ON THE PITMAN CLOSENESS OF SOME SEQUENTIAL ESTIMATORS

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For a general class of stopping rules, the connection between median unbiasedness and the Pitman closeness of statistical estimators is examined in the sequential case. It is also shown that in the light of the Pitman closeness, sequential shrinkage estimators of the multinormal mean vector dominate the classical maximum likelihood estimator.

1. Introduction. The Pitman closeness (or nearness) criterion (PCC) is an intrinsic measure of the comparative behavior of two estimators without requiring the finiteness of their second moment. This pair-wise comparison has been extended to suitable classes of (equivariant) estimators by Ghosh and Sen (1989) and Nayak (1989). In this context, ancillarity and median unbiasedness (MU) play a fundamental role and provide an easy access to the verification of the PC dominance in the classical nonsequential case. In the multivariate case, the conventional Stein-rule or shrinkage estimators may not belong to such a class of equivariant estimators, and hence, the characterizations mentioned above may not apply directly to such estimators. Nevertheless, Sen, Kubokawa and Saleh (1989) have shown that for the p($\geq$ 2)-variate normal distribution, the usual Stein-rule estimators of the mean vector dominate the classical maximum likelihood estimator (MLE) in the light of the PCC as well. Also, in the sequential case, under the usual quadratic error loss (incorporating the cost of sampling), the dominance of the shrinkage estimators of the multinormal mean vector over the classical MLE has been studied by Ghosh, Nickerson and Sen(1987). Our main objectives are to extend the results in Ghosh and Sen (1989) in the general

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sequential case (covering multivariate situations as well) and to establish the PC dominance of the sequential shrinkage estimators of the multinormal mean vector. These are accomplished in the next two sections.

2. PCD of sequential estimators. Note that for a parameter θ, if T_1 and T_2 are two rival estimators, then T_1 is said to be Pitman-closer than T_2 if

(2.1) $\quad P_\theta\{ |T_1 - \theta| \leq |T_2 - \theta| \} \geq 1/2$, for all θ ;

see Pitman (1937). Further, an estimator T of θ is said to be median unbiased (MU) if

(2.2) $\quad P_\theta\{ T \leq \theta \} = P_\theta\{ T \geq \theta \}$, for all θ ;

see Lehmann (1983,p.6). Consider the class **C** of all estimators of the form

(2.3) $\quad U = T + Z$, T is MU for θ , and T and Z are independently distributed.

Then, it follows from Ghosh and Sen (1989) that

(2.4) $\quad P_\theta\{| T - \theta | \leq |U - \theta| \} \geq 1/2$, for all θ and all $\mathbf{U} \in \mathbf{C}$.

Although not necessary, the sufficiency of T and ancillarity of Z lead to (2.3), and hence, the PC dominance holds for MU sufficient statistics under very general conditions. One can imagine an immediate extension of (2.4) to the sequential case provided (2.3) is shown to be valid in such a setup too.

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.) with a distribution function (d.f.) $F_\theta(x)$, $x \in R$, $\theta \in \Theta \subset R$. A sequential estimation procedure is governed by a stopping rule (N) and an estimation rule (T). The stopping number N is a positive integer valued r.v., such that the event [N=n] depends only on the outcome $X_1,\ldots,X_n$, for every $n \geq 1$. Further, given that N=n, the estimation rule prescribes an estimator , say, T_n, which is also based on $X_1,\ldots,X_n$, for every $n \geq 1$. Thus, a sequential estimator T_N depends on both the stopping number N and the estimation rule. Now, for every $n \geq 1$, consider the transformation :

(2.5) $\quad (X_1,\ldots,X_n) \rightarrow [\, T_n, \underset{\sim}{V}_n, \underset{\sim}{W}_n \,]$ (where $\underset{\sim}{V}_n$ could be null).

Recall that transitivity and sufficiency of T_n typically dictate the transformation in (2.5). Let $\mathcal{B}_T^{(n)}$ and $\mathcal{B}_W^{(n)}$ be the sigma subfield generated by T_n and $\underset{\sim}{W}_n$, respectively, for $n \geq 1$. We assume that the following three conditions hold :

(2.6) For every $n \geq 1$, $[N=n]$ is $\mathcal{B}_W^{(n)}$- measurable ;

(2.7) For every $n \geq 1$, $Z_n = v_n(\underset{\sim}{W}_n)$ is $\mathcal{B}_W^{(n)}$-measurable and T_n and $\underset{\sim}{W}_n$ are independently distributed ;

(2.8) For every $n \geq 1$, T_n is MU for θ .

Finally, let C^o be the class of all (sequential) estimators of the form $U_N = T_N + Z_N$, where the stopping number N satisfies (2.6), Z_N satisfies (2.7) and T_N (2.8).

THEOREM 1. Under (2.5) through (2.8),

(2.9) $P_\theta\{ |T_N - \theta| \leq |U_N - \theta| \} \geq 1/2$, for every $U_N \in C^o$ and $\theta \in \Theta$.

Outline of the proof. It suffices to show that $P_\theta\{ Z_N(T_N - \theta) \geq 0\} \geq 1/2, \ \forall \theta \in \Theta$, or equivalently,

(2.10) $\Sigma_{n\geq 1} P_\theta\{N=n\}. P_\theta\{ Z_n(T_n - \theta) \geq 0 \mid N = n \} \geq 1/2, \ \forall \theta \in \Theta$.

Now, (2.6), (2.7) and (2.8) ensure that for every $n(\geq 1)$, $P_\theta\{ Z_n(T_n - \theta) \geq 0 | N=n\}$ is $\geq 1/2$, and hence, (2.9) follows from (2.10). Q.E.D.

In passing, we may remark that (2.6), (2.7) and (2.8) imply (2.3) in the sequential case. If $\mathcal{B}_Z^{(n)}$ be the sigma subfield generated by Z_n, then by (2.7), $\mathcal{B}_Z^{(n)} \subset \mathcal{B}_W^{(n)}$, for every n, and hence, in (2.10), given N=n, Z_n and T_n are conditionally independent. In many applications, T_N may be identified as a function of a sufficient statistic (in a sequential setup), and we need to choose such a function in such a way that T_N is MU for θ. As an illustration, we consider the following simple example:

Let $\{X_i, i \geq 1\}$ be i.i.d.r.v. with the normal(θ, σ^2) distribution, where both θ and σ^2 are unknown. For every $n \ (\geq 2)$, let $s_n^2 = (n-1)^{-1}\Sigma_{i=1}^n(X_i - \bar{X}_n)^2$ be the sample variance (based on $X_1,\ldots,X_n$), and consider a stopping number $N = N_K$, defined by

(2.11) $N = \inf\{ n \geq n_o : \psi(n) \geq Ks_n^2 \}$, $K > 0$ (usually taken large) ,

where $\psi(n)$ is a monotone increasing function of n , and n_o (≥ 2) is the minimum sample size. Such a stopping number arises in the context of bounded-width confidence intervals for θ (where $\psi(n) \equiv n$) or minimum risk point estimation of θ (where $\psi(n) \equiv n^2$).

Let us consider the Helmert transformation :

(2.12) $W_i = [X_1+\ldots+X_{i-1} - (i-1)X_i]/[i(1-1)]^{\frac{1}{2}}$, $i \geq 2$, $W_1 = 0$.

Then, we have

(2.13) $T_n = \bar{X}_n = n^{-1}\Sigma_{i=1}^n X_i$ and $s_n^2 = (n-1)^{-1}\Sigma_{i=1}^n W_i^2$, for every $n \geq 2$.

Note that for every n, T_n is independent of $\underset{\sim}{W}_n = (W_1,\ldots,W_n)'$ [and hence of s_k^2, $k \leq n$]. Further, for every $n \geq 1$, T_n has a normal distribution with mean θ and variance $n^{-1}\sigma^2$, so that T_n is MU for θ . Hence, for any $Z_n = v_n(\underset{\sim}{W}_n)$, (2.6) through (2.8) hold. This leads to the PC dominance of $T_N = \bar{X}_N$ within the class C^o of estimators of the form $T_N + Z_N$ where $\{Z_n\}$ satisfies the $\mathcal{B}_W^{(n)}$- measurability condition. From considerations of equivariance, we may consider the following group of transformations :

(2.14) $G = \{ g_{a,b}(\underset{\sim}{X}_n) = a\underset{\sim}{1}_n + b\underset{\sim}{X}_n\}$, a real, b > 0,

so that G-equivariant estimators of θ [under a loss $L(x,\theta;\sigma) = \rho((x-\theta)/\sigma)$ for a suitable ρ] have the representation :

(2.15) $m_n(\underset{\sim}{X}_n) = \bar{X}_n + \phi(||\underset{\sim}{W}_n||^{-1}\underset{\sim}{W}_n)\upsilon(\underset{\sim}{W}_n)$, $n \geq 1$,

where $\underset{\sim}{X}_n = (X_1,\ldots,X_n)'$, $\phi(.)$ and $\upsilon(.)$ are suitable functions and $||.||$ stands for the Euclidean norm. Identifying Z_n with the second term on the right hand side of (2.15), we are led to the class C^o of G- equivariant estimators in (2.9). In the context of estimation of the scale parameter, typically, we have a nonnegative T_N , and in that case, we may set C^{o*} as the class of estimators of the form $U_N = T_N\{1 + Z_N\}$, where (2.6)-(2.8) pertain to the T_N and Z_N . Then along the same line as in Theorem 1, we obtain that within the class C^{o*} of estimators, the nonnegative, MU estimator T_N is the Pitman closest one. We may also remark that in the conventional nonsequential case, Brown, Cohen and Strawderman (1976) have shown that a MU estimator not solely based on a sufficient statistic can be dominated by a version of the sufficient statistic which is MU. In this respect, they confined to the class of all MU estimators of θ , whereas our U_n needs not be MU for θ . In that respect, we have a larger class. However, in passing we may note that the Brown-Cohen-Strawderman result [Corollary 4.1] extends directly to the sequential case under (2.6)-(2.8).

Let us now consider a multiparameter extension of Theorem 1. We conceive of a p-vector $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_p)'$, for some $p \geq 1$, so that in (2.5), $\underset{\sim}{T}_n$ is also a p-vector. The condition (2.6) stands as it is; in (2.7), $\underset{\sim}{Z}_n$ is also a p-vector, while, we may modify (2.8) as follows: For every $n \geq 1$ and arbitrary ℓ,

(2.8') $\ell'(T_n - \theta)$ is MU for 0.

Moreover, in this case, for a given positive semi-definite (p.s.d.) matrix Q, we define the quadratic norm $||d||_Q^2 = d'Qd$, and extend (2.1) as follows: T_1 is Pitman-closer than T_2, if $P_\theta\{ ||T_1 - \theta||_Q \leq ||T_2 - \theta||_Q \} \geq 1/2$, for every $\theta \in \Theta$. Then we have the following.

THEOREM 2. Under (2.6), (2.7) and (2.8'), for the class C^o of estimators of the form $U_N = T_N + Z_N$, we have for all p.s.d. Q,

(2.16) $P_\theta\{ ||T_N - \theta||_Q \leq ||U_N - \theta||_Q \} \geq 1/2$, $\forall\, U_N \in C^o$, $\theta \in \Theta$.

Outline of the proof. It suffices to show that

(2.17) $P_\theta\{ Z_N'Q(T_N - \theta) \geq 0 \} \geq 1/2$, for every $\theta \in \Theta$.

As in (2.10), we rewrite (2.17) as

(2.18) $\Sigma_{n \geq 1} P_\theta\{N = n\} P_\theta\{ Z_n'Q(T_n - \theta) \geq 0 \mid N=n \}$.

Now, $Z_n'Q$ is $\mathcal{B}_W^{(n)}$-measurable, so that by (2.6) and (2.8'), given $N = n$, $Z_n'Q(T_n - \theta)$ is MU for 0. Hence, (2.18) is $\geq 1/2$, and the proof is complete.

Remarks. Note that in Theorem 2, Q is allowed to be arbitrary, so that the PC dominance holds for all p.s.d. Q. On the other hand, (2.8') is more restrictive than the usual definition of MU. Often, it may be easier to verify (2.8') by using the possible diagonal symmetry of T_n around θ: T_n is diagonally symmetric about θ if $T_n - \theta$ and $\theta - T_n$ both have the same distribution. Note that this diagonal symmetry is not necessary for (2.8'). Further, (2.8') [or (2.8)] is also a sufficient condition for Theorem 2 to hold. However, without (2.8') verification of (2.17) may be highly dependent on the distribution of Z_n, given N=n. Thus, the simplicity of Theorem 2 crucially rests on (2.8'). As an illustration, we consider the following.

Let $\{X_i, i \geq 1\}$ be i.i.d.r.v. with the multinormal distribution with mean vector θ and dispersion matrix Σ (both unknown). In this case, in the context of sequential estimation of θ, Ghosh, Sinha and Mukhopadhyay (1976) and Woodroofe (1977) have considered a stopping number which may be defined as

(2.19) $N = \inf\{ n \geq n_o : \psi(n) \geq K[\,\mathrm{trace}(QS_n)\,] \}$, $K(>0)$ is usually large.

In this context, $\psi(n)$ may be defined as in (2.11) while $\underset{\sim}{S}_n$ is the sample covariance matrix based on $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$, for $n \geq 2$. The Helmert transformation in (2.12) extends directly to the p-variate case, so that the characterization in (2.13) also extends directly to the p-variate case ; here, $\underset{\sim}{S}_n = (n-1)^{-1}\Sigma_{i=1}^{n} \underset{\sim}{W}_i\underset{\sim}{W}_i'$, for $n \geq 2$. The equivariance in (2.14) also extends to the class of nonsingular linrear transformations $\underset{\sim}{X}$ to $\underset{\sim}{a} + \underset{\sim}{B}\underset{\sim}{X}$, where $\underset{\sim}{B}$ is nonsingular. Moreover, since $\bar{\underset{\sim}{X}}_n - \underset{\sim}{\theta}$ has a multinormal law with null mean vector, its distribution is diagonally symmetric about $\underset{\sim}{0}$, independently of the $\underset{\sim}{W}_i$ (and hence, $\underset{\sim}{Z}_n$ and N=n). Hence, (2.6), (2.7) and (2.8') all hold, and Theorem 2 yields the PC dominance of $\bar{\underset{\sim}{X}}_N$ within the class of equivariant estimators.

3. PCD of sequential shrinkage estimators. Let us refer to the example considered at the end of the last section. For p, the number of coordinates of $\underset{\sim}{\theta}$, greater than 2, the MLE $\bar{\underset{\sim}{X}}_n$ is known to be inadmissible [see, Stein (1956)] , and various other (shrinkage or Stein-rule) estimators have been considered in the literature which dominate the MLE in quadratic error loss. Ghosh, Nickerson and Sen (1987) showed that such a quadratic error risk dominance holds in the sequential case too. Also, for $p > 1$, Sen, Kubokawa and Saleh (1989) have shown that in the light of the PCC, the Stein-rule estimators dominate the MLE $\bar{\underset{\sim}{X}}_n$ in the conventional fixed sample size case. Note that the Stein-rule estimators may not belong to the class C^o in Theorem 2, and hence, the characterization in Theorem 2 may not apply to these estimators. Thus, there is a natural interest in studying the PC dominance of such Stein-rule estimators in the general sequential case. This will be done here.

For the sake of simplicity, we consider the most simple problem in the sequential estimation of a multivariate normal mean vector $\underset{\sim}{\theta}$ when the dispersion matrix $\underset{\sim}{\Sigma}$ is of the form $\sigma^2\underset{\sim}{I}_p$, where $\underset{\sim}{\theta}$ and $\sigma^2(> 0)$ are unknown. Based on n i.i.d. observations $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$, the MLE of $\underset{\sim}{\theta}$ is $\bar{\underset{\sim}{X}}_n = n^{-1}\Sigma_{i=1}^{n}\underset{\sim}{X}_i$, $n \geq 1$. Consider the class of James-Stein estimators $\underset{\sim}{\delta}_n^b(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n)$ of the form

$$(3.1) \qquad \underset{\sim}{\delta}_n^b = \underset{\sim}{\delta}_n^b(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n) = \{ 1 - bs_n^2(n||\bar{\underset{\sim}{X}}_n||^2)^{-1} \} \bar{\underset{\sim}{X}}_n , \quad n \geq 2,$$

where

$$(3.2) \qquad s_n^2 = (n\,p)^{-1}\Sigma_{i=1}^{n}(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}}_n)'(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}}_n) , \quad n \geq 2 ,$$

and b is a nonnegative shrinkage factor. In this setup, we conceive of the null pivot and the adjustments for any other given pivot θ_o are routine in nature. Also, in this setup, the stopping number N in (2.19) simplifies further, and we may consider a well defined stopping rule N, such that for every $n \geq 2$, $[N=n]$ depends only on the s_k^2, $k \leq n$. Finally, we define the class of sequential shrinkage estimators δ_N^b by (3.1) allowing $\delta_N^b = \delta_n^b$ when $N = n$, for $n \geq 2$. In view of the special structure of the dispersion matrix, one may consider here a simple quadratic error loss

(3.3) $\quad L(\delta_n, \theta) = (\delta_n - \theta)'(\delta_n - \theta) + cn$, $n \geq 1$, $c > 0$.

Then, it follows from Theorem 1 of Ghosh, Nickerson and Sen (1987) that under the loss in (3.1) and the stopping rule : $N = \inf\{ n \geq 2 : n \geq (p/c)^{\frac{1}{2}} s_n\}$, the risk of the sequential estimator δ_N^b is smaller than (or equal to) that of $\bar{X}_N$, for every $c > 0$ and every $b \in (0, 2(p-2))$, $p \geq 3$, $\theta \in \Theta \subset R^p$. It is customary to take c small, so that p/c is large, and this is then comparable to (2.11). Also, in a fixed sample size case, it follows from Theorem 1 of Sen, Kubokawa and Saleh (1989) that for every $b \in (0, (p-1)(3p+1)/2p)$, $p \geq 2$, δ_n^b dominates $\bar{X}_n$ in the light of the PC measure. Our basic goal is to extend the later result to the sequential case, so that it would provide a result complementary to Theorem 1 of Ghosh et al. (1987).

Note that δ_n^b in (3.1) does not belong to the class of estimators in Theorem 2. Moreover, when we compare δ_N^b and $\bar{X}_N$, both based on the same stopping number N, it is not necessary to incorporate the second term in (3.3) in this comparison. Hence, we shall use the conventional PCC: δ_N^b dominates $\bar{X}_N$ in the PC measure if

(3.4) $\quad P_{\theta,\sigma}\{ \ ||\delta_N^b - \theta|| \leq ||\bar{X}_N - \theta|| \ \} \geq 1/2, \quad \forall \theta, \sigma.$

For our further analysis, we define the stopping number N by (2.11) where s_n^2 is now defined by (3.2), and for the sake of simplicity of presentation, we take $\psi(n) = n$. In Section 4, we shall briefly mention the other cases.

THEOREM 3. <u>For the class of Stein-rule estimators in</u> (3.1) <u>and the stopping number in</u> (2.11), <u>the PC dominance in</u> (3.4) <u>holds, for every</u> $b \in (0, (p-1)(3p+1)/2p]$.

<u>Proof</u>. Note that by (3.1), for every $n \geq 2$,

(3.5) $\quad ||\delta_n^b - \theta||^2 = ||\bar{X}_n - \theta||^2 + n^{-2}b^2 s_n^4 ||\bar{X}_n||^{-2} - 2b(n||\bar{X}_n||^2)^{-1}\bar{X}_n'(\bar{X}_n - \theta).$

Hence, it suffices to show that

(3.6) $\quad P_{\underset{\sim}{\theta},\sigma}\{ N\bar{X}_N'(\bar{X}_N - \underset{\sim}{\theta}) \geq (b/2)s_N^2 \} \geq 1/2, \ \forall \underset{\sim}{\theta},\sigma$ and $b \in (0,(p-1)(3p+1)/2p]$.

Let us introduce the following notations. Let $\underset{\sim}{\mu} = \frac{1}{2}\underset{\sim}{\theta}$, $\lambda = \sigma^{-2}\underset{\sim}{\mu}'\underset{\sim}{\mu} = ||\underset{\sim}{\theta}||^2/4$, $b^o = b/2$, and let $G_p^{(\Delta)}(x) = 1 - \bar{G}_p^{(\Delta)}(x)$, $x \in R^+$ be the d.f. of a noncentral chi squared r.v. with p degrees of freedom (DF) and noncentrality parameter Δ (≥ 0). Then, on noting that $\bar{X}_n'(\bar{X}_n - \underset{\sim}{\theta}) = ||\bar{X}_n - \underset{\sim}{\mu}||^2 - \lambda\sigma^2$, we may rewrite the left hand side of (3.6) as

$$(3.7) \quad \begin{aligned} &P_{\underset{\sim}{\mu},\sigma}\{ N||\bar{X}_N - \underset{\sim}{\mu}||^2/\sigma^2 \geq N\lambda + b^o s_N^2/\sigma^2 \} \\ &= \Sigma_{n\geq 2} E\{ I_{[N=n]} \bar{G}_p^{(n\lambda)}(n\lambda + b^o s_n^2/\sigma^2) \} \\ &= E[\bar{G}_p^{(N\lambda)}(N\lambda + b^o s_N^2/\sigma^2)]. \end{aligned}$$

Therefore, it suffices to show that

(3.8) $\quad E[\bar{G}_p^{(N\lambda)}(N\lambda + bs_N^2/\sigma^2)] \geq 1/2$, for every $b \in (0,(p-1)(3p+1)/4p]$, $\lambda \geq 0$.

Note that

$$(3.9) \quad (\partial/\partial b)E[\bar{G}_p^{(N\lambda)}(N\lambda + bs_N^2/\sigma^2)] = -E[(s_N^2/\sigma^2)g_p^{(N\lambda)}(N\lambda + bs_N^2/\sigma^2)] \leq 0 \text{ , for every } b \geq 0,$$

where $g_p^{(\Delta)}(y)$ stands for the noncentral chi square pdf with p DF and noncentrality parameter Δ . Thus, if we verify that (3.8) holds for any b arbitrary close to $(p-1)(3p+1)/4p$, then it follows that it would also hold for all smaller (but positive) values of b. Thus, if we let $\kappa = (p-1)(3p+1)/4p$, then it suffices to show that

(3.10) $\quad E[\bar{G}_p^{(N\lambda)}(N\lambda + \kappa s_N^2/\sigma^2)] \geq 1/2$, for every $\lambda \geq 0$.

In this context, we may refer to Theorem 1 of Sen, Kubokawa and Saleh (1989) where it is shown that in the fixed sample size case, for every $n \geq 2$, $p \geq 2$,

(3.11) $\quad E[\bar{G}_p^{(n\lambda)}(n\lambda + bs_n^2/\sigma^2)] \geq 1/2$, for every $b \in (0, \kappa]$, $\lambda \geq 0$.

Thus, the crux of the problem is to verify that (3.11) holds in the sequential case. The actual proof is lengthy and complicated too. Hence, for the sake of simplicity of presentation, we shall provide a broad outline of the proof.

First, we consider the asymptotic setup of Chow and Robbins(1965) or Robbins (1959) where in (2.11) [with N designated as N_K] K is allowed to go to + . Note that $s_n^2/\sigma^2 \to 1$ almost surely (a.s.) as $n \to \infty$, and $N_K \to \infty$ a.s. as $K \to \infty$. Further,

$\bar{G}_p^{(N\lambda)}(N\lambda + \kappa s_N^2/\sigma^2)$ is a bounded r.v. assuming values in [0,1]. Hence, in this case, convergence in probability would ensure convergence in the first moment too. Finally, for any $n \geq 1$ and $\lambda \geq 0$, $\bar{G}_p^{(n\lambda)}(n\lambda + y)$ has a uniformly bounded and continuous first derivative (a.e. y), for all $p \geq 2$. Therefore, it suffices to show that for n sufficiently large, $\bar{G}_p^{(n\lambda)}(n\lambda + \kappa)$ is $\geq$ 1/2 . Now, by Lemma 2.2 of Sen et al. (1989), $\bar{G}_p^{(n\lambda)}(n\lambda + \kappa)$ is nonincreasing in $(n\lambda)$, while, proceeding as in Theorem 2 of Sen (1989), we obtain that $\lim_{(n\lambda)\to\infty} \bar{G}_p^{(n\lambda)}(n\lambda + \kappa) = 1/2$. Hence, $\bar{G}_p^{(n\lambda)}(n\lambda + \kappa)$ is $\geq 1/2$, for every $n, \lambda : n\lambda \geq 0$. This simple method may not work out for the case where K in (2.11) is held fixed, and hence, a more elaborate proof is necessary.

For every $n \geq 1$ and $\lambda \geq 0$, let $m_p^{(n\lambda)}$ be the median of the d.f. $G_p^{(n\lambda)}$, $p \geq 2$. It follows from Sen (1989) that $m_p^{(n\lambda)}$ is nondecreasing in n and λ , and further, $m_{p+2}^{(n\lambda)} \leq p + n\lambda$, $\forall\, n \geq 1$, $\lambda \geq 0$. Let K be defined as in (2.11), and let

$$(3.12)\qquad n^* = \begin{cases} \min\{ n : \kappa n \geq pK\sigma^2 \}, & \text{if } 2K\sigma^2 \geq \kappa ; \\ \min\{ n : \kappa(n-1) \geq (p-2)K\sigma^2\}, & \text{if } 2K\sigma^2 < \kappa . \end{cases}$$

Then the left hand side of (3.10) can be written as

$$(3.13)\quad E[\, \bar{G}_p^{(N\lambda)}(N\lambda + \kappa s_N^2/\sigma^2)\,] = \Sigma_{n\geq 2}\ E\{\, I_{[N=n]}\, \bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_n^2/\sigma^2)\,\}$$

$$= \Sigma_{n<n^*} E\{\, I_{[N\leq n]}[\, \bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_n^2/\sigma^2) - \bar{G}_p^{((n+1)\lambda)}((n+1)\lambda + \kappa s_{n+1}^2/\sigma^2)]\,\}$$

$$+\ E[\, \bar{G}_p^{(n^*\lambda)}(\, n^*\lambda + \kappa s_{n^*}^2/\sigma^2\,)\,] +$$

$$\Sigma_{n>n^*} E\{\, I_{[N\geq n]}[\, \bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_n^2/\sigma^2) - \bar{G}_p^{((n-1)\lambda)}((n-1)\lambda + \kappa s_{n-1}^2/\sigma^2)\,]\}\ .$$

At this stage, we may assume without any loss of generality that n^* is ≥ 2, as otherwise, the first sum on the right hand side becomes vacuous. The second term is [by (3.11)] bounded from below by 1/2. Thus, we need to show that each of the two sums in the right hand side of (3.13) is nonnegative. Towards this, we may note that

$$(3.14)\quad (\partial/\partial n)\bar{G}_p^{(n\lambda)}(n\lambda + y) = \lambda[\, g_{p+2}^{(n\lambda)}(n\lambda + y) - g_p^{(n\lambda)}(n\lambda + y)\,]$$

$$\lesseqgtr 0\ , \text{ according as } n\lambda + y \text{ is } \lesseqgtr m_{p+2}^{(n\lambda)}\ ,$$

where the last step follows from the unimodality results in Sen (1989). As a result, we obtain that

$$(3.15)\quad \bar{G}_p^{((n+1)\lambda)}((n+1)\lambda + y) - \bar{G}_p^{(n\lambda)}(n\lambda + y) \quad \begin{cases} \leq 0\ , & \text{if } (n+1)\lambda + y \leq m_{p+2}^{((n+1)\lambda)}, \\ \geq 0, & \text{if } n\lambda + y \geq m_p^{(n\lambda)}. \end{cases}$$

Further, note that $U_n^* = nps_n^2/\sigma^2$ has the central chi square d.f. with p(n-1)DF ($G_{p(n-1)}$) and $U_{n+1}^* = (n+1)ps_{n+1}^2/\sigma^2 = U_n^* + U_{n+1}$ where U_{n+1} has the d.f. G_p, independently of U_n^*. Also note that $(s_{n+1}^2 - s_n^2)/\sigma^2 = (n+1)^{-1}[\, p^{-1}U_{n+1} - s_n^2/\sigma^2\,]$, so that $E[(s_{n+1}^2 - s_n^2)/\sigma^2 \mid \mathcal{B}_n] = (n+1)^{-1}[\,1 - s_n^2/\sigma^2\,]$, for every $n \geq 2$. Finally, note that $[N \leq n] \iff [\, s_n^2 \leq n/K]$ is $\mathcal{B}_n$-measurable, and $[N \geq n] \iff [\, s_m^2 > m/K, \forall\, m \leq n-1]$ is $\mathcal{B}_{n-1}$-measurable. With these, we consider a typical term in the last sum on the right hand side of (3.13). Note that for any $n \geq n^*+1$,

$$\text{(3.16)}\quad E\{\, I_{[N\geq n]}[\, \bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_n^2/\sigma^2) - \bar{G}_p^{((n-1)\lambda)}((n-1)\lambda + \kappa s_{n-1}^2/\sigma^2)\,]\,\}$$
$$= E\{\, I_{[N\geq n]}[\, \bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_n^2/\sigma^2) - \bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_{n-1}^2/\sigma^2)\,]\,\} \;+$$
$$E\{\, I_{[N\geq n]}[\bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_{n-1}^2/\sigma^2) - \bar{G}_p^{((n-1)\lambda)}((n-1)\lambda + \kappa s_{n-1}^2/\sigma^2)\,]\} .$$

Now, by (3.15) and the fact that on $[N \geq n]$, $(n-1)\lambda + \kappa s_{n-1}^2/\sigma^2 > (n-1)\lambda + \kappa(n-1)/K\sigma^2 \geq m_{p+2}^{((n-1)\lambda)} \geq m_{p+2}^{(n^*\lambda)}$, we obtain that the second term on the right hand side of (3.16) is nonnegative. For the first term, we may note that

$$\text{(3.17)}\quad (\partial/\partial y)\bar{G}_p^{(n\lambda)}(n\lambda + y) = -g_p^{(n\lambda)}(n\lambda + y)\,,\; y \geq 0\,;$$

$$\text{(3.18)}\quad (\partial^2/\partial y^2)\bar{G}_p^{(n\lambda)}(n\lambda + y) = [\, g_p^{(n\lambda)}(n\lambda + y) - g_{p-2}^{(n\lambda)}(n\lambda + y)]/2$$
$$\geq 0\,,\; \text{for all}\;\; y : n\lambda + y \geq m_p^{(n\lambda)}\,,$$

where again the last step follows from the unimodality results in Sen (1989). Thus, $\bar{G}_p^{(n\lambda)}(n\lambda + y)$ is a convex function of y, for all $y \geq m_p^{(n\lambda)} - n\lambda$ ($\leq$ p-2). Note that for $n > n^*$, on the set $[N \geq n]$, $n\lambda + \kappa n^{-1}(n-1)s_{n-1}^2/\sigma^2 > n\lambda + (p-2) \geq m_p^{(n\lambda)}$, and $n\lambda + \kappa s_n^2/\sigma^2 = n\lambda + \kappa n^{-1}(n-1)s_{n-1}^2/\sigma^2 + \kappa U_n/np$, where U_n has the central chi square d.f. with p DF, independently of s_{n-1}^2 or $[N \geq n]$. Hence, using the convexity of $\bar{G}_p^{(n\lambda)}(n\lambda + \kappa n^{-1}(n-1)s_{n-1}^2/\sigma^2 + \kappa U_n/np)$ [in U_n] along with the Jensen inequality, we obtain that on $[N \geq n]$,

$$\text{(3.19)}\quad E[\, \bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_n^2/\sigma^2) \mid \mathcal{B}_{n-1}] \geq \bar{G}_p^{(n\lambda)}(n\lambda + \kappa n^{-1}(n-1)s_{n-1}^2/\sigma^2 + \kappa n^{-1})$$
$$\geq \bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_{n-1}^2/\sigma^2) + \kappa n^{-1}[\, s_{n-1}^2/\sigma^2 - 1]g_p^{(n\lambda)}(n\lambda + \kappa s_{n-1}^2/\sigma^2)\,,$$

where the last step follows from (3.17)-(3.18) and the fact that $\kappa s_{n-1}^2 n^{-1}(n-1)/\sigma^2$ is $\geq m_p^{(n\lambda)} - n\lambda$, for $N \geq n$. Finally, note that by (3.12), for $n > n^*$, $s_{n-1}^2/\sigma^2 \geq p/\kappa$ on $[N \geq n]$, and $\kappa < p-1 < p$. Hence, from (3.16) and (3.19), we obtain that

$$(3.20)\quad E\{ I_{[N\geq n]}[\bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_n^2/\sigma^2) - \bar{G}_p^{(n\lambda)}(n\lambda + \kappa s_{n-1}^2/\sigma^2)]\}$$
$$\geq E\{ I_{[N\geq n]}\, \kappa n^{-1}(s_{n-1}^2/\sigma^2 - 1)g_p^{(n\lambda)}(n\lambda + \kappa s_{n-1}^2/\sigma^2) \}$$
$$\geq 0, \text{ for every } n > n^*.$$

This shows that the last sum on the right hand side of (3.13) is nonnegative. The treatment for the first sum is very similar. Note that we would have two terms as in (3.16), where (3.15) would ensure that the second term is nonnegative. For the first term, we write $\bar{G}_p^{(n\lambda)}(.) = 1 - G_p^{(n\lambda)}(.)$, and note that (3.18) for the complementary part would ensure the convexity of $G_p^{(n\lambda)}$, and hence, the rest of the manipulations can be carried out as in (3.19) and (3.20). We therefore omit these details. Q.E.D.

4. Some general remarks. Our main result is contained in Theorem 3. For the sake of simplicity, we have made some assumptions which are possibly less general than they may appear actually in this context. For example, in (2.11), we have considered a general $\psi(n)$ while in the proof of Theorem 3, we have taken $\psi(n) \equiv n$. The treatment of $\psi(n) \equiv n^2$ (or other plausible forms) poses no extra regularity conditions but extra manipulations. The basic fact is that the stopping time N has a distribution governed by the sequence $\{ s_n^2 \}$, and the convergence properties of these s_n^2 provide the desired keys to the actual manipulations. Secondly, in (3.2), instead of the conventional divisor p(n-1), we have taken np. A similar modification was also made in Ghosh, Nickerson and Sen (1987) for dealing with the sequential shrinkage estimation in the light of the quadratic error loss functions. In the current case, it may not be necessary to have the divisor np ; p(n-1) would have been quite valid. But, then in the definition of n* in (3.12) we would have needed some adjustments. Thirdly, in the conventional fixed sample size case, Sen, Kubokawa and Saleh (1989) considered a more general form of Stein-rule estimators, where in (3.1), the scalar constant b $(0 < b \leq (p-1)(3p+1)/2p)$ was replaced by an arbitrary function $\phi(\bar{X}_n, s_n^2)$ such that for all $p \geq 2$, $n \geq 2$,

$$(4.1)\quad 0 < \phi(\bar{X}_n, s_n^2) \leq (p-1)(3p+1)/4p, \text{ for every } (\bar{X}_n, s_n^2) \text{ a.e.}$$

In our case too, we may extend (4.1) to the sequential setup by incorporating the stopping time N (i.e., taking $\phi(\bar{X}_N, s_N^2)$) , and the steps in (3.6) through (3.8) remain in tact [by virtue of the nonnegativity of $\phi(.)$ and its boundedness from above]. Thus, (3.10) again pertains to this more general situation, and the proof provided remains applicable too. Fourthly, under appropriate quadratic error loss, positive-rule versions of shrinkage estimators are known to dominate the usual shrinkage estimators, and in the conventional nonsequential case, this dominance result has also been established in the light of the Pitman closeness [viz., Sen, Kubokawa and Saleh (1989)]. For the particular model considered in Section 3, a positive-rule version of (3.1) is

$$(4.2)\qquad \underset{\sim}{\delta}_n^{b+} = \{1 - bs_n^2(n||\bar{\underset{\sim}{X}}_n||^2)^{-1}\}^+ \bar{\underset{\sim}{X}}_n \quad ; \qquad a^+ = \max(a,0) ,$$

where b, s_n^2 and the other notations are borrowed from (3.1). A natural sequential version of this positive rule estimator is given by $\underset{\sim}{\delta}_N^{b+}$, where the stopping number N is defined as in (2.11). Note that by (3.1) and (4.2)

$$(4.3)\qquad P_{\underset{\sim}{\theta},\sigma}\{ ||\underset{\sim}{\delta}_N^{b+} - \underset{\sim}{\theta}||^2 \le || \underset{\sim}{\delta}_N^{b} - \underset{\sim}{\theta}||^2 \}$$
$$\ge P_{\underset{\sim}{\theta},\sigma}\{ bs_N^2 \le N||\bar{\underset{\sim}{X}}_N||^2\} + P_{\underset{\sim}{\theta},\sigma}\{[bs_N^2 > N||\bar{\underset{\sim}{X}}_N||^2] \cap [\underset{\sim}{\theta}'\bar{\underset{\sim}{X}}_N \ge 0] \}.$$

When $\underset{\sim}{\theta} = \underset{\sim}{0}$, the right hand side of (4.3) is equal to 1, so we need to consider only the case of $\underset{\sim}{\theta} \neq \underset{\sim}{0}$. Let $\underset{\sim}{A}$ be an orthogonal matrix of order p , such that the first row of $\underset{\sim}{A}$ is $||\underset{\sim}{\theta}||^{-1}\underset{\sim}{\theta}'$. For every $n \ge 1$, let $\underset{\sim}{Y}_n = (Y_{1n},\ldots,Y_{pn})' = \underset{\sim}{A}\bar{\underset{\sim}{X}}_n$, so that $Y_{1n} = ||\underset{\sim}{\theta}||^{-1}\underset{\sim}{\theta}'\bar{\underset{\sim}{X}}_n$, for every $n \ge 1$. Then, the second term on the right hand side of (4.3) can be written as

$$(4.4)\qquad P_{\underset{\sim}{\theta},\sigma}\{ [bs_N^2 > N|| \underset{\sim}{Y}_N||^2] \cap [Y_{1N} \ge 0] \}$$
$$= \Sigma_{n\ge 2}\ P\{ I_{[N=n]}[bs_n^2 > n|| \underset{\sim}{Y}_n||^2] \cap [Y_{1n} \ge 0] \}.$$

Note that if $\mathcal{B}_n^0 = \mathcal{B}(s_k^2, k \le n)$ be the sigma subfield generated by the s_k^2, $k \le n$, then (i) [N=n] is $\mathcal{B}_n^0$-measurable, for every $n \ge 2$, (ii) given N = n, Y_{1n} has the normal distribution with mean $||\underset{\sim}{\theta}||$ and variance $n^{-1}\sigma^2$, and (iii) s_n^2 and $\underset{\sim}{Y}_n$ are independent. Thus, we may virtually repeat the proof in (2.6)-(2.7) of Sen, Kubokawa and Saleh (1989), and conclude that for every $n \ge 2$,

$$(4.5)\qquad P_{\underset{\sim}{\theta},\sigma}\{ I_{[N=n]} [bs_n^2 > n||\underset{\sim}{Y}_n||^2] \cap [Y_{1n} \ge 0] \}$$
$$\ge (1/2)P_{\underset{\sim}{\theta},\sigma}\{ I_{[N=n]}[bs_n^2 \ge n||\underset{\sim}{Y}_n||^2] \}$$
$$= (1/2)P_{\underset{\sim}{\theta},\sigma}\{ I_{[N=n]}[bs_n^2 > n||\bar{\underset{\sim}{X}}_n||^2]\}.$$

Thus, (4.4) is bounded from below by $(1/2)P_{\underset{\sim}{\theta},\sigma}\{ bs_N^2 > N||\bar{\underset{\sim}{X}}_N||^2 \}$, and hence, the right hand side of (4.3) is bounded from below by

(4.6) $$P_{\underset{\sim}{\theta},\sigma}\{ bs_N^2 \le N||\bar{\underset{\sim}{X}}_N||^2\} + (1/2)P_{\underset{\sim}{\theta},\sigma}\{ bs_N^2 > N||\bar{\underset{\sim}{X}}_N||^2 \}$$
$$= \tfrac{1}{2} + \tfrac{1}{2} P_{\underset{\sim}{\theta},\sigma}\{ bs_N^2 \le N||\bar{\underset{\sim}{X}}_N||^2\} \ge 1/2.$$

Therefore, $\underset{\sim}{\delta}_N^{b+}$ dominate the usual shrinkage estimator $\underset{\sim}{\delta}_N^{b}$ in the light of the Pitman closeness criterion for an arbitrary stopping rule (depending only on the s_n^2). Here also, p is ≥ 2, and again, we may replace the shrinkage factor b by a more general term $\phi(.)$ as in (4.1), and the PC dominance remains in tact. Fifthly, in the fixed sample size case, Sen, Kubokawa and Saleh (1989) considered the case of a normal distribution with mean vector $\underset{\sim}{\theta}$ and dispersion matrix $\underset{\sim}{V}\sigma^2$, where $\underset{\sim}{V}$ is a known positive definite matrix and σ^2 is unknown. In this setup, if we assume that if the $\underset{\sim}{X}_i$ are i.i.d.r.v.'s with $N(\underset{\sim}{\theta}, \underset{\sim}{V}\sigma^2)$ distribution, then the PC dominance in Theorem 3 also holds ; we only need to modify the shrinkage estimator in (3.1) allowing a possibly more general norm as in Theorem 2. Towards this, we let

(4.7) $$\underset{\sim}{\delta}_n^{\phi} = [\underset{\sim}{I} - \phi(\bar{\underset{\sim}{X}}_n,s_n^2)s_n^2(n\bar{\underset{\sim}{X}}'\underset{\sim}{V}^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}\bar{\underset{\sim}{X}}_n)^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{V}^{-1}]\bar{\underset{\sim}{X}}_n , \quad n \ge 2,$$

where $\underset{\sim}{Q}$ appears in the definition of the norm in Theorem 2 and all the other notations are borrowed from Section 3. As adapted to a stopping rule N, the shrinkage estimator in (4.7) is defined for the sequential case, i.e., $\underset{\sim}{\delta}_N^{\phi} = \underset{\sim}{\delta}_n^{\phi}$ if N = n . If we proceed as in (3.5) through (3.9) where we use the $\underset{\sim}{Q}$-norm $\underset{\sim}{d}'\underset{\sim}{Q}\underset{\sim}{d}$ instead of the Euclidean norm $\underset{\sim}{d}'\underset{\sim}{d}$, then we may show easily that (3.10) again provides the desired result. Therefore, we omit the details. Finally, we consider the general case of an arbitrary covariance matrix $\underset{\sim}{\Sigma}$ (positive definite but unknown) , and examine the PC dominance of sequential shrinkage estimators for stopping rules of the type in (2.19). In this general case, a Stein-rule estimator of $\underset{\sim}{\theta}$ [viz., Stein (1981)] is of the form

(4.8) $$\underset{\sim}{\delta}_n^{*} = [\underset{\sim}{I} - (n-p)^{-1}\phi(\bar{\underset{\sim}{X}}_n,\underset{\sim}{S}_n)d_n(n\bar{\underset{\sim}{X}}_n\underset{\sim}{S}_n^{-1}\bar{\underset{\sim}{X}}_n)^{-1}\underset{\sim}{Q}^{-1}\underset{\sim}{S}_n^{-1}]\bar{\underset{\sim}{X}}_n ,$$

where d_n = minimum characteristic root of $\underset{\sim}{Q}\underset{\sim}{S}_n$, $\underset{\sim}{S}_n = \Sigma_{i=1}^n(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}}_n)(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}}_n)'$ has a Wishart ($\underset{\sim}{\Sigma}$,p, n-1) distribution, and the other notations are borrowed from Sections 2, 3 and 4. For the fixed sample size case, the PC dominance of $\underset{\sim}{\delta}_n^{*}$ over $\bar{\underset{\sim}{X}}_n$ has been studied by Sen, Kubokawa and Saleh (1989), and in this cintext, it is assumed that $n > p$ (so that $\underset{\sim}{S}_n$ has a nondegenerate distribution). Thus, in the sequential case too, we would need that the initial sample size n_0 is $> p$. Let us define

(4.9) $\quad \underset{\sim}{V}_n = (n-p)^{-1}\underset{\sim}{S}_n$, for $n > p$,

and let κ be defined as in (3.10). Then, virtually repeating the first part of the proof of Theorem 2 of Sen, Kubokawa and Saleh (1989), we conclude that for the desired PC-dominance of $\underset{\sim}{\delta}_N^*$ over $\underset{\sim}{\bar{X}}_N$, it suffices to show that

(4.10) $\quad P_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\{ N(\underset{\sim}{\bar{X}}_N - \underset{\sim}{\theta})'\underset{\sim}{V}_N^{-1}\underset{\sim}{\bar{X}}_N \geq \kappa \} \geq 1/2 , \quad \forall (\underset{\sim}{\theta},\underset{\sim}{\Sigma}),$

where the stopping number N is defined by (2.19) [with $\underset{\sim}{S}_n$ being replaced by $\underset{\sim}{V}_n$]. In this context, we may set

(4.11) $\quad \underset{\sim}{\lambda} = \underset{\sim}{\Sigma}^{-\frac{1}{2}}\underset{\sim}{\theta}$, $\underset{\sim}{Z}_n = n^{\frac{1}{2}} \underset{\sim}{\Sigma}^{-\frac{1}{2}} (\underset{\sim}{\bar{X}}_n - \underset{\sim}{\theta})$ and $\underset{\sim}{V}_n^0 = \underset{\sim}{\Sigma}^{-\frac{1}{2}}\underset{\sim}{V}_n\underset{\sim}{\Sigma}^{-\frac{1}{2}}$, for $n > p$.

Note that for every $n \geq 1$, $\underset{\sim}{Z}_n$ has the normal law with null mean vector and dispersion matrix $\underset{\sim}{I}_p$, and $(n-p)\underset{\sim}{V}_n^0$ has the Wishart $(\underset{\sim}{I}_p, p, n-1)$ distribution. Thus, the left hand side of (4.10) can be written as

(4.12) $\quad P_{\underset{\sim}{0},\underset{\sim}{I}}\{ \underset{\sim}{Z}_N'(\underset{\sim}{V}_N^0)^{-1}\underset{\sim}{Z}_N \geq \kappa - N^{\frac{1}{2}} \underset{\sim}{\lambda}'(\underset{\sim}{V}_N^0)^{-1}\underset{\sim}{Z}_N \}$

$\quad = \Sigma_{n \geq n_0} P_{\underset{\sim}{0},\underset{\sim}{I}}\{ N=n \}. P_{\underset{\sim}{0},\underset{\sim}{I}}\{ \underset{\sim}{Z}_n'(\underset{\sim}{V}_n^0)^{-1}\underset{\sim}{Z}_n \geq \kappa - n^{\frac{1}{2}} \underset{\sim}{\lambda}'(\underset{\sim}{V}_n^0)^{-1}\underset{\sim}{Z}_n \mid N = n \},$

where the even $[N = n]$ depends on the $\underset{\sim}{V}_k^0$, $k \leq n$ and $\underset{\sim}{Q}$, introduced in (2.19), for $n \geq n_0$. Using the stochastic independence of the $\underset{\sim}{Z}_k$ and $\underset{\sim}{V}_k^0$, we may claim that given $\underset{\sim}{V}_n^0$ (and $N = n$), $\underset{\sim}{\lambda}'(\underset{\sim}{V}_n^0)^{-1}\underset{\sim}{Z}_n$ has a normal law with mean 0 , for any $\underset{\sim}{\lambda}$, and we can conceive of a set of independent standard normal variables $U_1,\ldots,U_p$, such that for every $n \geq n_0$,

(4.13) $\quad P_{\underset{\sim}{0},\underset{\sim}{I}}\{ \underset{\sim}{Z}_n'(\underset{\sim}{V}_n^0)^{-1}\underset{\sim}{Z}_n \geq \kappa - n^{\frac{1}{2}}\underset{\sim}{\lambda}'(\underset{\sim}{V}_n^0)^{-1}\underset{\sim}{Z}_n \mid N=n , \underset{\sim}{V}_n^0 \}$

$\quad = P_{\underset{\sim}{0},\underset{\sim}{I}}\{ \Sigma_{j=1}^p d_{nj}U_j^2 \geq \kappa - n^{\frac{1}{2}} \Sigma_{j=1}^p d_{nj}^*U_j \mid N=n, \underset{\sim}{V}_n^0 \}$

where the d_{nj} are the characteristic roots of $(\underset{\sim}{V}_n^0)^{-1}$ and the d_{nj}^* depend on $\underset{\sim}{\lambda}$ as well as the d_{nj} (which are held fixed) ; the U_j are independent of the d_{nj} and d_{nj}^*. Although this form is quite appealing, it does not lead us to the deisred result. The main difficulty is caused by the fact that in (2.19) N involves the sequence $K(\text{trace}(\underset{\sim}{Q}\underset{\sim}{V}_n))$, $n \geq n_0$ and K (> 0) given. With the transformation in (4.11), we may write $\underset{\sim}{Q}^o = \underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{Q}\underset{\sim}{\Sigma}^{\frac{1}{2}}$, so that trace$(\underset{\sim}{Q}\underset{\sim}{V}_n)$ = trace$(\underset{\sim}{Q}^o\underset{\sim}{V}_n^o)$ = $\Sigma_{j=1}^p d_{nj}^o$, say, where the d_{nj}^o are the characteristic roots of $\underset{\sim}{Q}^o\underset{\sim}{V}_n^o$, for $n \geq n_o$. The basic difficulty is caused by the fact that the stopping number N is determined by the sequence $\{d_{nj}^o\}$ while in (4.13) the d_{nj} and d_{nj}^* may not be in general proportional to the d_{nj} , and this creates a problem in adapting the breakup in (3.13). Thus, the method outlined in Section 3 may not work out for this general problem. This is not at all surprising. In the sequential shrinkage estimation (under quadratic error loss), a very similar problem cropped up, and hence, in all the works of Ghosh et al. (1987), Nickerson(1987) and Sriram and Bose (1988), the case of $\underset{\sim}{\Sigma} = \sigma^2\underset{\sim}{V}$ was considered. Also, if we go back to the classical multivariate sequential problem treated in Ghosh, Sinha and Mukhopadhyay (1976) and Woodroofe (1977), there also, a stopping rule of the type (2.19) was considered,allowing K to go to $+\infty$. In this asymptotic case, we have no difficulty in claiming that $\underset{\sim}{\delta}_N^*$ dominates $\bar{\underset{\sim}{X}}_N$ in the PC criterion. This follows from the fact that $\underset{\sim}{V}_n \to \underset{\sim}{\Sigma}$ a.s., as $n \to \infty$, so that as $K \to \infty$, N_K [defined as in (2.19) with K] goes to $+\infty$ a.s. As such, in (4.8) (for n = N), we may replace $(N-p)\underset{\sim}{S}_N^{-1}$ by $\underset{\sim}{\Sigma}^{-1}$, and with the resulting estimators, the case reduces to that of a known $\underset{\sim}{\Sigma}$, for which our earlier results would directly apply. However, we may remark that as $K \to \infty$, the PC-dominance of $\underset{\sim}{\delta}_N^*$ over $\bar{\underset{\sim}{X}}_N$ is perceptible only in a small neighborhood of the pivot , i.e.,

(4.14) $\qquad \underset{\sim}{\theta} \in \Theta_K = \{ K^{-1}\underset{\sim}{\lambda} : \underset{\sim}{\lambda} \in \text{Compact } C \subset R^p\}.$

This asymptotic feature of the shrinking neighborhood pertaining to the sequential shrinkage estimators (under quaratic loss) has already been noticed in Sen (1987), and in the PC-dominance, the same remains pertinent. The problem of establishing the dominance (in quadratic error loss or in the PC-criterion) for an arbitrary $\underset{\sim}{\Sigma}$ and for any fixed K > 0 remains largely as open.

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