SOME RESULTS ON THE BOMBER PROBLEM

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ABSTRACT

The problem of optimally allocating partially effective, defensive weapons against randomly arriving enemy aircraft so that a bomber maximizes its probability of reaching its designated target is considered in the usual continuous–time context, and in a discrete-time context. The problem becomes that of determining the optimal number of missiles $K(n,t)$ to use against an enemy aircraft encountered at time (distance) t away from the target when n is the number of remaining weapons (missiles) in the bomber's arsenal. Various questions associated with the properties of the function K are explored including the long-standing, unproven conjecture that it is a nondecreasing function of its first variable.

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1. Introduction and summary. Consider a bomber aircraft t time units away from its designated target with n missiles to defend itself against enemy aircraft. Assume the bomber encounters enemy aircraft, on its way to the target, that arrive according to a Poisson process with known intensity. Without loss of generality, this will be taken to be constant and equal to one. When an encounter occurs, the bomber can use, simultaneously, as many of its remaining missiles as it sees fit. Each missile hits the enemy aircraft with probability $1 - q$, $0<q<1$, independently of the other missiles fired. A single hit is lethal. If none of the missiles hits the enemy aircraft, the bomber is destroyed by the enemy with probability u, $0<u\leq 1$, with the case $u = 1$ of particular interest. The bomber seeks to use its missiles optimally so as to maximize its probability of reaching the target.

Let $K(n,t)$ denote the optimal number of missiles to be used when encountering an enemy aircraft at time t (t time units away from the target) with n missiles at hand. It seems intuitively clear that the function K will satisfy several monotonicity conditions:

(A) For fixed n, $K(n,t)$ is nonincreasing in t.

(B) For fixed t, $K(n,t)$ is nondecreasing in n.

(C) For fixed t, the number of missiles held back, $n - K(n,t)$, is nondecreasing in n.

Of these three "conjectures", only the third has a trivial proof. Allen Klinger and Thomas A. Brown (1968) showed that Conjecture A holds *providing conjecture* B *holds.* This unsatisfactory state of affairs was partly relieved when Ester Samuel (1970) showed Conjecture A holds, whether or not B holds. Conjecture B remains unproven.

While Conjecture C seems to have received little or no *direct* attention, it has been stated in a slightly disguised form and proven by Klinger and Brown, as their Lemma 2.2.7, and also by Samuel. Both sets of authors found it useful.

Conjecture B seems (at least to us) just as believable as A and C. The question is: Why should its proof be so much more illusive?

Our first objective in studying the "bomber problem" was to find a proof of Conjecture

B, or at least to shed some light on its veracity and, possibly, on the difficulty of its proof. To these ends, we decided to introduce a discrete–time version of the problem: An enemy aircraft is encountered at integer time t with probability r (independent of previous encounters), $t = 1,2,\cdots$. In what follows, we shall refer to the original problem as the "continuous–time bomber problem" (CBP), and the discrete version as the "discrete–time bomber problem" (DBP).

Conjectures A, B and C still seem reasonable for the DBP. Moreover, a proof of Conjecture B for the DBP should provide a proof for the the CBP: let r go to zero while adjusting the units of time accordingly.

Conjecture C is still easily proven for the DBP. And the Klinger–Brown proof that Conjecture B implies Conjecture A is also readily adapted to the DBP. Surprisingly, Samuel's proof of Conjecture A, alone, *does not adapt.* Her argument very crucially uses the constancy of K(n,t) over intervals of time t (for fixed n), *and the continuity in* t of the conditional optimal survival probability function H(n,t) (a probability conditioned by an encounter with the enemy at time t to which a response is required). Such continuity is not possible in the discrete–time context, at least not directly: we have found a way around the difficulty by making the encounter probability r time–dependent. See Theorem 1.

The unexpected difficulty we encountered in extending Samuel's argument to a discrete-time setting may shed some light on why a proof of Conjecture B is so illusive. Conjecture B is concerned with *discrete* changes in the number of available missiles. The discreteness could be the problem. One can imagine the replacement of missiles by some other kind of defensive weapon which can be used in continuous amounts. Then both K(n,t) and H(n,t) should be continuous in the variable n. We have not attempted to prove Conjecture B in such a setting. But such an investigation should be interesting, given the history of Conjecture B.

The study of the DBP offers several advantages. The most obvious is that the recursive description of the optimal missile allocation rule is simplified. (See Section 2.) A more important advantage is that it is quite easy to obtain the optimal allocation rule numerically, and to *rapidly* assess various conjectures. Already, we have numerically "verified" Conjecture B for tens of thousands of randomly generated pairs (q,r). Mostly, these were checked for $t \leq 12$ and $n \leq 20$. But some larger values of t and n were checked when q is not too small. The truth of Conjecture B was always supported, except for a very few instances when unavoidable difficulties with round–off errors were clearly indicated, because of an extreme value of q or r.

We have been able to study the CBP numerically with a very accurate, but slower and much more complicated, computer program. Our program has sustained Conjecture B for $n \leq 18$, and *all times* t, at about 1000 different values of q in the range .25 to .99, and for comparable numbers of q values in the range .01 to .25, but with lesser values of n. So the numerical evidence that Conjecture B holds for the CBP seems quite strong as well.

Our numerical work raises doubts concerning the validity of an important conjecture of Klinger and Brown; they conjecture that the optimal survival probability function $P(n,t)$ (defined in Section 2) is logconcave in n, i.e., $P(n+1,t)/P(n,t)$ is nonincreasing in n. It definitely fails to hold for the DBP, and we suspect it fails for the CBP. The importance of this is that a very simple "proof" of Conjecture B is known that depends on the validity of the logconcavity. Thus our numerical studies provide quite strong evidence for the veracity of Conjecture B, while showing that a very promising approach to its proof is not feasible, at least for the DBP. The logconcavity begins to break down at $t = 3$, when $n = 8$ and q is slightly larger than r, e.g., when $(q,r) = (.6,.58)$. It must be emphasized that we have *not* succeeded in finding a comparable counterexample for the CBP.

The CBP raises other interesting questions:

(i) Is the inequality $K(n,t-) - K(n,t) \geq 2$ possible? (Note, that Samuel showed $K(n,\cdot)$ is nonincreasing and right–continuous.) Conjecture A does not rule this out. (The analogous inequality $K(n,t-1) - K(n,t) \geq 2$ *is* possible for the DBP.) It is easily seen that $K(n,0+) = n$, and that $K(n,t) = 1$ for all sufficiently large t. But there is no proof that $K(n,\cdot)$

has n–1 points of discontinuity and, consequently, range $\{1,2,\cdots,n\}$. However, we have excellent numerical evidence for this. It should be remarked that Klinger and Brown showed that Conjecture B implies that $K(n,\cdot)$ has n–1 points of discontinuity.

(ii) Let y_{ni} denote the i–th point of discontinuity, *change point*, of $K(n,\cdot)$, $n \geq 2$. The complete set of change points $\{y_{ni}\}$ $(n \geq 2)$ characterize the optimal allocation rule providing $K(n,y_{ni}-) = n-i+1$ and $K(n,y_{ni}) = n-i$, $i = 1,2,\cdots,n-1$, as (i) suggests. Surprisingly, it is possible to give a precise formula for about a third of the change points, specifically when $i \leq (n+1)/3$. See Theorem 2 in Section 4 for details. We also know that the *last* change point (beyond which $K(n,t) = 1$) grows with n no faster than $O(n)$ as $n \to \infty$. See Proposition 2 in Section 4.

(iii) Suppose a military commander is planning a long-distance bombing run and wants the bomber to carry enough missiles to give it a specified survival probability α. How many missiles will be needed (approximately)? Crude approximations on our part indicate that $(\log(1/q))^{-1}\{t\log t - t\log(\frac{1}{u}\log\frac{1}{\alpha})\}$ is about right; it seems clear that this is within "o(t)" of being enough as $t \to \infty$ (see Proposition 3 in Section 4). Can this be pared down to "O(1)"? We don't know.

(iv) How fast does $K(n,t)$ grow as n goes to infinity with t fixed? Again, we don't know the answer.

Section 2 discusses notation and recursive relationships. A proof of Conjecture A for the DBP, with time–dependent encounter probabilities r_t, is given in Section 3. Several topics associated with the CBP are covered in Section 4. Section 5 discusses an extension of the CBP which sheds additional light of Conjecture B. Computing and numerical results are described in Section 6. Our numerical investigations uncovered some minor errors in some formulas of Klinger and Brown, which we correct.

It is perhaps worth noting that Klinger and Brown introduced the present model in a fairly general non–military context, but stated that it first arose in a (classified) military context. Thus the topic is relevant in a broader context. However, we shall continue to use the graphic terminology of the military setting.

2. Notation and recursive formulas. Let $H(n,t)$ denote the optimal probability of survival, for the bomber, when it meets an enemy aircraft at time t with n missiles at hand. Further, let $G_n(i,t)$ denote the probability of survival when meeting an enemy aircraft at time t and using i of n available missiles, and then proceeding optimally. Finally, let $P(n,t)$ denote the unconditional optimal probability of survival at time t with n missiles at hand (with no awareness of enemy aircraft).

For the CBP,

$$H(n,t) = \max_{1 \leq i \leq n} G_n(i,t), \quad G_n(i,t) = a_i P(n-i,t), \tag{1}$$

where

$$P(n,t) = e^{-t}\{ \int_0^t H(n,v)\, e^v\, dv + 1 \}, \tag{2}$$

and $a_i = 1 - uq^i$. Here, q is the probability that a given missile misses the enemy aircraft at which it is fired, u is the probability that the bomber is destroyed by a surviving enemy aircraft, and a_i is the probability that the bomber survives an encounter with an enemy aircraft after firing i of its available missiles. Then the "optimal allocation function" $K(n,t)$ defined by

$$K(n,t) = \min\{i\colon G_n(i,t) = H(n,t)\} \quad (t>0,\ n=1,2,\cdots), \tag{3}$$

describes the optimal number of missiles the bomber should fire if it encounters an enemy aircraft at time t with n missiles at hand.

The modifications needed for the DBP are:

$$H(n,t) = \max_{0 \leq i \leq n} G_n(i,t), \; G_n(i,t) = a_i P(n-i,t-1),$$
$$P(n,t) = r_t H(n,t) + \bar{r}_t P(n,t-1), \tag{4}$$

where r_t is the probability of encountering an enemy aircraft at time t and $\bar{r}_t = 1-r_t$. We may simply drop the subscript from r_t when it is independent of time. Likewise, we will add variables to G, H, P and K when their dependence on r_t needs to be made clear. Thus $G_n(i,t)$ and $P(n,t)$ *may* be written, respectively, as $G_n(i,t,r_1,\cdots,r_{t-1})$ and $P(n,t,r_1,\cdots,r_t)$.

It follows from (4) that

$$P(n,t) = r_t H(n,t) + \bar{r}_t r_{t-1} H(n,t-1) + \cdots + \bar{r}_t \bar{r}_{t-1} \cdots \bar{r}_2 r_1 H(n,1) + \bar{r}_t \bar{r}_{t-1} \cdots \bar{r}_1.$$

Thus

$$P(n,t,r_1,\cdots,r_t) = E\, H(n,\tau,r_1,\cdots,r_{\tau-1}), \qquad (H(n,0):=1), \tag{5}$$

where

$$\tau = \max\{0 \text{ and } s, 1 \leq s \leq t\text{: an enemy aircraft is encountered at time } s\}. \tag{6}$$

3. The discrete–time bomber problem. Here, the notation and definitions of the previous section are assumed. Of primary interest are equations (3)–(6) and the formula $a_i = 1 - uq^i$, $i=0,1,\cdots$, for the probability the bomber survives an encounter with an enemy aircraft after firing i missiles at it. Observe that a_i is strictly positive except when $i = 0$ and $u = 1$. Also, observe that G, H, and P are nonincreasing functions in every variable except i and n. (For the variable t, one has $P(n,t,r_1,\cdots,r_t) \leq P(n,t-1,r_1,\cdots,r_{t-1})$, $0 \leq r_t \leq 1$, with similar inequalities for G and H.)

A peculiarity of the DBP, not arising with the CBP, is the possible optimality of "nonengagement", i.e., of using no missile when they are available. We do not know whether nonengagement is ever a *uniquely* optimal strategy when $n \geq 1$, but there are two known ways it can arise nonuniquely:

(i) The situation may be hopeless at time t because $u = 1$ and too many future encounters are predictable, due to "ones" in the set $\{r_1, \cdots, r_{t-1}\}$.

(ii) The situation, without being hopeless, may be ambivalent between using a missile now, at time t, or, instead, for a *predictable* encounter in the future, e.g., when $n = 1$, $r_{t-1} = 1$, and $u < 1$. (It might be optimal *not* to use the missile at either time.)

When nonengagement is *an* optimal strategy, the convention, prescribed by (3), is to choose it, i.e., to set $K(n,t,r_1,\cdots,r_{t-1}) = 0$. Though arbitrary, this convention is compatible with Conjectures A and C; some possible conventions would not be.

Allowing some of the r_j's to be one poses a technical problem for Samuel's approach, which is based on the study of ratios such as $G_n(i,t)/G_n(i+1,t)$, since the denominator may be zero. This technical difficulty does not arise for the CBP. It can be overcome, here, without imposing artificial restrictions. They are not needed. But some care is.

Conjecture C for the DBP can be stated as follows.

Lemma 1. $K(n+1,t,r_1,\cdots,r_{t-1}) \leq K(n,t,r_1,\cdots,r_{t-1}) + 1$, $(n \geq 0,\ t \geq 1)$.

Proof. Let $k = K(n,t,r_1,\cdots,r_{t-1})$. The lemma is obvious if $k = n$. Likewise, the lemma is obvious if $k = 0$ and $u = 1$: this occurs iff the number of ones in the set $\{r_1,\cdots,r_{t-1}\}$ is n or larger, so the mission is completely hopeless (since $u = 1$). Then, necessarily, $K(n+1,t,r_1,\cdots,r_{t-1}) \leq 1 = k + 1$, and the lemma holds. If $k > 0$ or $u < 1$, all a_i's $(i \geq 1)$ are strictly positive and the ratios a_{i+1}/a_i are nonincreasing in the index i. If, in addition, $k < n$, then for every ℓ such that $k+1 < \ell \leq n+1$, one has

$$G_{n+1}(\ell,t,r_1,\cdots,r_{t-1}) = \frac{a_\ell}{a_{\ell-1}} G_n(\ell-1,t,r_1,\cdots,r_{t-1})$$

$$\leq \frac{a_\ell}{a_{\ell-1}} G_n(k,t,r_1,\cdots,r_{t-1})$$

$$\leq \frac{a_{k+1}}{a_k} G_n(k,t,r_1,\cdots,r_{t-1})$$
$$= G_{n+1}(k+1,t,r_1,\cdots,r_{t-1}).$$

Thus $K(n+1,t,r_1,\cdots,r_{t-1}) \leq k + 1$. □

Conjecture A for the DBP is a corollary of the following variant of Conjecture A:

Theorem 1. $K(n,t,r_1,\cdots,r_{t-1})$ *is nonincreasing in* r_{t-1} $(t\geq 2)$.

Corollary 1. $K(n,t+1,r_1,\cdots,r_t) \leq K(n,t,r_1,\cdots,r_{t-1})$.

Proof of Corollary 1. Clearly,

$$K(n,t,r_1,\cdots,r_{t-1}) = K(n,t+1,r_1,\cdots,r_{t-1},0) \geq K(n,t+1,r_1,\cdots,r_t). \qquad \square$$

Notice, for the special case of constant encounter probabilities, $r_t = r$, that the corollary asserts: $K(n,t+1) \leq K(n,t)$. Thus, for fixed n, $K(n,t)$ is nonincreasing in t.

The proof of Theorem 1, which is modeled on Samuel's (1970) proof of Conjecture A for the CBP, is based on the following lemma.

Lemma 2. *Let* $f(i,v)$ *be nonnegative, nonincreasing and continuous functions of* v, $0 \leq v \leq 1$, *for* $i = 0,\cdots,n$ *such that*

$$f(i_1,v_1)f(i_2,v_2) \leq f(i_1,v_2)f(i_2,v_1),\ i_1<i_2,\ v_1<v_2. \tag{7}$$

Further, let

$$k(v) = \min\{j\colon f(j,v) = \max_i f(i,v)\}. \tag{8}$$

Then $k(v)$ *is non–increasing and right–continuous.*

PROOF. If $f(k(v_1),v_1) = 0$, then (by (8)) $f(i,v_1) = 0$ for all i, and $k(v_1) = 0$. In turn, $f(i,v_2) = 0$ for all i, and $k(v_2) = 0 = k(v_1)$ $(v_2>v_1)$. If $f(k(v_1),v_1) > 0$, the argument proceeds as in Samuel's lemma (1970). (The strong assumption made here that f(i,v) is nonincreasing in v is not a burden; it replaces Samuel's [not fully suitable] requirement of strictly positive functions.) □

According to Samuel Karlin (1968), a nonnegative function $f(\cdot,\cdot)$, defined in a rectangular region, which satisfies the ordering condition

$$f(n_1,v_1)f(n_2,v_2) \geq f(n_1,v_2)f(n_2,v_1),\ n_1<n_2,\ v_1<v_2, \qquad (9)$$

is said to be *totally positive of order 2* (TP_2). More generally, if it satisfies condition (7) or condition (9), it is said to be *sign regular of order 2* (SR_2). We shall distinguish between the two cases, (9) and (7) by, respectively, using the notation "TPP" and "TPN" ("P" for "positive", "N" for "negative"). (See Cambanis, Simons and Stout (1976) for similar concepts concerned with sums rather than products.)

PROOF OF THEOREM 1. It will be proven by induction on a variable T that, for $t \leq T$, $G_n(i,t,r_1,\cdots,r_{t-1})$ is TPN in the variables i and r_{t-1} (for every $n\geq 0$), and $H(n,t,r_1,\cdots,r_{t-1})$ is TPP in the variables n and r_{t-1}, $T \geq 2$. Since, clearly, $G_n(i,t,r_1,\cdots,r_{t-1})$ is nonnegative and continuous in r_{t-1}, the desired conclusion will follow immediately from Lemma 2. It is convenient to view $G_n(i,1) = a_i$ and $H(n,1) = a_n$ as constant functions in a fictitious variable r_0 and to begin the induction argument with $T = 1$: let $v = r_0$, observe that $f(i,v) := G_n(i,1) = a_i$ $(0\leq i\leq n,\ 0\leq v\leq 1)$ satisfies the TPN condition (7) trivially, and $f(n,v) := H(n,1) = a_n$ $(n\geq 0,\ 0\leq v\leq 1)$ satisfies the TPP condition (9) trivially. The induction step is carried out in two stages. The first stage uses the induction hypothesis for $H(n,t,r_1,\cdots,r_{t-1})$, $t \leq T$, to extend the induction to $G_n(i,t,r_1,\cdots,r_{t-1})$, $t \leq T+1$. This is done using Lemmas 3 and 4 below. The second stage extends the induction to $H(n,t,r_1,\cdots,r_{t-1})$, $t \leq T+1$, by using the result of the first stage. This is done in Lemma 5 below. □

As a technical point, we remark that we needed to base our induction on t (i.e., time), rather than on n (as Samuel did for her continuous–time version of Theorem 1), to avoid artificial restrictions. (Since induction on t is impossible in a continuous–time setting, it was fortunate that K(n,t), $n \geq 1$, is never zero *in her setting.* Her induction would have been impossible without this.)

Lemma 3. *If* $H(n,t,r_1,\cdots,r_{t-1})$ *is TPP in the variables* n *and* r_{t-1}, $t \leq T$, *then*

$$H(n_1,T)H(n_2,t) \leq H(n_2,T)H(n_1,t),\ n_1 \leq n_2,\ t \leq T. \tag{10}$$

Consequently, for any random variable τ *taking values on the integers* $0,1,\cdots,T$,

$$H(n_1,T)EH(n_2,\tau) \leq H(n_2,T)EH(n_1,\tau),\ n_1 \leq n_2. \tag{11}$$

An appropriately chosen τ *yields the inequality*

$$H(n_1,T)P(n_2,T-1) \leq H(n_2,T)P(n_1,T-1),\ n_1 \leq n_2. \tag{12}$$

Proof. Inequality (12) easily follows from (5), (6) and (11). The task is to derive (10). This is trivial if $H(n_1,T) = 0$. If $H(n_1,T) > 0$, then $H(n_1,t) > 0$, for $t \leq T$, and it is enough to show $H(n_2,t)/H(n_1,t)$ is nondecreasing in t, $t \leq T$. By assumption,

$$H(n_1,t,r_1,\cdots,r_{t-2},v_1)H(n_2,t,r_1,\cdots,r_{t-2},v_2) \geq H(n_1,t,r_1,\cdots,r_{t-2},v_2)H(n_2,t,r_1,\cdots,r_{t-2},v_1),$$

for $n_1 < n_2$, $v_1 < v_2$, $t \leq T$. Setting $v_1 = 0$ and $v_2 = r_{t-1}$, one obtains, for $t \geq 2$,

$$H(n_1,t-1,r_1,\cdots,r_{t-2})H(n_2,t,r_1,\cdots,r_{t-1}) \geq H(n_1,t,r_1,\cdots,r_{t-1})H(n_2,t-1,r_1,\cdots,r_{t-2}), \tag{13}$$

since $H(n_i,t,r_1,\cdots,r_{t-2},0) = H(n_i,t-1,r_1,\cdots,r_{t-2})$, i=1,2. Inequality (13) is true trivially when $t = 1$ since $H(n,0) = 1$. Thus $H(n_2,t)/H(n_1,t)$ is nondecreasing in t, $t \leq T$.

□

Lemma 4. *If* $G_n(i,t,r_1,\cdots,r_{t-1})$ *is TPN in the variables* i *and* r_{t-1}, $t \leq T$, *and if inequality* (12) *holds, then it is TPN in the variables* i *and* r_{t-1}, $t \leq T+1$.

Proof. The task is to show $f(i,v) := G_n(i,t,r_1,\cdots,r_{t-2},v)$, satisfies (7) when $t = T+1$. According to (4), $f(i,v) = a_i\{vH(n-i,T) + (1-v)P(n-i,T-1)\}$, a linear function of v. So (7) is equivalent to $a_{i_1}a_{i_2}(v_2-v_1)\{H(n-i_2,T)P(n-i_1,T-1) - H(n-i_1,T)P(n-i_2,T-1)\} \leq 0$. Thus (7) follows from (12), as asserted. □

Lemma 5. *If* $G_n(i,t,r_1,\cdots,r_{t-1})$ *is TPN in the variables* i *and* r_{t-1}, $t \leq T$, *then* $H(n,t,r_1,\cdots,r_{t-1})$ *is TPP in the variables* n *and* r_{t-1}, $t \leq T$.

Proof. The task is to infer the TPP condition in (9) for the functions $h(n,v) := H(n,t,r_1,\cdots,r_{t-2},v)$ from the TPN in (7) for the functions $g(i,v) := G_{n_2}(i,t,r_1,\cdots,r_{t-2},v)$. Fix n_1, $n_2 > n_1$ and $t \leq T$. It follows from Lemma 2, that there exist an integer $m \geq 1$, $0=\alpha_0<\alpha_1<\cdots<\alpha_m=1$, and integers $\beta_1,\cdots,\beta_m$, $\gamma_1,\cdots,\gamma_m$, such that for $\alpha_{j-1} \leq r_{t-1} < \alpha_j$,

$$K(n_1,t,r_1,\cdots,r_{t-1}) = \beta_j,\ K(n_2,t,r_1,\cdots,r_{t-1}) = \gamma_j,\ j=1,\cdots,m.$$

By Lemma 1, $\gamma_j \leq \beta_j + n_2 - n_1$. So, for $\alpha_{j-1} \leq v_1 < v_2 < \alpha_j$, the TPP condition for $h(n,v)$ is equivalent to (see (3)) $G_{n_1}(\beta_j,..,v_1)G_{n_2}(\gamma_j,..,v_2) \geq G_{n_1}(\beta_j,..,v_2)G_{n_2}(\gamma_j,..,v_1)$, which trivially is true if $a_{\beta_j} = 0$, and otherwise is equivalent to (see (4)) $G_{n_2}(\beta_j+n_2-n_1,..,v_1) \cdot G_{n_2}(\gamma_j,..,v_2) \geq G_{n_2}(\beta_j+n_2-n_1,..,v_2)G_{n_2}(\gamma_j,..,v_1)$. The latter follows immediately from the TPN condition for the functions $g(i,v)$. Since the function $H(n,t,r_1,\cdots,r_{t-2},v)$ is continuous at the boundary points $v = \alpha_j$, $j = 0,1,\cdots,m$, the TPP condition holds for all v, $0 \leq v \leq 1$.

□

4. The continuous–time bomber problem. Here, the notation and definitions of Section 2 are assumed. Of primary interest are equations (1)–(3) and the formula $a_i = 1 - uq^i$, $i=0,1,\cdots$, for the probability the bomber survives an encounter with the enemy after firing i missiles at it. Here, the functions G, H, and P are positive, continuous and nonincreasing in t. The function K is nonincreasing and right–continuous in t (see Samuel (1970)), and the values of $K(n,t)$ are a subset of $\{1,2,\cdots,n\}$ (see Klinger and Brown's (1968) Theorem 4.1.1).

If Conjecture B is correct, the optimal allocation policy is completely specified by the locations of the *change points* (discontinuities) of $K(n,\cdot)$, $n \geq 2$: then $K(n,0+) = n$, $K(n,\infty) = 1$ ($K(n,t) \to 1$ as $t \to \infty$), $K(n,\cdot)$ has $n-1$ change points $0 < y_{n1} < y_{n2} < \cdots < y_{n(n-1)} < \infty$, and $K(n,y_{ni}-) = n-i+1$, $K(n,y_{ni}) = n-i$, for $i = 1,2,\cdots,n-1$. Also, because Conjecture C holds (see Lemma 1), the change points would necessarily be interlaced, i.e., exactly one of the n change points of $K(n+1,\cdot)$ would occur in each of the n intervals determined by the n–1 change points of $K(n,\cdot)$. While this tidy picture has not been established, some of its parts have been put in place: it can be shown that the range of $K(n,\cdot)$ includes at least the values $\{1,2,n-1,n\}$, for $n \geq 2$. The following theorem gives very explicit information on about one third of the (potential) y_{ni}–values for the case $u = 1$. Similar results appear in Klinger and Brown (1967, 1968), but they *assume the validity of Conjecture* B.

THEOREM 2. *Let* $z(n,s) = 0, q^{n+1-2s}/(1-q^{n+1-2s}), \infty$, *respectively, for* $s = 0$, $0 < s < \frac{n+1}{2}$, *and* $s \geq \frac{n+1}{2}$ $(n \geq 0)$, *and let* $u = 1$. *Then:*

(a) $K(n,z(n,i)) = n - i$, $K(n,z(n,i)-) = n - i + 1$, *so that* $y_{ni} = z(n,i)$, *for* $1 \leq i \leq \lfloor\frac{n+2}{3}\rfloor$, $n \geq 2$;

(b) *for* $n \geq 0$ *and finite* $t \leq z(n,1)$,

$$P(n,t) = e^{-t}\{1 + (1-q^n)t\}; \tag{14}$$

(c) *for* $n \geq 0$ *and finite* $t \leq z(n,\lfloor\frac{n+2}{3}\rfloor)$,

$$P(n,t) = e^{-t}\{1 + (1-q^n)t + \int_0^t (t-v)(1-q^{K(n,v)})(1-q^{n-K(n,v)})\,dv\}. \tag{15}$$

REMARKS. When $u = 1$, it is more convenient to work with the functions $\tilde{G}_n(i,t) = e^t G_n(i,t)$, $\tilde{H}(n,t) = e^t H(n,t)$, and $\tilde{P}(n,t) = e^t P(n,t)$. Then $\tilde{H}(n,t) = \max_{1 \le i \le n} \tilde{G}_n(i,t)$, $\tilde{G}_n(i,t) = a_i \tilde{P}(n-i,t)$, and $\tilde{P}(n,t) = 1 + \int_0^t \tilde{H}(n,v)\,dv$. (See (1) and (2).) Integration by parts yields $\tilde{P}(n,t) = 1 + \tilde{H}(n,0+)t + \int_0^t (t-v)\,\tilde{H}(n,dv) = 1 + (1-q^n)t + \int_0^t (t-v)\,\tilde{G}_n(K(n,v),dv)$, which leads to

$$\tilde{P}(n,t) = 1 + (1-q^n)t + \int_0^t (t-v)(1-q^{K(n,v)})\cdot\tilde{H}(n-K(n,v),v)\,dv. \tag{16}$$

Equation (15) is simply a special case of (16), as we shall see.

PROOF. This theorem will be proven by induction based on the index n. The proofs of parts (a) and (c) require a common induction. The proof of part (b) can stand alone.

Note, $z(n,s)$ is strictly decreasing in n and strictly increasing in s when $0 < s < \frac{n+1}{2}$.

PROOF OF (b). Part (b) holds for $n = 0, 1$ since $K(0,v) = 0$ and $K(1,v) = 1$, so that equation (16) reduces to (14). Since $\tilde{P}(n,t) = 1 + \int_0^t \tilde{H}(n,v)\,dv$ and $\tilde{H}(n,v) \ge 1 - q^n$, part (b) is equivalent to the assertion "$\tilde{H}(n,t) = 1 - q^n$ for finite $t \le z(n,1)$." By induction, $\tilde{H}(n,t) = \max_{1 \le i \le n} (1-q^i)\tilde{P}(n-i,t) = \max_{1 \le i \le n} (1-q^i)\{1 + (1-q^{n-i})t\} = 1 - q^n$ when $t \le z(n,1)$ and finite. (Note that $z(\cdot,1)$ is nonincreasing.) This establishes (b).

PROOF OF (a) AND (c). The validity of part (c) for $n = 0,1$ is like that for part (b); equation (16) reduces to (15) since $K(0,v) = 0$ and $K(1,v) = 1$. For $n \ge 2$, part (c) holds for index n if (a) holds for index n. To see this, observe that (c) holds for index n if $\tilde{H}(n-K(n,v),v)$, appearing in (16), equals $1-q^{n-K(n,v)}$ $(0 \le v \le t \le z(n,\lfloor\frac{n+2}{3}\rfloor))$. According to (b), this occurs when $v \le z(n-K(n,v),1)$. If (a) holds for index n, then (using Conjecture A) $K(n,v) \ge K(n,z(n,\lfloor\frac{n+2}{3}\rfloor)) = n - \lfloor\frac{n+2}{3}\rfloor$. Consequently, $n - K(n,v) \le n + 2 - 2\lfloor\frac{n+2}{3}\rfloor$, so that $v \le z(n,\lfloor\frac{n+2}{3}\rfloor) \le z(n-K(n,v),1)$. Thus (c) holds.

Part (a) can be viewed as valid for $n = 0,1$, since no assertion is being made. Now assume (a) and (c) are valid up to index $n-1$ for some $n \geq 2$. It remains to show that (a) holds for index n. To show part (a) for n, it is enough to show (i) $K(n,z(n,\lfloor\frac{n+2}{3}\rfloor)) = n - \lfloor\frac{n+2}{3}\rfloor$, so that (according to Conjecture A) $K(n,t) \geq n - \lfloor\frac{n+2}{3}\rfloor$ when $t < z(n,\lfloor\frac{n+2}{3}\rfloor)$, and to show (ii) for $0 \leq i < \lfloor\frac{n+2}{3}\rfloor$, and for $z(n,i) < t < z(n,i+1)$ (recall $z(n,0) = 0$), that $\tilde{G}_n(n-i,t) > \tilde{G}_n(n-m,t)$ for $\{m \neq i: 0 \leq m \leq \lfloor\frac{n+2}{3}\rfloor\}$. While part (ii) is a simple consequence of part (b), part (i) requires a careful use of the induction hypothesis.

PROOF OF PART (ii). Part (b) yields

$$\tilde{G}_n(n-m,t) = (1-q^{n-m})\{1+(1-q^m)t\} \text{ for finite } t \leq z(m,1), \tag{17}$$

so $\tilde{G}_n(n-i,t)-\tilde{G}_n(n-m,t) = (q^m-q^i)(t-z(n,\frac{i+m+1}{2}))(1-q^{n-i-m})$, when $i + m < n$, for finite $t \leq \min(z(i,1),z(m,1))$, so that part (ii) easily follows, i.e, $\tilde{G}_n(n-i,t) > \tilde{G}_n(n-m,t)$ for $z(n,i) < t < z(n,i+1)$ when $0 \leq i < \lfloor\frac{n+2}{3}\rfloor$ and $\{m \neq i: 0 \leq m \leq \lfloor\frac{n+2}{3}\rfloor\}$. (Recall, z is increasing in its second argument.)

PROOF OF PART (i). By the same reasoning, $\tilde{G}_n(n-\lfloor\frac{n+2}{3}\rfloor,z(n,\lfloor\frac{n+2}{3}\rfloor)) \geq \tilde{G}_n(n-m,z(n,\lfloor\frac{n+2}{3}\rfloor))$ when $m < \lfloor\frac{n+2}{3}\rfloor$. Thus $K(n,z(n,\lfloor\frac{n+2}{3}\rfloor)) \leq n-\lfloor\frac{n+2}{3}\rfloor$. This is an equality if

$$\tilde{G}_n(n-\lfloor\tfrac{n+2}{3}\rfloor,z(n,\lfloor\tfrac{n+2}{3}\rfloor)) > \tilde{G}_n(n-m,z(n,\lfloor\tfrac{n+2}{3}\rfloor)) \text{ for } \lfloor\tfrac{n+2}{3}\rfloor < m \leq n-1. \tag{18}$$

Equation (18) holds for $n = 2$ since it asserts nothing. For $n > 2$, the index $m = n - 1$ can be ignored, since: $z(n,\lfloor\frac{n+2}{3}\rfloor) \leq z(2,1)$ and $K(2,z(2,1)-) = 2$ so that $K(n,z(n,\lfloor\frac{n+2}{3}\rfloor)) \geq K(n,z(2,1)) \geq 2$. (See Lemma 7.) There are two cases to consider. For $\lfloor\frac{n+2}{3}\rfloor < m \leq n + 2 - 2\lfloor\frac{n+2}{3}\rfloor$, equation (17) is applicable, so

$$\tilde{G}_n(n-\lfloor\tfrac{n+2}{3}\rfloor,z(n,\lfloor\tfrac{n+2}{3}\rfloor)) - \tilde{G}_n(n-m,z(n,\lfloor\tfrac{n+2}{3}\rfloor)) = (q^{i_0}-q^m)\{q^{n-i_0-m} - (1-q^{n-i_0-m})z(n,i_0)\} > 0,$$

where $i_0 = \lfloor\frac{n+2}{3}\rfloor$. For $n+2-2\lfloor\frac{n+2}{3}\rfloor < m \le n-2$, equation (17) is inapplicable; but the induction hypothesis for part (c) yields

$$\tilde{G}_n(n-m,t) = (1-q^{n-m})\{1+(1-q^m)t + \int_0^t (t-v)(1-q^{K(m,v)})(1-q^{m-K(m,v)})dv\} \qquad (19)$$

for $t \le z(m,\lfloor\frac{m+2}{3}\rfloor)$. This is suitable for $t = z(n,\lfloor\frac{n+2}{3}\rfloor)$, and for all $m \le n-2$, since the inequality $z(n,\lfloor\frac{n+2}{3}\rfloor) \le z(m,\lfloor\frac{m+2}{3}\rfloor)$ is equivalent to $2(\lfloor\frac{n+2}{3}\rfloor - \lfloor\frac{m+2}{3}\rfloor) \le n-m$.

Letting $i_0 = \lfloor\frac{n+2}{3}\rfloor$ and applying (17) and (19) to the left and right sides of (18), respectively, one finds that (18) is valid if

$$\frac{q^{n-m}(1-q^{m-i_0})(1-q^{m+1-i_0})}{1-q^{n+1-2i_0}} > (1-q^{n-m})\int_0^{z(n,i_0)} (z(n,i_0)-v)(1-q^{K(m,v)})(1-q^{m-K(m,v)})dv. \qquad (20)$$

Since $n+2-2i_0 < m$, one finds $1-q^{n-m} < 2(1-q^{i_0-1})$ and $z(m,1) < z(n,i_0)$. Moreover, for $0 < v < z(m,1)$, one has $m - K(m,v) = 0$. Thus $(1-q^{K(m,v)})(1-q^{m-K(m,v)}) = 0$ for $0 < v < z(m,1)$, and, of course, it is $\le (1-q^{m/2})^2$ for $z(m,1) \le v \le z(n,i_0)$. So the right side of (20) is strictly bounded above by $(1-q^{i_0-1})(1-q^{m/2})^2(z(n,i_0)-z(m,1))^2$.

Therefore, since

$$z(n,i_0) - z(m,1) = \frac{q^{n+1-2i_0}(1-q^{m+2i_0-n-2})}{(1-q^{n+1-2i_0})(1-q^{m-1})},$$

one can establish the validity of (18) for all q, $0 < q < 1$, by showing

$$\frac{q^{n-m}(1-q^{m-i_0})(1-q^{m+1-i_0})}{1-q^{n+1-2i_0}} > (1-q^{i_0-1})(1-q^{m/2})^2\left\{\frac{q^{n+1-2i_0}(1-q^{m+2i_0-n-2})}{(1-q^{n+1-2i_0})(1-q^{m-1})}\right\}^2$$

The key facts needed are: $1 - q^{m+2i_0-n-2} \le 1 - q^{m-i_0} < 1 - q^{m+1-i_0}$ (since $3i_0 \le n+2$); $(1-q^{m/2})^2 \le (1-q^{m-1})^2$ (since $m > (n-2i_0)+2 \ge 2$); and $1 - q^{i_0-1} \le 1 - q^{n+1-2i_0}$ (since $3i_0 \le$

$n + 2)$. Using these, one is left with the inequality $q^{n+2+m-4i_0} \leq 1$ to validate. This holds because the exponent of q exceeds $2(n + 2 - 3i_0) \geq 0$ (since $n + 2 - 2i_0 < m$). This completes the proof of part (i). □

The following is a slight strengthening of Conjecture C for the CBP.

LEMMA 6. $K(n+1,t-) \leq K(n,t) + 1$ *for* $t > 0$, *where* $K(n+1,t-) = \lim_{s \uparrow t} K(n+1,s)$.

PROOF. Proceeding as in the proof of Lemma 1, one can show for $k := K(n,t) < n$ and $k+1 < \ell \leq n+1$, that $G_{n+1}(\ell,t) < G_{n+1}(k+1,t)$. Thus, for $k+1 < \ell \leq n+1$,

$$G_{n+1}(\ell,t) < H(n+1,t) = \lim_{s \uparrow t} H(n+1,s) = \lim_{s \uparrow t} G_{n+1}(K(n+1,s),s) = G_{n+1}(K(n+1,t-),t).$$

So $G_{n+1}(\ell,t) < G_{n+1}(K(n+1,t-),t)$. Hence, $K(n+1,t-) \leq k + 1 = K(n,t) + 1$. □

LEMMA 7. $K(n+1,t) > K(n,t-)/2$ *for* $t > 0$. *Thus, if* $K(n,t-) \geq 2$, *then* $K(n+1,t) \geq 2$.

PROOF. Let $k = K(n,t-)$. The lemma clearly holds if $k = 1$. Assume $k \geq 2$. By repeated use of Lemma 6, one obtains $k-j \leq K(n-j,t-)$, $j = 0,1,\cdots,k-1$. So $k-j \leq K(n-j,s)$, $0<s<t$, $j = 0,1,\cdots,k-1$ ($K(n-j,\cdot)$ is nonincreasing). Equation (1) yields, for $j = 0,1,\cdots,k-2$,

$$H(n-j,s) = \max_{i \geq 1} G_{n-j}(i,s) = \max_{i \geq k-j} G_{n-j}(i,s) = \max_{i \geq k-j} \frac{a_i}{a_{i-1}} G_{n-j-1}(i-1,s) \leq \frac{a_{k-j}}{a_{k-j-1}} H(n-j-1,s),$$

since the ratio $\frac{a_i}{a_{i-1}}$ is nonincreasing in i. Then (2) and the simple inequality $\frac{a_{k-j}}{a_{k-j-1}} > 1$ yield $P(n-j,t) < \frac{a_{k-j}}{a_{k-j-1}} P(n-j-1,t)$. This, along with (1), yields $G_{n+1}(j+1,t) < \frac{a_{j+1}}{a_{j+2}} \cdot \frac{a_{k-j}}{a_{k-j-1}} \cdot G_{n+1}(j+2,t)$, $j = 0,1,\cdots,k-2$. Thus $G_{n+1}(j+1,t) < G_{n+1}(j+2,t)$ if $j+2 \leq k-j$ and $0 \leq j \leq k-2$, i.e., for $1 \leq j+1 \leq k/2$. So $K(n+1,t) > k/2 = k(n,t-)/2$. □

Let $P_L(n,t)$ denote the probability the bomber reaches its target if it were to use one missile per encounter with an enemy aircraft, as long as they last. And let $P_U(n,t)$ denote the probability the bomber reaches its target if the number of enemy aircraft in $(0,t)$ were known in advance, and this information is used optimally. Then, clearly, $P_L(n,t) \leq P(n,t) \leq P_U(n,t)$. These inequalities are excellent when t is large:

Lemma 8. *For* $n = 1, 2, \cdots,$

$$P_L(n,t) = \sum_{m=0}^{n} \frac{(a_1 t)^m e^{-t}}{m!} + a_1^n \cdot \sum_{m=n+1}^{\infty} \frac{a_0^{m-n} t^m e^{-t}}{m!} \tag{21}$$

and

$$P_U(n,t) = e^{-t} + \sum_{m=1}^{\infty} a_{\lfloor \frac{n}{m} \rfloor + 1}^{\langle n,m \rangle} \cdot a_{\lfloor \frac{n}{m} \rfloor}^{m - \langle n,m \rangle} \cdot \frac{t^m e^{-t}}{m!} \tag{22}$$

where $\lfloor \frac{n}{m} \rfloor$ *is the integer part of* $\frac{n}{m}$, *and* $\langle n,m \rangle = n \bmod m$. (*Interpret* 0^0 *as* 1). *The coefficients of* t^m *in* (21) *and* (22) *are equal for each* $m \geq n$. *Thus,*

$$P(n,t) = P_L(n,t) + O(t^{n-1} e^{-t}) = P_L(n,t)(1 + O(t^{-1})) \quad \text{as } t \to \infty. \tag{23}$$

Proof. The proof of (21) is straightforward, and (23) easily follows from (21) and (22). The proof of (22) uses the fact that $a(\cdot)$ is logconcave, so that the product $\prod_{j=1}^{m} a_{i_j}$, for nonnegative integers i_j adding up to n, is maximized when the i_j's are close to being equal. (Here, i_j is the number of missile allocated to the j–th of m enemy aircraft.) Specifically, one wants $\langle m,n \rangle$ of them equal to $\lfloor \frac{n}{m} \rfloor + 1$, and the remainder equal to $\lfloor \frac{n}{m} \rfloor$. The equality of the coefficients of t^m, for $m \geq n$, is easily checked. □

Since $G_n(i,t)$ is TPN in the variables i and t (see Samuel (1970)), the ratio $\frac{G_n(i,t)}{G_n(1,t)}$ is a nonincreasing function of t when $i \geq 2$. The limiting value can be found, and it is less than one:

Lemma 9. *For* $i = 2, 3, \cdots, n$ $(n \geq 2)$,

$$\lim_{t\to\infty} \frac{G_n(i,t)}{G_n(1,t)} = \frac{a_i \cdot a_0{}^{i-1}}{a_1{}^i}.$$

Thus $K(n,t) = 1$ *for all sufficiently large* t.

PROOF. The asserted limit is easily justified using Lemma 8. To begin with,

$$\frac{G_n(i,t)}{G_n(1,t)} = \frac{a_i P(n-i,t)}{a_1 P(n-1,t)} = \frac{a_i P_L(n-i,t)}{a_1 P_L(n-1,t)} (1 + O(t^{-1})).$$

If $u = 1$, then $a_0 = 0$, and (21) yields $\frac{P_L(n-i,t)}{P_L(n-1,t)} = O(t^{1-i})$; so the asserted limit follows. If $u < 1$, then $a_0 > 0$, and (21) yields $\frac{P_L(n-i,t)}{P_L(n-1,t)} = (\frac{a_0}{a_1})^{i-1}(1 + O(t^{n-2}e^{-(1-u)t}))$; so the asserted limit follows. In both cases, the limiting value $\frac{a_i \cdot a_0{}^{i-1}}{a_1{}^i} < 1$. Thus, for all sufficiently large t, $G_n(1,t) = \max_{1 \le i \le n} G_n(i,t)$, so that $K(n,t) = 1$. □

Let T_n be the largest change point of $K(n,\cdot)$ for $n \geq 2$, and for convenience, let $T_1 = 0$.

PROPOSITION 1. *For* $n \geq 2$, $T_{n-1} < T_n$, $K(n,T_n-) = 2$, *and* $K(n,T_n) = 1$.

PROOF. The proposition holds for $n = 2$. Suppose it holds up to $n-1$ ($n \geq 3$). From Lemma 9, $K(n,t) = 1$ for large t. So $K(n,T_n) = 1$ and $K(n,T_n-) \geq 2$. Also, $K(n-1,T_n-) = 1$, by Lemma 7. So $T_{n-1} < T_n$, since $K(n-1,\cdot)$ is nonincreasing and $K(n-1,T_{n-1}-) = 2 > 1 = K(n-1,T_n-)$. Finally, by Lemma 6, $K(n,T_n-) \leq K(n-1,T_n) + 1 = 2$. So $K(n,T_n-) = 2$. □

PROPOSITION 2. *For* $u = 1$, $T_n = O(n)$ *as* $n \to \infty$. *In particular,*

$$\limsup_{n\to\infty} n^{-1} T_n \leq \frac{1+q+(q+q^2)^{\frac{1}{2}}}{1-q}. \tag{24}$$

PROOF. Samuel proved that $\frac{G_n(1,t)}{G_n(2,t)}$ is strictly increasing in t. This result, along with Proposition 1, indicates that T_n is the unique t satisfying $G_n(1,t) = G_n(2,t)$. Moreover,

$G_n(1,t) \geq G_n(2,t)$ iff $t \geq T_n$. Since $G_n(1,t) = (1-q)P(n-1,t) \geq (1-q)P_L(n-1,t)$ and $G_n(2,t) = (1-q^2)P(n-2,t) \leq (1-q^2)P_U(n-2,t)$, the desired conclusions easily follow from Lemma 10 below. □

Lemma 10. *For* $u = 1$ *and for* $j = 0, 1, \cdots,$ *as* $n \to \infty$,

$$P_L(n-j,nx) \sim \left\{\frac{(1-q)^{-j}x^{-j}}{1-(1-q)^{-1}x^{-1}}\right\} \cdot \frac{(1-q)^n(nx)^n e^{-nx}}{n!}, \quad x > (1-q)^{-1}; \tag{25}$$

and

$$P_U(n-j,nx) \sim \left\{\frac{(1-q)^{-j}x^{-j}}{1-\frac{1+q}{1-q}x^{-1}}\right\} \cdot \frac{(1-q)^n(nx)^n e^{-nx}}{n!}, \quad x > \tfrac{1+q}{1-q}. \tag{26}$$

Proof. Equation (21) immediately yields

$$P_L(n-j,nx)/\{(1-q)^n(nx)^n e^{-nx}/n!\} = \sum_{k=0}^{n-j} \frac{n!}{k!n^{n-k}}(1-q)^{k-n}x^{k-n} = \sum_{i=j}^{n} \frac{n!}{(n-i)!n^i}(1-q)^{-i}x^{-i}.$$

The i–th term converges to $(1-q)^{-i}x^{-i}$ as $n \to \infty$ $(i \geq j)$, and when $x > (1-q)^{-1}$, the latter sum converges to $\sum_{i=j}^{\infty}(1-q)^{-i}x^{-i} = \frac{(1-q)^{-j}x^{-j}}{1-(1-q)^{-1}x^{-1}}$. This establishes (25). The proof of (26) is similar with the i–th term converging to $(1+q)^{i-j}(1-q)^{-i}x^{-i}$ as $n \to \infty$. □

It appears one could use Lemma 10, with the roles of P_L and P_U reversed, to show T_n is of exact order n. But this does not work out. Nevertheless, we suspect $n^{-1}T_n$ is bounded below, and, in fact, that $n^{-1}T_n \to \frac{1+q}{1-q}$ as $n \to \infty$. We have insufficient numerical evidence to support such a conjecture, but this is suggested by Lemma 10 providing the ratio $\frac{P(n-1,nx)}{P(n-2,nx)}$ is well approximated either by the same ratio formed when the P's are both replaced by P_L's, or when both are replaced by P_U's. The slight numerical evidence we have, for $n \leq 18$, suggests that $n^{-1}T_n$ increases with n. But the ratios $n^{-1}T_n$, for such n, are quite a bit smaller than $\frac{1+q}{1-q}$.

Finally, we return to the question, raised in Section 1, of a military commander who wants his bombers to reach their designated targets with a specified probability α. For $t > 0$ and $0 < \alpha < 1$, define $N(t,\alpha) = \inf\{n: P(n,t) \geq \alpha\}$.

PROPOSITION 3. *For fixed* α, $0 < \alpha < 1$, and $t > 0$, *as* $t \to \infty$,

$$N(t,\alpha) = (\log(1/q))^{-1}\{t \log t - t \log(\tfrac{1}{u} \log \tfrac{1}{\alpha}) + o(t)\}.$$

PROOF. Let $M(t,\alpha) = \lfloor(\log(1/q))^{-1}\{t \log t - t \log(\frac{1}{u} \log \frac{1}{\alpha})\}\rfloor$. Clearly, it is enough to show for fixed δ, $0 < \delta < \min(\alpha,1-\alpha)$, that $M(t,\alpha-\delta) \leq N(t,\alpha) \leq M(t,\alpha+\delta)$ for large t. These inequalities hold if for large t,

$$P(M(t,\alpha-\delta),t) + \delta/2 \leq \alpha \leq P(M(t,\alpha+\delta),t) - \delta/2. \tag{27}$$

Since the number of enemy aircraft occurring between times 0 and t is asymptotically normal with mean t and variance t, there exist a constant $c > 0$ such that for large t, the number of such enemy aircraft is between $t - c\sqrt{t}$ and $t + c\sqrt{t}$ with probability exceeding $1 - \delta/2$. Clearly, for large t,

$$P(M(t,\alpha-\delta),t) < \delta/2 + P_U'(t),$$

where $P_U'(t)$ is the probability the bomber reaches its target when it encounters exactly $\lfloor t-c\sqrt{t}\rfloor$ enemy aircraft and $M(t,\alpha-\delta)$ missiles are allocated as evenly as possible. But it is easily seen that $P_U'(t)$ converges to $\alpha - \delta$ as $t \to \infty$. This establishes the left inequality in (27).

Now, consider the strategy that the bomber fires $\lfloor M(t,\alpha+\delta)/(t+c\sqrt{t})\rfloor$ missiles at each encountered enemy aircraft, *as long as they last.* Clearly, for large t,

$$P(M(t,\alpha+\delta),t) \geq (1 - \delta/2)P_L'(t),$$

where $P_L'(t)$ is the probability the bomber reaches its target when it encounters exactly $\lfloor(t+c\sqrt{t})\rfloor$ enemy aircraft, and the above strategy is adopted. It is easily verified that $P_L'(t)$ converges to $\alpha + \delta$ as $t \to \infty$. So $P(M(t,\alpha+\delta),t) \geq \alpha + \delta/2$, for large t, which is the inequality on the right side of (27). □

5. A continuous–time extension. This short section provides additional insight into why Conjecture B is more difficult to establish than either Conjecture A or C.

For n *ordered* defensive missiles, let q_i denote the probability the i–th missile fails to hit its target. Assume the missiles have to be fired in the reverse order, i.e., missile j must be fired no later than missile i for $i < j$. Further, let $K(n,t,q_1,\cdots,q_n)$ denote the optimal number of missiles to be fired at an enemy aircraft encountered at time t. This reduces to K(n,t), in the old notation, when $q_1 = q_2 = \cdots = q_n = q$.

It seems intuitively plausible that $K(n,t,q_1,\cdots,q_n)$ is nonincreasing in the first variable q_1, and nondecreasing in the "current variable" q_n. The first intuition is directly linked to Conjecture B and the second is directly linked to Conjecture C. For if $q_1 = q_2 = \cdots = q_n = q$, then, in the first circumstance,

$$K(n,t) = K(n,t,q_1,\cdots,q_n) \geq K(n,t,1,q_2,\cdots,q_n) = K(n-1,t),$$

which establishes Conjecture B, while, in the second circumstance,

$$K(n,t) = K(n,t,q_1,\cdots,q_n) \leq K(n,t,q_1,\cdots,q_{n-1},1) = K(n-1,t) + 1,$$

which establishes Conjecture C.

However, working with the first variable q_1 is much more difficult than working with the current variable q_n; we have no idea of how to show $K(n,t,q_1,\cdots,q_n)$ is nonincreasing in q_1. In contrast, it is quite easy to show that it is nondecreasing in q_n: Let $n \geq 2$ and assume, only for the sake of simplicity, that $q_n < q_n' < 1$, or that $u < 1$ and $q_n < q_n'$. Further, let $k = K(n,t,q_1,\cdots,q_{n-1},q_n')$. Then (using notation with an obvious meaning), we have for $i > k$,

$$\frac{G_n(i,t,q_1,\cdots,q_n)}{G_n(k,t,q_1,\cdots,q_n')} \leq \frac{G_n(i,t,q_1,\cdots,q_n)}{G_n(i,t,q_1,\cdots,q_n')} = \frac{1 - uq_nq_{n-1}\cdots q_{n-i+1}}{1 - uq_n'q_{n-1}\cdots q_{n-i+1}}$$

$$\leq \frac{1 - uq_nq_{n-1}\cdots q_{n-k+1}}{1 - uq_n'q_{n-1}\cdots q_{n-k+1}} = \frac{G_n(k,t,q_1,\cdots,q_n)}{G_n(k,t,q_1,\cdots,q_n')}.$$

So $K(n,t,q_1,\cdots,q_n) \leq k = K(n,t,q_1,\cdots,q_{n-1},q_n')$.

A close analogy can be drawn with the discrete–time bomber problem (described in Section 3). In order to establish Conjecture A, the function K was shown to be monotone in the "current variable" r_{t-1}. This is relatively easy, certainly much easier than showing monotonicity in the "first variable" r_1, or in any of the other r–variables. (The monotonicity seems plausible for all of them.) Conjectures A and C require relatively simple recursive arguments because they are linked to "current variables". Since "first variables" are deeply embedded in the recursive structure of the bomber problem, and are much less accessible mathematically, one should reasonably expect Conjecture B to be difficult to establish. Experience to date confirms this assessment.

6. Numerical work. All of our numerical work is limited to the important case $u = 1$, so that any surviving enemy aircraft is definitely lethal. While the recursive formulas for the discrete bomber problem (DBP) are very easily programmed, the programming for the continuous bomber problem (CBP) is much more complicated.

6.1. Computations for the DBP. Two significant things should be said about our numerical work with the DBP. Firstly, our program has confirmed Conjecture B for literally tens of thousands of randomly generated pairs (q,r), usually for time points $t \leq 12$ and (missile) arsenal sizes $n \leq 20$. Here, q is the missile failure probability, and r is the probability an enemy aircraft is encountered at a given time point t. The range of (n,t) values for which reliable calculations are possible is restricted further for some extreme values of (q,r). Secondly, we discovered that an important logconcavity conjecture, described by Klinger and Brown (1968) for the CBP, does not hold for the DBP. Specifically, it *is* possible for $P(n+1,t)P(n-1,t) - (P(n,t))^2$ to be strictly positive. Counterexamples are easily found in settings that can not possibly be attributed to round-off error. A simple case is $(q,r) = (.6,.58)$, which produces a counterexample when $(n,t) = (8,3)$. If similar examples hold for the CBP, which we suspect, then a very attractive, simple proof of Conjecture B, based on the logconcavity, is ruled out. However, a fairly extensive numerical search has failed, so far, to confirm this suspicion.

6.2. The computer program for the CBP. Our computer program for the CBP is designed to find the continuous, piecewise *polynomial* functions $\tilde{P}(n,\cdot)$, described in the remark surrounding equation (16). This is done recursively in n, starting with $\tilde{P}(0,t) \equiv 1$ and using the variants of equations (1) and (2) described in the remark (rather than the variant of (1) and equation (16), which might have worked just as well). The main tasks are to locate and keep track of the nodal points where one polynomial function switches to another, to determine the proper polynomial function, among a number of possible candidates, for each adjacent pair of nodes, and to keep track of its coefficients. In this process, one starts with a node on the left side and must locate the proper node on the right side of the interval by using a root finder and a table of previously identified nodes. Integrations are performed *formally* (rather than numerically), using the coefficients of relevant, previously determined polynomials, so as to maintain good numerical precision; several possibly relevant polynomials must be integrated and compared to determine which is actually relevant for the recursion and nodes under consideration. A further complicating factor is caused by the fact that the number of nodes builds up fairly rapidly with n: Nodes, relevant to the recursion index n, occur at the change points y_{ni} (defined in Section 2) at which $K(n,\cdot)$ changes values, and, sometimes, at previous change points y_{mj}, $m < n$.

Some care is required to properly identify new nodes. "Close calls" occur. (According to Theorem 2, some of the change points y_{ni} are repeated for larger values of n. Some are not.) This task is made feasible by our use of formal integrations.

We do not assume a priori that any of the conjectured change points y_{ni} exists; the computer program discovers their existence and their values.

Our program is written in the Gauss programming language, which greatly simplifies the heavy bookkeeping burden. Fortran would be an extremely awkward language to use!

It should be stressed, for the CBP (when $u = 1$), that it is possible to determine the functions $P(n,t)$, implicitly, *for all values of* t (by finding the values of polynomial coefficients rather than the values of the polynomials at selected values of t). In contrast, numerical calculations for the DBP are necessarily restricted to a finite range of t values.

6.3 Numerical results for the CBP. We are able, for most values of q, to carry the recursion to $n = 18$. This takes about five minutes on an IBM–AT. Carefully testing the validity of Conjecture B (without making unwarranted assumptions) takes additional time. Such calculations have been made for several thousand cases of randomly generated values of q. In no case was a contradiction found to Conjecture B. One is restricted to a smaller upper bound on n (than 18) when q is small.

Our numerical results do not agree with the formula for $\tilde{P}(4,t)$ given by Klinger and Brown (1968). The constant term of the polynomial $\Phi(4,x)$, in the range $\frac{q^3}{1-q^3} \leq x < \frac{q}{1-q}$, should be $1 + \frac{q^6/2}{1+q+q^2}$; in the range $\frac{q}{1-q} \leq x < c(q)$, should be $1 + \frac{q^6/2}{1+q+q^2} + \frac{q^3}{3}(1-\frac{q}{2})$; and in the range $x \geq c(q)$, should be:

$$1 + \frac{q^6/2}{1+q+q^2} + \frac{q^3}{3}(1-\frac{q}{2}) + q(1-q)[1 - \frac{q}{2} + \frac{2}{3}q^2 - \frac{q^3}{2(1+q)} - \frac{2\sqrt{2q}}{3}]c(q)$$
$$+ \frac{1}{4}q(1-q)^2(2-q)c^2(q) + \frac{1}{3!}q(1-q)^3c^3(q) - \frac{1}{4!}(1-q)^4c^4(q).$$

The coefficient of x in the range $x \geq c(q)$ is $(1-q)[1 + q^2 - \frac{q^3}{3!} + \frac{q^4}{2(1+q)} + \frac{2q\sqrt{2q}}{3}]$.

6.4 An example: the case q = .5. We shall consider the special case $q = .5$, $u = 1$, for $n \leq 10$. Recall that the time t at which $K(n,\cdot)$ switches values the i–th time is denoted by y_{ni}. Since everything for this case is consistent with conjectures B and C, this switch is from "n–i+1" to "n–i", $1 \leq i \leq n-1$. These change points are shown in Table 1 below.

Change Points (y_{ni}–values) for $K(n,\cdot)$

n \ i	1	2	3	4	5	6	7	8	9
2	1								
3	.333333	3							
4	.142857	1	5						
5	.066667	.333333	1.31662	7.15190					
6	.032258	.142857	.681605	2.33729	9.24704				
7	.015873	.066667	.333333	1	2.72119	11.4278			
8	.007874	.032258	.142857	.452728	1.25383	3.58515	13.5640		
9	.003922	.015873	.066667	.265421	.671897	1.50398	4.25024	15.7565	
10	.001957	.007874	.032258	.142857	.369038	.877618	2.09098	4.96562	17.9118

Table 1

Some of the entries are easily recognizable decimal expansions of simple fractions, as are all of the polynomial coefficients described below.

As described in Theorem 2, the table has many repetitions, which show up as (chess) "knight moves": $y_{(n+2)(i+1)} = y_{ni}$ for certain pairs (n,i). Another feature is the nesting described earlier: $y_{n(i-1)} < y_{(n+1)i} < y_{ni}$.

For fixed n, the degree of the piecewise polynomials describing $\tilde{P}(n,t)$ increase with i. The number of these grows rapidly with n, but the salient features of these are apparent for

$n \leq 7$: $\tilde{P}(0,t) = 1$; $\tilde{P}(1,t) = 1 + .5t$; $\tilde{P}(2,t) = 1 + .75t$, and $= 1.125 + .5t + .125t^2$ to the left and right of the change point y_{21}, respectively; $\tilde{P}(3,t) = 1 + .875t$, $= (1.020833\cdots) + .75t + .1875t^2$, and $= (1.5833\cdots) + .5625t + .125t^2 + (.020833\cdots)t^3$ over the intervals $(0,y_{31})$, (y_{31},y_{32}), and (y_{32},∞), respectively. This pattern continues for $n = 4$. For $n = 5$, one finds that an additional node is encountered in each of the intervals (y_{52},y_{53}) and (y_{53},y_{54}): for the subinterval $(y_{52}=.33\cdots,1)$, $\tilde{P}(5,t) = (1.01145833\cdots) + .875t + .328125t^2$, and for the subinterval $(1,y_{53}=1.317-)$, $\tilde{P}(5,t) = .975 + .984375t + .21875t^2 + (.03645833\cdots)t^3$. Likewise, the interval $(y_{53}=1.317-,y_{54}=7.152-)$ has an additional node, at $t = 3$. We emphasize that these additional nodes do not affect the value of $K(5,t)$, but they do signal a change in the form of $\tilde{P}(5,t)$. There is only one additional node when $n = 6$. Then there are two additional nodes when $n = 7$, both occurring between the *same* pair of change points, y_{75} and y_{76}. So the basic pattern seems clear, but the details are complicated.

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