{\sim}{t},u)$ and $S_n^{*}(\underset{\sim}{t},u)$, the $\psi((Y_i-n^{-\frac{1}{2}}\underset{\sim}{c}_i^{\prime}\underset{\sim}{t})/e^{un^{-\frac{1}{2}}})$ by $\psi(Y_i-n^{-\frac{1}{2}}(\underset{\sim}{c}_i^{\prime}\underset{\sim}{t}+uY_i))$. With this replacement, we may proceed as in (4.7) through (4.8), with the only change that in the right hand side of (4.8), we need to replace $\psi_{\delta_n}^{\prime\prime}(Y_i)$ by $(Y_i^2+1)\psi_{\delta_n}^{\prime\prime}(Y_i)$. Once this has been done, we appeal to the second condition in (4.3) , and again by reference to (3.2) and the Markov law of large numbers, we conclude that the right hand side of (4.8) as ammended here is $O_p(1)$. This completes the proof of the theorem.

5. Some General Remarks.

It may be remarked that in Theorems 2.1 through 4.2, we have not attempted to establish the weak convergence of the appropriate stochastic processes related to the $S_n(\underset{\sim}{t})$ or $S_n(\underset{\sim}{t},u)$ to some multiparameter Gaussian (or related) processes. If we were to do so, then in addition to the established uniform boundedness, we would have to show that (i) the finite dimensional distributions (f.d.d.) converge, and (ii) the stochastic processes under consideration are compact or tight. The first aspect is relatively simple, and can be done along the lines of Jurečková and Sen (1981 a,b, 1984) and others. The second aspect is, however, relatively more involved. Recall that a process may be uniformly bounded (in probability) without being tight (although tightness implies the uniform boundedness in probability). For Theorems 2.1 and 2.2, we need to multiply both sides of (2.4) or (2.9) by $n^{\frac{1}{4}}$, and then the tightness can be proved along the lines of Jurečková and Sen (1987a). For the processes related to Theorems 3.1 and 3.2, note that ψ'' exists and is equal to 0 excepting at the jump points of ψ'(which are finite in number), and hence, the proof of tightness is not that involved. In the case of Theorems 4.1 and 4.2, the situation is slightly different. Here ψ'' may not be bounded, and even so, it may not be equicontinuous. As such, we may need an additional condition [related to (4.3)] under which tightness can be established in a relatively simpler manner. Let us introduce a compactness condition on ψ'' by defining

(5.1) $\psi''^{*}_{\delta}(y) = \sup\{ |\psi''(y+u) - \psi''(y)| \ : \ |u| \leq \delta \}, \ \delta > 0,$

and replace the first condition in (4.3) by

(5.2) $E[|Y\psi''(Y)|^{\nu}] < \infty$ and $E[|Y\psi''^{*}_{\delta}(Y)|^{\nu}] < \infty$, for all $\delta : 0 < \delta \leq \delta_0$;

a very similar modification can be posed for the second condition in (4.3) and (4.2). In addition to (5.2), we may also need $E[|Y\psi'(Y)|^{\nu}] < \infty$ (or $E[|Y\psi'(Y)|^{\nu}] < \infty$ for Theorem 4.2) to apply the Markov law of large numbers on the leading term, and then proceeding very much in the same line as in Section 4, we can establish the tightness property .

Our main interest in this study is to show that as regards the uniform boundedness (in probability) result is concerned, such additional regularity conditions may not be needed, and detailed results can be obtained under appropriate regularity conditions pertaining to the nature of the score function, ψ. The case of the studentized M-estimators is of especial interest, as here we have shown precisely the effect of studentization on the uniform linearity result for M-statistics, and this is very useful in the study of the asymptotic properties of one-step M-estimators [viz., Jurečková and Sen (1984)].

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