Asymptotic Distributions for Vectors of Power Sums *

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We show that the recent 'probabilistic approach' to the asymptotic distribution of sums of independent and identically distributed random variables and of lightly trimmed versions of these sums, given in [7] and [5], can be extended to vectors of sums of fixed powers of such random variables. A special case when the underlying distribution has regularly varying tails appears as an extension of an equivalent form of the main result of Szeidl [9, 11] which, in turn, was motivated by earlier work of Zolotarev [15, 16] and others on polynomial statistics of a fixed degree. The result is also illustrated by an example of elementary symmetric polynomials having Poisson limits.

1. RESULTS AND DISCUSSION

Let $X, X_1, X_2, \ldots$ be independent random variables with a common distribution function $F(x) = P\{X \leq x\}$, $x \in \mathbb{R}$, and quantile function

$$Q(s) = \inf\{x : F(x) \geq s\}, \quad 0 < s \leq 1, \quad Q(0) = Q(0+).$$

We also introduce the quantile function of $|X|$ defined as

$$K(s) = \inf\{x : P\{|X| \leq x\} \geq s\}, \quad 0 < s \leq 1, \quad K(0) = K(0+),$$

and note that for any power $p > 0$, the quantile functions of $|X|^p$ and $|X|^p \operatorname{sgn} X$, where sgn stands for the sign function, are seen by elementary considerations to be

(1.1) $\qquad Q_p^{(1)}(s) = \inf\{x : P\{|X|^p \leq x\} \geq s\} = K^p(s), \quad 0 < s \leq 1,$

and
(1.2)
$$Q_p^{(2)}(s) = \inf\{x : P\{|X|^p \operatorname{sgn} X \leq x\} \geq s\} = |Q(s)|^p \operatorname{sgn} Q(s), \quad 0 < s \leq 1,$$

and we again put $Q_p^{(l)}(0) = Q_p^{(l)}(0+)$, $l = 1, 2$. Note that $Q_1^{(2)} = Q$. Setting

$$\sigma_p^{(l)}(s) = \left(\int_s^{1-s} \int_s^{1-s} (u \wedge v - uv) dQ_p^{(l)}(u) dQ_p^{(l)}(v) \right)^{1/2},$$

$0 < s < 1/2$, $l = 1, 2$, where $u \wedge v = \min(u, v)$, define

(1.3) $$a_p^{(l)}(n) = n^{1/2} \sigma_p^{(l)}(1/n), \quad l = 1, 2,$$

and note that these quantities are non-zero for all n large enough, provided the distribution of X is not degenerate, and this we will assume throughout to avoid trivialities. Note that $a(n) = a_1^{(2)}(n)$ is the "natural" normalizing sequence of [7, 5].

For fixed integers $m \geq 0$, $k \geq 0$ and a power $p > 0$, introduce

$$S_{n,p}^{(l)}(m, k) = \begin{cases} \sum_{j=m+1}^{n-k} |X_{j,n}|^p, & \text{if } l = 1, \\ \sum_{j=m+1}^{n-k} |X_{j,n}|^p \operatorname{sgn} X_{j,n}, & \text{if } l = 2, \end{cases}$$

where $X_{1,n} \leq \ldots \leq X_{n,n}$ are the order statistics pertaining to the sample $X_1, \ldots, X_n$. The aim of the present paper, motivated by Szeidl [9-12], is to investigate the asymptotic distribution of the vector

$$(S_{n,p_1}^{(l_1)}(m_1, k_1), \ldots, S_{n,p_d}^{(l_d)}(m_d, k_d)),$$

where $d \geq 1$, $m_1 \geq 0$, $k_1 \geq 0$, ..., $m_d \geq 0$, $k_d \geq 0$ are fixed integers, $p_1 > 0$, ..., $p_d > 0$ are fixed powers and

(1.4) $$\mathcal{L} = \{l_1, \ldots, l_d\}, \quad l_i = 1 \quad \text{or} \quad 2, \quad i = 1, \ldots, d,$$

is a fixed set, and where we shall generally allow n to go to infinity along a subsequence of the sequence $\{n\}$ of the positive integers.

Let α_n be any sequence of positive numbers such that $\alpha_n \downarrow 0$ and $n\alpha_n \to 0$ as $n \to \infty$, and introduce the functions

$$\psi_{1,p}^{(l)}(n, s) = \begin{cases} Q_p^{(l)}\left(\frac{s}{n}+\right)/a_p^{(l)}(n), & 0 < s \leq n - n\alpha_n, \\ Q_p^{(l)}((1 - \alpha_n)+)/a_p^{(l)}(n), & n - n\alpha_n < s < \infty, \end{cases}$$

and

$$\psi_{2,p}^{(l)}(n, s) = \begin{cases} -Q_p^{(l)}\left(1 - \frac{s}{n}\right)/a_p^{(l)}(n), & 0 < s \leq n - n\alpha_n, \\ -Q_p^{(l)}(\alpha_n)/a_p^{(l)}(n), & n - n\alpha_n < s < \infty, \end{cases}$$

For both $l = 1$ and $l = 2$, for any $p > 0$, and for all n large enough, these are non-decreasing and right-continuous functions on $(0, \infty)$, and setting

(1.5) $$\psi_j(n, \cdot) = \psi_{j,1}^{(2)}(n, \cdot), \quad j = 1, 2, \quad a(n) = a_1^{(2)}(n),$$

we obtain the two main operational sequences of [7, 5]. Let $\Rightarrow$ denote weak convergence of functions on the half line $(0, \infty)$, that is, pointwise convergence on $(0, \infty)$ in every continuity point of the limiting function. Of course, we have

(1.6) $$\lim_{n \to \infty} \psi_{1,p}^{(1)}(n, s) = \psi_{1,p}^{(1)}(s) = 0, \quad s > 0, \quad \text{for any} \quad p > 0.$$

It will assist the presentation of our main result in the theorem below if we first formulate some preliminary results, using the notation in (1.5). All the proofs are in the next section.

LEMMA. *(i) If there exist a subsequence $\{n^*\} \subset \{n\}$ and non-positive, non-decreasing, right-continuous functions ψ_1 and ψ_2 on $(0,\infty)$ such that*

$$(1.7) \qquad \psi_j(n^*,\cdot) \Rightarrow \psi_j(\cdot) = \psi_{j,1}^{(2)}(\cdot), \quad \text{as} \quad n^* \to \infty, \quad j = 1,2,$$

and at least one of ψ_1 and ψ_2 is not identically zero, then for any $p_1,\ldots,p_d > 0$ one can find a (generally different) subsequence $\{n'\} \subset \{n\}$ and non-positive, non-decreasing, right-continuous functions $\psi_{j,p_1}^{(2)},\ldots,\psi_{j,p_d}^{(2)}$, $j = 1,2$, such that

$$(1.8) \qquad \psi_{j,p_i}^{(2)}(n',\cdot) \Rightarrow \psi_{j,p_i}^{(2)}(\cdot), \quad \text{as} \quad n' \to \infty, \quad j = 1,2; \; i = 1,\ldots,d.$$

(ii) If there exist a subsequence $\{n^*\} \subset \{n\}$ and a non-positive, non-decreasing, right-continuous function $\psi_{2,1}^{(1)}$ on $(0,\infty)$ such that $\psi_{2,1}^{(1)}$ is not identically zero and

$$(1.9) \qquad \psi_{2,1}^{(1)}(n^*,\cdot) \Rightarrow \psi_{2,1}^{(1)}(\cdot) \quad \text{as} \quad n^* \to \infty,$$

then for any $p_1,\ldots,p_d > 0$ one can find a (generally different) subsequence $\{n'\} \subset \{n\}$ and non-positive, non-decreasing, right-continuous functions $\psi_{2,p_1}^{(1)},\ldots,\psi_{2,p_d}^{(1)}$ such that

$$(1.10) \qquad \psi_{2,p_i}^{(1)}(n',\cdot) \Rightarrow \psi_{2,p_i}^{(1)}(\cdot), \quad \text{as} \quad n' \to \infty \quad i = 1,\ldots,d.$$

(iii) If (1.7) and (1.9) hold along the same $\{n^*\}$, then $\{n'\}$ can be constructed so that (1.8) and (1.10) both hold along this $\{n'\}$.

(iv) If for some subsequence $\{n'\} \subset \{n\}$,

$$(1.11) \qquad \psi_j(n',s) \to 0, \quad s > 0, \quad j = 1,2,$$

as $n' \to \infty$, then for the same subsequence $\{n'\}$ and all $0 < p < 1$,

$$(1.12) \qquad \psi_{j,p}^{(2)}(n',s) \to 0, \quad s > 0, \quad j = 1,2,$$

as $n' \to \infty$.

(v) If for some subsequence $\{n'\} \subset \{n\}$,

$$(1.13) \qquad \psi_{2,1}^{(1)}(n',s) \to 0, \quad s > 0,$$

as $n' \to \infty$, then for the same subsequence $\{n'\}$ and all $0 < p < 1$,

$$(1.14) \qquad \psi_{2,p}^{(1)}(n',s) \to 0, \quad s > 0,$$

as $n' \to \infty$.

(vi) If in (i)-(v) above we have the corresponding assumptions (1.7), (1.9), (1.11), or (1.13) for the functions $a_1^{(l)}(\overline{n})\psi_{j,1}^{(l)}(\overline{n},\cdot)/A_{\overline{n}}$, where $\overline{n} = n^*$ or $\overline{n} = n'$, $A_{\overline{n}} > 0$ and $a_1^{(l)}(\overline{n})/A_{\overline{n}} \to 0$ as $\overline{n} \to \infty$, with the corresponding super- and subscripts l and j, then we have the same conclusions for the respective renormalized functions $a_p^{(l)}(n')\psi_{j,p}^{(l)}(n',\cdot)/A_{n'}^p$.

We note that by Theorem 5 in [7] the alternative conditions in this lemma, corresponding to a component of our vector above is necessary for the convergence in distribution, to be denoted by $\to_D$, of any (linearly) centered and normalized version of that component to hold along some subsequence of $\{n\}$.

Recalling the notation in (1.4), we now formulate two conditions. Let $P = \{p_1, \ldots, p_d\}$ be a set of fixed positive powers, not necessarily different.

Condition (C_1). If $2 \in \mathcal{L}$ and $1 \notin \mathcal{L}$, then assume either (1.7), with at least one of ψ_1 and ψ_2 not identically zero, or (1.11). Then we either have (1.8) along some $\{n'\}$ and all powers p_i in P for which $l_i = 2$, or we have (1.12) for all powers p_i in P for which $p_i < 1$ and $l_i = 2$. In the latter case, under (1.11), if there are $p_i \in P$ for which $p_i > 1$ and $l_i = 2$, then we *assume* that (1.12) holds along the same $\{n'\}$ for all such p_i.

If $1 \in \mathcal{L}$ and $2 \notin \mathcal{L}$, then assume either (1.9) with a $\psi_{2,1}^{(1)}$ not identically zero, or (1.13). Then we either have (1.10) along some $\{n'\}$, or we have (1.14) for all powers $p_i \in P$ for which $p_i < 1$ and $l_i = 1$. In the latter case, under (1.13), if there are $p_i \in P$ for which $p_i > 1$ and $l_i = 1$, then we *assume* that (1.14) holds along the same $\{n'\}$ for all such p_i.

If $1 \in \mathcal{L}$ and $2 \in \mathcal{L}$, then we assume either (1.7) with $\psi_1 + \psi_2 \not\equiv 0$ or (1.11) and also either (1.9) with $\psi_{2,1}^{(1)} \not\equiv 0$ or (1.13) in such a way that (1.12) and (1.14) together with the additional *assumptions* for powers larger that 1, as described in the preceding two paragraphs, hold along the subsequence $\{n'\}$ given by case (i) or (ii) or (iii) of the Lemma.

Condition(C_2). This is the same condition as (C_1), only we start out from the respective conditions for the renormalized ψ functions, with some $A_{n^*} > 0$ such that $a_1^{(l)}(n^*)/A_{n^*} \to 0$ as $n^* \to 0$, or $A_{n'} > 0$ such that $a_1^{(l)}(n')/A_{n'} \to 0$ as $n' \to \infty$, where $l = 1$ or $l = 2$, as described in case (vi) of the Lemma.

Under condition (C_1) or (C_2) all the functions $\psi_{j,p_i}^{(l_i)}(n', \cdot)$ or $a_{p_i}^{(l_i)}(n')\psi_{j,p_i}^{(l_i)}(n', \cdot)/A_{n'}^{p_i}$ converge weakly along the same $\{n'\}$ to corresponding limiting functions $\psi_{j,p_i}^{(l_i)}(\cdot)$, $j = 1, 2$; $p_1, \ldots, p_d \in P$, where $l_i = 1$ or $l_i = 2$. Now we brake up P into the union $P_0 \cup P_1$ of disjoint sets P_0 and P_1 in the following way. If for $p_i \in P$ we have $\psi_{1,p_i}^{(l_i)} = \psi_{2,p_i}^{(l_i)} \equiv 0$, then let $p_i \in P_0$, otherwise let $p_i \in P_1$. Let the cardinality of P_0 be d_0, where $0 \leq d_0 \leq d$, and re-index the p_i if necessary so that $P_0 = \{p_1, \ldots, p_{d_0}\}$ and $P_1 = \{p_{d_0+1}, \ldots, p_d\}$. Of course, if $d_0 = 0$ then P_0 is empty and if $d_0 = d$ then P_1 is empty.

Having thus fixed P and $\mathcal{L}$ in (1.4), the notations

(1.15)
$$Q_i(\cdot) = Q_{p_i}^{(l_i)}(\cdot), \quad S_{n'}^{(i)}(m_i, k_i) = S_{n',\mathcal{P}_i}^{(l_i)}(m_i, k_i), \quad \psi_j^{(i)}(n',\cdot) = \psi_{j,\mathcal{P}_i}^{(l_i)}(n',\cdot)$$

and

$$\psi_j^{(i)}(\cdot) = \psi_{j,\mathcal{P}_i}^{(l_i)}(\cdot), \quad j = 1, 2,; \quad i = 1, \ldots, d \quad (\text{where} \quad \psi_j^{(i)} \equiv 0 \text{ if } i = 1, \ldots, d_0)$$

are fully meaningful under (C_1) or (C_2). Note that by (1.6), if the case $l_i = 1$ occurs, then $\psi_1^{(i)} \equiv 0$ for all such i. Aslo, based on the notation in (1.3), we further introduce

$$\sigma_i(s) = \sigma_{p_i}^{(l_i)}(s), \quad 0 < s < 1/2, \quad a_i(n) = a_{p_i}^{(l_i)}(n) = n^{1/2}\sigma_i(1/n),$$

(1.16)
$$\mu_n^{(i)}(m_i, k_i) = \int_{(m_i+1)/n}^{1-(k_i+1)/n} Q_i(u)\,du, \quad i = 1, \ldots, d,$$

$$\sigma_{ij}(s) = \int_s^{1-s} \int_s^{1-s} (u \wedge v - uv)\,dQ_i(u)\,dQ_j(v), \quad 0 < s < 1/2, \quad i,j = 1, \ldots, d,$$

where, of course, $\sigma_{ii}(\cdot) = \sigma_i^2(\cdot)$ for any $i = 1, \ldots, d$. The final non-random quantities that we need, depending on the subsequence $\{n'\}$ constructed in condition (C_1) or (C_2), are

$$\underline{s}_{ij} = \underline{s}_{ij}(\{n'\}) = \lim_{h\to\infty} \liminf_{n'\to\infty} \frac{\sigma_{ij}(h/n')}{\sigma_i(1/n')\sigma_j(1/n')}$$

(1.17) and

$$\bar{s}_{ij} = \bar{s}_{ij}(\{n'\}) = \lim_{h\to\infty} \limsup_{n'\to\infty} \frac{\sigma_{ij}(h/n')}{\sigma_i(1/n')\sigma_j(1/n')},$$

which are well defined and satisfy $0 \le \underline{s}_{ij} \le \bar{s}_{ij} \le 1$ for all $i,j = 1, \ldots, d$.

Consider now two independent sequences $\{E_i^{(j)}\}_{i=1}^{\infty}$, $j = 1, 2$, of independent exponentially distributed random variables with mean 1 and the corresponding partial sums $Y_n^{(j)} = E_1^{(j)} + \ldots + E_n^{(j)}$, $n \ge 1$, $j = 1, 2$, as jump-points of the independent standard left-continuous Poisson processes

$$N_j(s) = \sum_{n=1}^{\infty} I(Y_n^{(j)} < s), \quad s \ge 0, \quad j = 1, 2,$$

where $I(\cdot)$ is the indicator function, and define the random variables

$$(1.18) \quad V_j^{(i)}(k) = \int_{Y_{k+1}^{(j)}}^{\infty} (N_j(s) - s)d\psi_j^{(i)}(s) - \int_1^{Y_{k+1}^{(j)}} s\,d\psi_j^{(i)}(s)$$
$$+ k\psi_j^{(i)}(Y_{k+1}^{(j)}) - \int_1^{k+1} \psi_j^{(i)}(s)ds - \psi_j^{(i)}(1),$$

where $j = 1, 2$, $k \geq 0$, and $i = d_0 + 1, \ldots, d$. These are well defined both under (C_1) and (C_2) because of (1.19) and (1.20) below and $V_j^{(i)}(k)$ is non-degenerate if $\psi_j^{(i)} \not\equiv 0$. Fix the integers $m_1, \ldots, m_d, k_1, \ldots, k_d \geq 0$. Then, setting

$$W_n^{(i)}(m_i, k_i) = (S_n^{(i)}(m_i, k_i) - \mu_n^{(i)}(m_i, k_i))/a_i(n)$$

and

$$\overline{W}_n^{(i)}(m_i, k_i) = (S_n^{(i)}(m_i, k_i) - \mu_n^{(i)}(m_i, k_i))/A_n^{p_i},$$

$i = 1, \ldots, d$, our main result is the following.

THEOREM. *(1) Suppose condition.(C_1). Then, necessarily,*

$$(1.19) \quad \int_\epsilon^\infty (\psi_j^{(i)}(s))^2 ds < \infty \quad \text{for any} \quad \epsilon > 0, \quad j = 1, 2; \quad i = 1, \ldots, d,$$

and there exists a subsequence $\{n''\} \subset \{n'\}$ such that

$$(W_{n''}^{(1)}(m_1, k_1), \ldots, W_{n''}^{(d)}(m_d, k_d))$$
$$\to_D (Z_1, \ldots, Z_{d_0}, W_{d_0+1}(m_{d_0+1}, k_{d_0+1}), \ldots, W_d(m_d, k_d))$$

as $n'' \to \infty$, where

$$W_i(m_i, k_i) = -V_1^{(i)}(m_i) + Z_i + V_2^{(i)}(k_i), \quad i = d_0 + 1, \ldots, d,$$

and where $N_1(\cdot)$, $(Z_1, \ldots, Z_d)$, $N_2(\cdot)$ are independent and $(Z_1, \ldots, Z_d)$ has a (possibly degenerate) d-dimensional normal distribution with zero mean vector and covariance matrix $(s_{ij})_{i,j=1}^d$ with some numbers s_{ij} such that

$$\underline{s}_{ij} \leq s_{ij} \leq \overline{s}_{ij}, \quad \text{for all} \quad i, j = 1, \ldots, d,$$

and

$$\underline{s}_{ii} = s_{ii} = \overline{s}_{ii} = 1 \quad \text{for} \quad i = 1, \ldots, d_0.$$

Furthermore, if $\underline{s}_{ij} = \overline{s}^{ij}$ for all $i, j = 1, \ldots, d$, then the above convergence in distribution takes place along the original subsequence $\{n'\}$ constructed in

condition (C_1).

(2) Suppose condition (C_2). Then, necessarily,

$$(1.20) \qquad \psi_j^{(i)}(s) = 0 \quad \text{for all} \quad 1 \le s < \infty, \quad j = 1,2; \quad i = 1,\dots,d,$$

and, along the original subsequence $\{n'\}$ constructed in condition (C_2) and with the sequence $\{A_{n'}\}$ of that condition,

$$(W_{n'}^{(1)}(m_1, k_1), \dots, W_{n'}(m_d, k_d))$$
$$\to_{\mathcal{D}} (0, \dots, 0, W_{d_0+1}(m_{d_0+1}, k_{d_0+1}), \dots, W_d(m_d, k_d))$$

as $n' \to \infty$, where

$$W_i(m_i, k_i) = -V_1^{(i)}(m_i) + V_2^{(i)}(k_i), \quad i = d_0 + 1, \dots, d.$$

We note that for each $i = 1,\dots,d$, the random variable $W_i(0,0)$ is infinitely divisible and, in fact, any infinitely divisible random variable can be represented in this form according to Theorem 3 in [7].

In Corollary 2 below we construct an example, in which the original properly centered and normalized sums $S_{n',1}^{(2)}(0,0) = X_1 + \dots + X_{n'}$ converge in distribution along a genuine subsequence $\{n'\}$ to a Poisson random variable, that will illustrate both cases (1) and (2) of the theorem. (In principle, there is an easily formulated third case in which one stipulates the convergence of the original $\psi^{(1)}$ functions jointly with the convergence of the renormalized $\psi^{(2)}$ functions or *vice versa*. We did not include this because we don't have an example for such a situation.) Presently, we exhibit what is perhaps the most important special case of part (1) of the theorem.

Assume for the underlying quantile function that for some constants $\gamma > 0$ and $\delta_1, \delta_2 \ge 0$, $\delta_1 + \delta_2 > 0$, and a non-negative function $L(\cdot)$ defined on $(0,1)$ and slowly varying at zero we have

$$-Q(s+) = s^{-1/\gamma}L(s)(\delta_1 + o(1))$$

(1.21) and
$$Q(1 - s) = s^{-1/\gamma}L(s)(\delta_2 + o(1)) \quad \text{as} \quad s \downarrow 0.$$

This is equivalent to the assumption of Szeidl [9-12] that for some constants $a_1, a_2 \ge 0$, $a_1 + a_2 > 0$, and a function $l(\cdot)$ on $[0, \infty)$, slowly varying at infinity, $F(-x) = (a_1 + o(1))x^{-\gamma}l(x)$ and $1 - F(x) = (a_2 + o(1))x^{-\gamma}l(x)$ as $x \to \infty$. Furthermore, (1.21) is equivalent (cf. Corollary 3 in [7] and Proposition A. 3 in [3]) to the three conditions

$$K(1 - s) = s^{-1/\gamma}L(s), \quad 0 < s < 1,$$

$$\lim_{s \downarrow 0} Q_1(1-s)/K(1-s) = (1-r)^{1/\gamma},$$

$$\lim_{s \downarrow 0} Q_2(1-s)/K(1-s) = r^{1/\gamma},$$

where $Q_1(1-s) = \max(-Q(s),0)$, $Q_2(1-s) = \max(Q(1-s),0)$ and $0 \le r \le 1$ is some constant.

Consider now a set $P = \{p_1, \ldots, p_d\}$ of positive powers, where we assume that $0 < p_1 \le \ldots \le p_d$. Define

$$d_0 = \begin{cases} \max\{1 \le i \le d : p_i \le \gamma/2\}, & \text{if } p_1 \le \gamma/2, \\ 0, & \text{if } p_1 > \gamma/2. \end{cases}$$

With $\mathcal{L}$ in (1.4) fixed, the notation in (1.15) and (1.16) is in force, and, indeed, it will turn out in the proof of the following Corollary 1 that $\psi_j^{(i)} \equiv 0$ if $i = 1, \ldots, d_0$; $j = 1, 2$, and by (1.6), $\psi_1^{(i)} \equiv 0$ if $l_i = 1$, $i = d_0 + 1, \ldots, d$. Also, we will see that with some constant $c_2^{(i)} > 0$,

$$(1.22) \quad \psi_2^{(i)}(s) = c_2^{(i)} s^{-p_i/\gamma}, \quad \text{for all} \quad s > 0, \quad \text{if} \quad l_i = 1, \quad i = d_0 + 1, \ldots, d,$$

and that with some constants $c_1^{(i)}$, $c_2^{(i)} \ge 0$, $c_1^{(i)} + c_2^{(i)} > 0$,

$$(1.23)$$

$$\psi_j^{(i)}(s) = c_j^{(i)} s^{-p_i/\gamma}, \quad \text{for all} \quad s > 0, \quad \text{if} \quad l_i = 2, \quad i = d_0 + 1, \ldots, d; \quad j = 1, 2.$$

Substitute all these functions into (1.18) and define

$$(1.24) \qquad \tilde{W}_i(m_i, k_i) = -V_1^{(i)}(m_i) + V_2^{(i)}(k_i), \quad i = d_0 + 1, \ldots, d,$$

in terms of the obtained random variables. Note that by the Remark following Corollary 3 in [7], for each $i = d_0 + 1, \ldots, d$, $\tilde{W}_i(0,0)$ is a stable random variable with index $\gamma/p_i < 2$, expressed in terms of the same two independent Poisson processes $N_1(\cdot)$ and $N_2(\cdot)$. Furthermore,

$$(1.25) \qquad \sigma_i(0) < \infty \quad \text{if} \quad p_i < \gamma/2, \quad i = 1, \ldots, d_0,$$

while $\sigma_i(0)$ may or may not be finite if $p_i = \gamma/2$, $i = 1, \ldots, d_0$. In this way we arrive at the following.

COROLLARY 1. *If (1.21) holds, then*

$$(W_n^{(1)}(m_1, k_1), \ldots, W_n^{(d)}(m_d, k_d))$$

$$\to_{\mathcal{D}} (Z_1, \ldots, Z_{d_0}, \tilde{W}_{d_0+1}(m_{d_0+1}, k_{d_0+1}), \ldots, \tilde{W}_d(m_d, k_d))$$

as $n \to \infty$, where $N_1(\cdot)$, $(Z_1, \ldots, Z_{d_0})$, $N_2(\cdot)$ are independent and $(Z_1, \ldots, Z_{d_0})$ has a d_0-dimensional normal distribution with zero mean vector and covariance matrix $(s_{ij})_{i,j=1}^{d_0}$, where $s_{11} = \ldots = s_{d_0 d_0} = 1$, $s_{ij} = 1$ if $i < j$ and $p_i = p_j = \gamma/2$, $s_{ij} = \sigma_{ij}(0)/(\sigma_i(0)\sigma_j(0))$ if $i < j$ and $p_j < \gamma/2$, or $p_i < p_j = \gamma/2$ but $\sigma_j(0) < \infty$, and $s_{ij} = 0$ if $i < j$, $p_i < p_j = \gamma/2$ and $\sigma_j(0) = \infty$.

In this corollary, the normalizing constants $a_i(n)$ are asymptotically proportional to $\sqrt{n}$ whenever $1 \leq i \leq d_0$ and $p_i < \gamma/2$, or $p_i = \gamma/2$ but $\sigma_i(0) < \infty$. If $1 \leq i \leq d_0$, $p_i = \gamma/2$, and $\sigma_i(0) = \infty$, then the function $\sigma_i(\cdot)$ in $a_i(n) = \sqrt{n}\sigma_i(1/n)$ is slowly varying at zero according to the proof in the next section and to Corollary 1 in [7]. Also it is clear from [3] that for each $i = d_0 + 1, \ldots, d$, the normalizing sequence $a_i(n)$ is asymptotically proportional to $(n^{1/\gamma}L(1/n))^{p_i}$. When $m_1 = \ldots = m_d = k_1 = \ldots = k_d = 0$, the corollary is equivalent to the main result of Szeidl (Theorem in [9] and Theorem 1 in [11]), who describes the limiting random vector by giving its characteristic function (and makes his classical-type Fourier-analytic proof simpler by assuming that $L \equiv 1$).

One motivation for the Corollary can be the study of self-normalized sums (cf. the references in [4] and Zolatarev [14] as an additional reference). Indeed, the joint convergence of the first two components in relation (5) of [4] follows from the above Corollary 1 with obvious generalizations coming from the Theorem, from which results on self-normalized sums converging in distribution along subsequences of $\{n\}$ can be deduced.

The main motive, however, for results of the type in the Theorem and Corollary 1 is to study the asymptotic distribution of symmetric polynomials of $X_1, \ldots, X_n$ with a fixed degree d. This problem is very nicely exposed by Zolotarev [16]. (The main results of this paper have been announced without proof in [15].) Szeidl [9] illustrates his main result by describing the possible limiting distributions of symmetric polynomials with $d = 2$, proves some general results for arbitrary d in [11] with illustrations for $d = 3$ and $d = 4$, specifically investigates the case $d = 5$ in [10], and provides an analytic description of the limiting characteristic function for a general d in [12]. All these are done by him under the tail condition (1.21). (For a number of earlier and related references, see [15, 9-12] and Avram and Taqqu [1].) Following Szeidl [7-9], but starting out from the Theorem rather than Corollary 1, a more general description of subsequential limits of general symmetric polynomials is possible, replacing stable random variables by some infinitely divisible ones. We do not stop on the details, some of which can be found in [13].

Instead of this, here we present a concrete example which, as promised above, illustrates both cases of the Theorem and at the same time exhibits an interesting possibility of subsequential asymptotic distribution of elementary symmetric polynomials of an arbitrary degree $d \geq 1$. This example is a special case of a construction by one of us and Dodunekova [6], exhibiting a different and even more curious behavior concerning subsequential Poisson limits of a single sum.

Set

$$t_k = 2^{-2^k} \quad \text{and} \quad b_k = (t_k - t_{k+1})^{-1} = \frac{2^{2^k}}{1 - 2^{-2^k}}, \quad k = 0, 1, 2, \ldots,$$

so that $t_k/t_{k-1} \to 0$ and $b_k/b_{k-1} \to \infty$ as $k \to \infty$, and consider the random variable X given by the distribution

$$(1.26) \quad P\{X = 0\} = \frac{1}{2} \quad \text{and} \quad P\{X = b_k\} = t_k - t_{k+1}, \quad k = 0, 1, 2, \ldots.$$

The quantile function Q of X is given by

$$Q(s) = \begin{cases} 0, & \text{if } 0 \le s \le 1 - t_0 = 1/2, \\ b_k, & \text{if } 1 - t_k < s \le 1 - t_{k+1}, \quad k = 0, 1, 2, \ldots, \end{cases}$$

and if for a $\lambda > 0$ and $k = 0, 1, 2, \ldots$, we let

$$(1.27) \quad n_k = n_k(\lambda) = \lceil \lambda 2^{2^k} \rceil = \lceil \lambda/t_k \rceil = \min\{j : j \text{ integer}, \ j \ge \lambda/t_k\},$$

then it is shown in [6] by elementary considerations that

$$Q\left(1 - \frac{s}{n_k}\right) = \begin{cases} b_k, & \text{if } s < \lambda, \\ b_{k-1}, & \text{if } s \ge \lambda, \end{cases}$$

for all k large enough. Hence, if $A_{n_k} = b_k \sim 2^{2^k}$, then

$$(1.28) \quad Q(s/n_k)/A_{n_k} \to 0 \quad \text{and} \quad -Q(1 - s/n_k)/A_{n_k} \to \psi_\lambda(s), \quad s > 0,$$

as $k \to \infty$, where $\psi_\lambda(s) = -1$ if $s < \lambda$ and $\psi_\lambda(s) = 0$ if $s \ge \lambda$. This means that (1.7) or, what here is the same, (1.9) are satisfied along $\{n_k\}$. Our set of powers will be $P = P_1 = \{1, \ldots, d\}$, where $d \ge 2$. There is no need to construct a new subsequence for (1.8) or (1.10) to hold, since by (1.1) or (1.2) the quantile function $Q_i(\cdot)$ of X^i is $Q^i(\cdot)$, and hence by (1.28) we have

$$Q_i(s/n_k)/A_{n_k}^i \to 0 \quad \text{and} \quad -Q_i(1 - s/n_k)/A_{n_k}^i \to \psi_\lambda(s), \quad s > 0.$$

Extending the elementary computations from $i = 1$ to $2 \le i \le d$, given in [6], it is straightforward to see that for the $\sigma_i(\cdot)$ and $a_i(n_k)$ in (1.16) belonging to the present $Q_i(\cdot) = Q^i(\cdot)$, we have

$$a_i(n_k)/A_{n_k}^i \to 0 \quad \text{if} \quad \lambda \le 1$$

and for any $h > \lambda$,

$$a_i(n_k)/A_{n_k}^i \to \sqrt{\lambda} \quad \text{and} \quad \sigma_i(h/n_k)/\sigma_i(1/n_k) \to 0 \quad \text{if} \quad \lambda > 1$$

for each $1 \leq i \leq d$ as $k \to \infty$. Therefore, if $\lambda \leq 1$, we are in case (2) of the Theorem, that is Condition (C_2) is satisfied along $\{n_k\}$. On the other hand, if $\lambda > 1$, then we are in case (1) of the Theorem, that is, Condition (C_1) is satisfied along $\{n_k\}$ and, furthermore, the possible normal components of the limiting variables all disappear. Taking into account the computation with ψ_λ given in [6] concerning the limiting random variable and, though the statement could be given with nonzero (m_i, k_i) easily, simplifying the notation in (1.16) by setting here

$$S_i(n_k) = S_{n_k,i}(0,0) = \sum_{j=1}^{n_k} X_j^i$$

and

$$(1.29) \qquad \mu_i(u_k) = \mu_{n_k}^{(i)}(0,0) = n_k \int_{1/2}^{1-1/n_k} Q^i(u)du, \quad i = 1, \ldots, d,$$

for independent copies $X_1, X_2, \ldots$ of X above, the first statement of the following corollary follows from the Theorem, while the second one will be proved to follow from the first in the next section. It is interesting to see that the limit of the polynomials is constant times the limit of single sums.

COROLLARY 2. *For the distribution given in (1.26),*

$$\left(\frac{S_1(n_k) - \mu_1(n_k)}{A_{n_k}}, \ldots, \frac{S_d(n_k) - \mu_d(n_k)}{A_{n_k}^d} \right)$$
$$\to_D (N(\lambda) + (1 \wedge \lambda) - \lambda, \ldots, N(\lambda) + (1 \wedge \lambda) - \lambda),$$

as $k \to \infty$, where $P\{N(\lambda) = m\} = \lambda^m e^{-\lambda}/m!$, $m = 0, 1, 2, \ldots$. Furthermore, for any fixed $d \geq 1$,
(1.30)

$$\frac{\sum_{1 \leq j_1 < \ldots < j_d \leq \lceil \lambda 2^{2^k} \rceil} X_{j_1} \ldots X_{j_d} - C_{\lceil \lambda 2^{2^k} \rceil}^{(d)}}{k^{d-1} 2^{d2^k}} \to_D \frac{\lambda^{d-1}}{(d-1)!} \{N(\lambda) + (1 \wedge \lambda) - \lambda\}$$

as $k \to \infty$, where, with $\mu_i(n_k) = \mu_i(\lceil \lambda 2^{2^k} \rceil)$ in (1.29),

$$(1.31) \qquad C_{\lceil \lambda 2^{2^k} \rceil}^{(d)} = (-1)^d \sum_{(i_1,\ldots,i_d)}^* \frac{(-1)^{i_1 + \ldots + i_d}}{i_1! \ldots i_d!} \prod_{l=1}^d \left(\frac{\mu_l(\lceil \lambda 2^{2^k} \rceil)}{l} \right)^{i_l},$$

the sum $\sum^*$ being extended for all vectors $(i_1, \ldots, i_d)$ that have non-negative integer-valued components such that $i_1 + 2i_2 + \ldots + di_d = d$.

We close this section with a general methodological remark. Our 'probabilistic approach' may appear unusual at first sight and one meets the superficial difficulty or disadvantage to have to deal with quantile functions and associated quantities such as the "truncated variance function" $\sigma^2(\cdot)$ rather than more customary distribution functions, truncated expectations or characteristic functions. However, this approach or method has several advantages. The main one in the present setting, as will be clear from the indication of the proof, is that after a quantile transformation all the components of the vectors in the Theorem are expressed in distribution by means of a single sequence of a specifically constructed Uniform $(0,1)$ random variables so that the problem of convergence in distribution becomes the problem of component-wise convergence in probability for which the already existing theory can be used. Therefore, we don't really have to worry about joint distributions while in the usual characteristic-function approach this is the basic problem (cf. the proof in [9]). Second, a "fine-structure" theorem could be stated in an obvious manner as in [7], showing what portions of the sums of the components contribute the various ingredients of the limiting vector, in particular, the independence properties become trivial and heuristically clear. Third, the influence of the extreme terms of the sums are nicely delineated. Fourth, the possibility for the universal choices of the centering and normalizing sequences makes things easier. Finally, the probabilistic representation of the limits provides some complementary intuition to the more usual Fourier-, Laplace-, or Mellin-transform descriptions.

2. PROOFS

Proof of the Lemma. (i) Since the remaining cases are easier, we only consider the case when Q is negative near enough to zero and positive near enough to one. In this case, the function $H(s) = H_1(s)$, where for $p > 0$, $H_p(s) = |Q(s+)|^p + |Q(1-s)|^p$, $0 < s < 1$, is non-increasing in a right-side neighborhood of zero. Now (1.7) implies that for all $p > 0$,

$$H_p(\cdot/n^*)/a^p(n^*) \Rightarrow \varphi_p(\cdot) = |\psi_1(\cdot)|^p + |\psi_2(\cdot)|^p$$

on $(0,\infty)$ as $n^* \to \infty$. By the condition there is a continuity point $0 < s_0 < 1$ of $\varphi_1(\cdot)$ such that $\varphi_1(s_0) > 0$, and hence

$$H_p(\cdot/n^*)/H_p(s_0/n^*) \Rightarrow \varphi_p(\cdot)/\varphi_p(s_0) \quad \text{as} \quad n^* \to \infty.$$

Independently of $p > 0$, choose now the subsequence $\tilde{n} = \lfloor n^*/s_0 \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the usual integer-part function. Then, exactly as in the proof of Corollary 8 in [7], one can see that

$$\limsup_{\tilde{n}\to\infty} H_p(s/\tilde{n})/a_p^{(2)}(\tilde{n}) < \infty, \quad s > 0,$$

for any $p > 0$. Repeated applications of the Helly selection theorem now yield (1.8).

(ii) This part follows exactly as part (i), or can in fact be considered as a consequence of (i) applied to the quantile function $K(\cdot)$.

(iii) On the basis of the proof of (i), this is obvious.

(iv) Recalling the notation above (1.3), if $(\sigma_p^{(2)}(0))^2 = \text{Var}(|X|^p sgn X) < \infty$, then the statement clearly follows from Corollary 1 in [7]. If $\sigma_p^{(2)}(0) = \infty$, then a lengthier but fairly routine consideration based on formula (2.58) and some ideas from [7] is needed, the details of which are given in [13].

(v) The relationship of parts (iv) and (v) is the same as that of (i) and (ii).

(vi) This is obvious. ∎

Proof of the Theorem. We work on the specially constructed probability space described at the beginning of Section 2 in [7]. With the special uniform order statistics $U_{1,n} \leq \ldots U_{n,n}$ carried by that space we have the distributional equality

$$(S_n^{(1)}(m_1, k_1), \ldots, S_n^{(d)}(m_d, k_d)) =_D \left(\sum_{j=m_1+1}^{n-k_1} Q_1(U_{j,n}), \ldots, \sum_{j=m_d+1}^{n-k_d} Q_d(U_{j,n}) \right)$$

for each n. From now on one works with the right-side representation of the vector, and Theorem 2 in [7], applied component-wise, gives the present theorem in case (2). Also, an application of Theorem 1 in [7] as augmented in Theorem 1* of [5], using the same $\{r_{n'}\}$ sequence in each component, leads to case (1) of the present theorem at least component-wise but again with the same two Poisson processes $N_1(\cdot)$ and $N_2(\cdot)$ in the expressions of the non-normal terms of the limits of the different components. Furthermore, the normal term in the limit of the i-th component is the limit of the normally distributed random variables

$$Z_i(n') = \int_{(r_{n'}+1)/n'}^{1-(r_{n'}+1)/n'} B_{n'}(s)dQ_i(s)/\sigma_i(1/n'), \quad i = 1, \ldots, d,$$

where $B_{n'}(\cdot)$ is a sequence of Brownian bridges, the same for all $i = 1, \ldots, d$. Hence $EZ_i(n') = 0$ and, with the notation in (1.16),

$$EZ_i(n')Z_j(n') = s_{ij}(n') := \frac{\sigma_{ij}((r_{n'}+1)/n')}{\sigma_i(1/n')\sigma_j(1/n')}, \quad i, j = 1, \ldots, d,$$

for each n'. Modifying now the proof of Theorem 1^* in [5] in a trivial fashion, it is clear that the sequence $\{r_{n'}\}$ can be constructed in such a way that

$$\liminf_{n'\to\infty} s_{ij}(n') = \underline{s}_{ij} \quad \text{and} \quad \limsup_{n'\to\infty} s_{ij}(n') = \overline{s}_{ij}, \quad i,j = 1,\ldots,d,$$

where the $\underline{s}_{ij}$ and $\overline{s}_{ij}$ are given in (1.17). This completes the proof, some obvious details of which are spelled out in [13]. ∎

Proof of Corollary 1. First we note that if $\gamma \geq 2$ in (1.21) then F is in the domain of attraction of the normal distribution. Indeed, if $\gamma > 2$ then it is easy to see that

$$EX^2 = \int_0^1 Q^2(s)ds < \infty,$$

while if $\gamma = 2$ then an application of Karamata's theorem ([2], p.26) for the slowly varying function $L^2(\cdot)$ shows that condition (1.26 c) of Corollary 1 in [7] is satisfied. Also, if $\gamma < 2$, then by Corollary 3 in [7] F is in the domain of attraction of a stable distribution with exponent γ. Using now (1.1) and (1.2), we see that for each $i = 1,\ldots,d$, Q_i also satisfies (1.21) with $L(\cdot)$ replaced by $L^{p_i}(\cdot)$, δ_1 and δ_2 replaced by appropriate constants $\delta_1(i)$, $\delta_2(i) \geq 0$, $\delta_1(i) + \delta_2(i) > 0$, and γ replaced by γ/p_i. Hence by Corollary 1 in [7] we indeed have $\psi_j^{(i)} \equiv 0$ if $i = 1,\ldots,d_0$; $j = 1,2$, and (1.22) and (1.23) are also satisfied by Corollary 3 in [7], and by these same corollaries the sequences of functions $\psi_j^{(i)}(n,\cdot)$, $j = 1,2$; $i = 1,\ldots,d$, converge to these limiting functions along the whole $\{n\}$.

Therefore, by the theorem it remains to show that for $\underline{s}_{ij}$ and $\overline{s}_{ij}$ defined in (1.7), presently along the whole $\{n\}$, we have $\underline{s}_{ij} = s_{ij} = \overline{s}_{ij}$, $i,j = 1,\ldots,d_0$, where the common values s_{ij} are as given in the formulation of the corollary. Noting that $\sigma_{ij}(0)$ is finite when both $\sigma_i(0)$ and $\sigma_j(0)$ are finite, and that by the opening remark of the present proof (1.25) holds true, this is clear except for the case when $1 \leq i < j \leq d_0$, $p_i < \gamma/2$, $p_j = \gamma/2$, and $\sigma_j(0) = \infty$.

In the latter case it is of course sufficient to show that $\sigma_{ij}(0) < \infty$. The first step towards this, extending somewhat formula (2.58) in [7], is to derive by elementary computation that for all $1 \leq i,j \leq d$

$$\sigma_{ij}(s) = sQ_i(s)Q_j(s) + sQ_i(1-s)Q_j(1-s) + \int_s^{1-s} Q_i(u)Q_j(u)du$$

$$- \{sQ_i(s) + sQ_i(1-s) + \int_s^{1-s} Q_i(u)du\}\{sQ_j(s) + sQ_j(1-s)$$

$$+ \int_s^{1-s} Q_j(u)du\}.$$

Substituting now (1.21) for Q_i and Q_j as described above, and assuming $p_i < \gamma/2$ and $p_j = \gamma/2$, we obtain by a somewhat lengthy computation, the details of which are given in [13], that the asymptotic equality

$$\sigma_{ij}(s) \sim a_{ij} \int_s^{1/2} u^{-(\frac{p_i}{\gamma}+\frac{1}{2})} L^{p_i+\frac{7}{2}}(u)\,du + b_{ij}s^{\frac{1}{2}-\frac{p_i}{\gamma}} L^{p_i+\frac{7}{2}}(s)$$

holds as $s \downarrow 0$, where a_{ij} and b_{ij} are positive constants. This implies that $\sigma_{ij}(0) < \infty$ and hence the corollary. ∎

Proof of Corollary 2. We only have to prove the second statement. First we note that for $\mu_i(n_k)$ in (1.29), where n_k is as in (1.27), we obtain by easy calculations that

$$\mu_l(n_k) \sim \begin{cases} \lambda 2^{l2^k} & ,\lambda > 1 \\ 2^{2^k}\sum_{j=1}^{k} 2^{(l-1)2^{j-1}}/(1-2^{-2^{j-1}})^{l-1} & ,\lambda = 1, \\ \lambda 2^{2^k} 2^{(l-1)2^{k-1}} & ,\lambda < 1, \end{cases}$$

and hence

(3.1) $$\mu_l(n_k) \begin{cases} \sim \lambda k A_{n_k}, & \text{if } l = 1, \\ \leq \lambda A_{n_k}^l, & \text{if } 2 \leq l \leq d. \end{cases}$$

Next, recalling the meaning of $\sum^*$ in (1.31), introducing the notation

$$a_{i_1,\dots,i_d} = (-1)^d (-1)^{i_1+\dots+i_d}/(i_1!\dots i_d!),$$

and using Lemma A. 2 in [1] (and hence correcting a misprint in [16] where, in our notation, the factor $(-1)^d$ is missing) for the elementary symmetric polynomials in question we obtain

$$P_{n_k}^{(d)} := \sum_{1 \leq j_1 < \dots < j_d \leq n_k} X_{j_1}\dots X_{j_d} = \sum_{(i_1,\dots,i_d)}^* a_{i_1,\dots,i_d} \prod_{l=1}^{d} \left(\frac{S_l(n_k)}{l}\right)^{i_l}$$

Now, dropping n_k everywhere in the notation and writing

$$S_l^{i_l} = \left(A^l \frac{S_l - \mu_l}{A^l} + \mu_l\right)^{i_l} = D_l + \mu_l^{i_l},$$

where, agreeing that an empty sum is zero and an empty product is one,

$$D_l = D_l(n_k) = \sum_{r=1}^{i_l} \binom{i_l}{r} \left(\frac{S_l - \mu_l}{A^l}\right)^r A^{lr} \mu_l^{i_l - r}$$

by the binomial formula, and using a well-known formula for the product of two-term sums (cf. [8], p.446), we obtain, with $C_{n_k}^{(d)}$ as given in (1.31),

$$P_{n_k}^{(d)} = C_{n_k}^{(d)} + \sum_{(i_1,\ldots,i_d)}^{*} a_{i_1,\ldots,i_d} \sum_{m=1}^{d} \frac{D_m}{m^{i_m}} \left(\prod_{l=1}^{m-1} (\frac{\mu_l}{l})^{i_l} \right) \left(\prod_{l=m+1}^{d} (D_l + \mu_l^{i_l}) \frac{1}{i_l!} \right)$$

Of course we have to keep in mind that D_l depends on i_l as well.

From the sum $\sum^{*}$ we now separate the term pertaining to the vector $(d, 0, \ldots, 0)$ and what remains we denote by R_{n_k}. The separated term, since $i_1 = d$ in D_1 and $i_l = 0$ in D_l if $l \geq 2$, is

$$\frac{(-1)^{2d}}{d!} \left\{ D_1 \prod_{l=2}^{d} (D_l + 1) + \sum_{m=2}^{k} D_m \mu_1^d \prod_{l=m+1}^{d} (D_l + 1) \right\} =$$

$$= \frac{1}{d!} D_1 = \frac{1}{d!} \sum_{r=1}^{d} \binom{d}{r} \left(\frac{S_1 - \mu_1}{A} \right)^r A^r \mu_1^{d-r}$$

$$= \frac{1}{(d-1)!} \left(\frac{S_1 - \mu_1}{A} \right) A \mu_1^{d-1} + T_{n_k}.$$

Therefore, since

(3.2) $$A = A_{n_k} = b_k \sim 2^{2^k} \quad \text{as} \quad k \to \infty,$$

we see that (1.30) will be proved once we show that, as $k \to \infty$,

(3.3) $$d! T_{n_k} = \sum_{r=2}^{d} \binom{d}{r} \left(\frac{S_1 - \mu_1}{A} \right)^r A^r \mu_1^{d-r} = o_P(k^{d-1} 2^{d2^k})$$

and

(3.4) $$R_{n_k} = o_P(k^{d-1} 2^{d2^k}).$$

Since by the already proved first statement of the corollary

(3.5) $$\max_{1 \leq l, r \leq d} |(S_l(n_k) - \mu_l(n_k))/A_{n_k}^l|^r = O_P(1),$$

we have

$$d! |T_{n_k}| = O_P(1) A_{n_k}^2 \mu_1^{d-2}(n_k),$$

which by (3.1) and (3.2) implies (3.3).

On the other hand, denoting by $\sum^{**}$ the sum $\sum^{*}$ with the vector $(d, 0, \ldots, 0)$ excluded, by (3.1) we get

$$|R_{n_k}| \leq K_1 \sum\nolimits^{**}_{(i_1,\ldots,i_d)} |D_1| \left(\prod_{l=2}^{d} (|D_l| + A^{l i_l}) \right)$$

$$+ K_2 \sum\nolimits^{**}_{(i_1,\ldots,i_d)} \mu_1^{i_1} \sum_{m=2}^{d} |D_m| \left(\prod_{l=2}^{m-1} A^{l i_l} \right) \left(\prod_{l=m+1}^{d} (|D_l| + A^{l i_l}) \right)$$

$$=: R_{n_k}^{(1)} + R_{n_k}^{(2)},$$

where K_1 and K_2 are some positive constants depending only on λ and d. Since by (3.1) and (3.5) again

$$(3.6) \qquad\qquad D_m = D_m(n_k) = A_{n_k}^{m i_m} O_P(1) \quad \text{if} \quad m \geq 2,$$

using the same relations repeatedly we obtain

$$R_{n_k}^{(1)} = O_P(1) \sum\nolimits^{**}_{(i_1,\ldots,i_d)} \left(\sum_{r=1}^{i_1} \binom{i_1}{r} A^r \mu_1^{i_1-r} \right) \left(\prod_{l=2}^{d} A^{l i_l} \right)$$

$$= O_P(1) \sum\nolimits^{**}_{(i_1,\ldots,i_d)} A^{2 i_2 + \ldots + d i_d} \mu_1^{i_1}$$

$$= O_P(1) A_{n_k}^2 \mu_1^{d-2}(n_k),$$

because i_1 in the sum $\sum^{**}$ can at most be $d - 2$ and it is exactly the vector $(d - 2, 1, 0, \ldots, 0)$ for which the summand is the largest. Hence (3.4) is satisfied for $R_{n_k}^{(1)}$, and we only have to show it for $R_{n_k}^{(2)}$. But, again by (3.6),

$$R_{n_k}^{(2)} = O_P(1) \sum\nolimits^{**}_{(i_1,\ldots,i_d)} \mu_1^{i_1} \sum_{m=2}^{d} A^{m i_m} \left(\prod_{l=2}^{m-1} A^{l i_l} \right) \left(\prod_{l=m+1}^{d} A^{l i_l} \right)$$

$$= O_P(1) \sum\nolimits^{**}_{(i_1,\ldots,i_d)} \mu_1^{i_1} A^{2 i_2 + \ldots + d i_d}$$

$$= O_P(1) A_{n_k}^2 \mu_1^{d-2}(n_k)$$

and thus (1.30) is completely proved. ∎

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