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Another Look at Among and Within Class Regressions in Analysis of Covariance

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ABSTRACT

In a recent paper, Monlezun and Blouin (1988) proposed an analysis of covariance for a split plot experiment. They suggest a model consisting of five regression coefficients, one for each of the effects - whole-plot treatment means, whole-plot errors, split plot-treatment means, whole-plot and split-plot treatment interactions and the experimental errors. In this paper we critically examine the analysis suggested by Monlezun and Blouin (1988). We relate their analysis to the within class (internal) and among class (external) regressions considered by Wishart and Sanders (1934) and Smith (1957) and present some of the issues that are not usually addressed in text books.

1. INTRODUCTION

Analysis of covariance is widely used by various scientists, including agronomists, biologists, chemists, psychologists, sociologists and statisticians. The primary use of the analysis of covariance is to increase precision in randomized experiments. Cochran (1957) and Snedecor and Cochran (1976) present a summary of the principal use of the analysis of covariance. Fisher (1934) expresses that "the analysis of covariance combines the advantages and reconciles the requirements of the two very widely applicable procedures known as regression and analysis of variance." Standard textbooks include the traditional analysis of covariance, which consists of (a) comparison of adjusted means, (b) a test for the treatment effects after adjusting for the covariate, (c) a test for the covariate effects after adjusting for the treatments and (d) a test for the homogeneity of slopes. However, a comparison of treatment ("between class," "among class," "external") and error ("within class," "internal") regressions is not usually presented. In this paper, we discuss some of the uses and the interpretations of the between class and within class regressions.

In a recent paper, Monlezun and Blouin (1988) (it will be referred to as MB in this paper) considered the following model for an analysis of covariance of a split plot experiment:

$$
\begin{aligned}
y_{ijk} = \mu &+ \beta_A(\bar{x}_{.j.} - \bar{x}_{...}) + \alpha_j + \tau_A(\bar{x}_{ij.} - \bar{x}_{.j.}) + \delta_{ij} \\
&+ \beta_B(\bar{x}_{..k} - \bar{x}_{...}) + \theta_k + \beta_{AB}(\bar{x}_{.jk} - \bar{x}_{.j.} - \bar{x}_{..k} + \bar{x}_{...}) \\
&+ \gamma_{jk} + \tau_B(x_{ijk} - \bar{x}_{.jk} - \bar{x}_{ij.} + \bar{x}_{.j.}) + e_{ijk},
\end{aligned}
\tag{1.1}
$$

$j=1,\ldots,a$: whole-plot treatments; $k=1,\ldots,b$: split-plot treatments; $i=1,\ldots,n_j$, where $\{\delta_{ij}\}$ and $\{e_{ijk}\}$ are independent mean zero normal random variables with variances σ_δ^2 and σ^2 respectively. In this model, β_A, β_B and β_{AB} are "among" class regressions and τ_A and τ_B are the "within" class regressions at the whole-plot and split-plot levels, respectively. Based on model (1.1), MB suggests tests for H_A: no whole-plot treatment effect, H_B: no split-plot treatment effect and H_{AB}: no interaction between the whole-plot and split-plot treatments, that utilize the treatment sums of squares (with one degree of freedom less than the traditional analysis of covariance) after adjusting for both among and within class regressions. Most of commonly used textbooks (Kempthorne (1952), Snedecor and Cochran (1976) and Steel and Torrie (1980)) use models with one ($\beta_A = \tau_A = \beta_B = \beta_{AB} = \tau_B$) or two slopes ($\beta_A = \tau_A$; $\beta_B = \beta_{AB} = \tau_B$) for the covariate. MB suggest modifications to their test statistics for "special cases" $\beta_A = \tau_A$; $\beta_B = \beta_{AB} = \tau_B$ and $\beta_A = \tau_A = \beta_B = \beta_{AB} = \tau_B$. In this paper, we will relate the above analysis to the among class and within class regressions considered by Wishart and Sanders (1934) and Smith (1957). Smith (1957) used the terms external and internal regressions for among class (more commonly referred to as between class) and within class regressions.

In Section 2, we discuss the among and within-class regression for the one way analysis of covariance. In Section 3, we present the extensions to split-plot analysis of covariance. We summarize our results in Section 4.

2. ONE WAY ANALYSIS OF COVARIANCE

2.1. The Traditional Model:

Consider the usual analysis of covariance model

$$y_{ij} = \mu_i + \beta(x_{ij} - \bar{x}_{..}) + e_{ij} , \tag{2.1}$$

i=1,...,a treatments; j=1,...,n: subjects; $e_{ij} \sim NID(0,\sigma^2)$. Let $\mu = a^{-1}\sum_{i=1}^{a}\mu_i$ and $\alpha_i = \mu_i - \mu$. We assume that $\sum_{i=1}^{a}\sum_{j=1}^{n}(x_{ij} - \bar{x}_{i.})^2 > 0$. (The results discussed here can easily be extended to the case where unequal number n_i of subjects are used for different treatments. We chose to present the case n_i=n only for the sake of notational simplicity). The model (2.1) can be expressed in matrix form as

$$\mathbf{Y} = X\mathbf{b} + \mathbf{e} \tag{2.2}$$

where $\mathbf{b} = (\mu, \alpha_2, \ldots, \alpha_a, \beta)'$, $X = (\mathbf{1}, \varphi, \mathbf{w})$, $\mathbf{1} = (1,\ldots,1)'$, $\mathbf{w} = (x_1' - \bar{x}_{..}\mathbf{1}', \ldots, x_a' - \bar{x}_{..}\mathbf{1}')'$ and φ is an na x (a-1) matrix given by

$$\varphi = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix} .$$

Note that φ and $\mathbf{w}$ are pairwise orthogonal to $\mathbf{1}$, but not to each other. The traditional analysis of covariance sums of squares are summarized in Table 1. (See Snedecor and Cochran (1976) for the R(·) notation.)

Table 1. Traditional One Way Analysis of Covariance

Source	d.f.	Type I SS	Type III SS
Treatments	a-1	$R(\alpha\|\mu)$	$R(\alpha\|\mu,\beta)$
Covariate	1	$R(\beta\|\mu,\alpha)$	$R(\beta\|\mu,\alpha)$
Error	an-a-1	$\mathbf{y}'\mathbf{y} - R(\mu,\alpha,\beta)$	

It is easy to show that

$$R(\alpha|\mu) = \mathbf{y}'P_{[\varphi]}\mathbf{y} = A_{yy} = n\Sigma_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..})^2 ,$$

$$R(\alpha|\mu,\beta) = \mathbf{y}'P_{[\varphi_w]}\mathbf{y} = A_{yy} - S_{xx}^{-1} S_{xy}^2 + E_{xx}^{-1} E_{xy}^2 ,$$

$$R(\beta|\mu) = \mathbf{y}'P_{[w]}\mathbf{y} = S_{xx}^{-1} S_{xy}^2 ,$$

$$R(\beta|\mu,\alpha) = \mathbf{y}'P_{[Z_2]}\mathbf{y} = E_{xx}^{-1} E_{xy}^2 ,$$

and

$$\mathbf{y}'\mathbf{y} - R(\mu,\alpha,\beta) = E_{yy} - E_{xx}^{-1} E_{xy}^2 ,$$

where A_{xx}, A_{xy}, A_{yy}; E_{xx}, E_{xy}, E_{yy} and S_{xx}, S_{xy}, S_{yy} are the among, within and total sums of squares and cross products, respectively (see Steel and Torrie (1980)); $\varphi_w = [I - P_{[w]}]\varphi$,

$$Z_2 = \Big((x_1' - \bar{x}_{1.}1'), \ \ldots, \ (x_a' - \bar{x}_{a.}1')\Big)' = \Big[I - P_{[\varphi]}\Big] w$$

and

$$P_{[A]} = A(A'A)^{-}A' .$$

The null hypothesis H_0: there are no treatment effects (after adjusting for the covariate) is tested using the F-statistic $F_1 = R(\alpha|\mu,\beta)/(a-1)MSE$, where MSE is the mean square error given by $[\mathbf{y}'\mathbf{y} - R(\mu,\alpha,\beta)]/[an-a-1]$.

2.2. The "New" Model:

We now consider the model for the one way analysis of covariance that is similar in concept to the model (1.1). Consider

$$y_{ij} = \mu_i^* + \beta_1(\bar{x}_{i.} - \bar{x}_{..}) + \beta(x_{ij} - \bar{x}_{i.}) + e_{ij} , \qquad (2.3)$$

$$= \mu_i^* + (\beta_1 - \beta)(\bar{x}_{i.} - \bar{x}_{..}) + \beta(x_{ij} - \bar{x}_{..}) + e_{ij} . \qquad (2.4)$$

Writing model (2.4) in matrix notation, we get

$$\mathbf{y} = W\, \tau + \mathbf{e} , \qquad (2.5)$$

where $W = (\mathbf{1}, \mathbf{Z}_1, \varphi, \mathbf{w})$; $\tau = (\mu^*, \beta_1\text{-}\beta, \alpha_2^*, \ldots, \alpha_a^*, \beta)'$; $\mathbf{Z}_1$ is an nabx1 vector with $Z_{1ij} = \bar{x}_{.j} - \bar{x}_{..}$; $\mu^* = a^{-1} \sum_{i=1}^a \mu_i^*$ and $\alpha_i^* = \mu_i^* - \mu^*$. In model (2.4), if $\beta_1 = \beta$, we get the traditional model (2.1). Note, however, that the column $\mathbf{Z}_1$ is a linear combination of columns of φ, ($\mathbf{Z}_1 = \varphi\mathbf{c}$, $\mathbf{c} = (\bar{x}_{2.} - \bar{x}_{..}, \ldots, \bar{x}_{a.} - \bar{x}_{..})'$) and hence the rank of W (=a+1) is the same as the rank of X. In model (2.4), a linear parametric function $\lambda'\tau = \sum_{i=1}^a \lambda_i \mu_i^* + \lambda_{a+1}(\beta_1 - \beta) + \lambda_{a+2}\beta$ is (linearly) estimable if and only if $\lambda_{a+1} = \sum_{i=1}^a (\bar{x}_{i.} - \bar{x}_{..})\lambda_i$. (See Searle (1971) for a definition of estimability.) Therefore, the parameter $\beta_1 - \beta$ in (2.4) is not estimable. Since the inference on estimable functions is not affected by imposing restrictions on a nonestimable function, one can without loss of generality assume that $\beta_1 - \beta = 0$. Therefore, the models (2.1) and (2.4) are equivalent.

Based on the model (2.4), MB partition the treatment sums of squares $R(\alpha|\mu,\beta)$ of model (2.1) into two parts and suggest the analysis of covariance given in Table 2. (MB did not consider a one way analysis of covariance directly, but their analysis of whole plot treatments can be considered as a one way analysis of covariance.)

Table 2. Analysis of Covariance for Model (2.4)

Source	d.f.	Type I SS	Type III SS
Among Class Regression	1	$R(\beta_1 \mid \mu)$	$R^*(\beta_1 - \beta)$
Treatments (after Among Class regression)	a-2	$R(\alpha^* \mid \mu, \beta_1)$	$R(\alpha^* \mid \mu, \beta_1)$
Within Class Regression	1	$R(\beta \mid \mu, \alpha)$	$R(\beta \mid \mu, \alpha)$
Error	an-a-1	$y'y - R(\mu, \alpha, \beta)$	

The sums of squares in Table 2 are given by

$$R(\beta_1 \mid \mu) = y' P_{[Z_1]} y = A_{xx}^{-1} A_{xy}^2$$

$$R(\alpha^* \mid \mu, \beta_1) = R(\alpha \mid \mu) - R(\beta_1 \mid \mu)$$

$$= y' P_{[\varphi_x]} y = A_{yy} - A_{xx}^{-1} A_{xy}^2$$

and

$$R^*(\beta_1 - \beta) = R(\alpha \mid \mu, \beta) - R(\alpha^* \mid \mu, \beta_1)$$

$$= y' P_{[\varphi_w]} y - y' P_{[\varphi_x]} y$$

$$= y' P_{[Z_1^*]} y$$

$$= W_{xx}^{-1} W_{xy}^2 - S_{xx}^{-1} S_{xy}^2 + A_{xx}^{-1} A_{xy}^2 ,$$

where $\varphi_x = [I - P_{[Z_1]}] \varphi$ and $Z_1^* = [I - P_{[w]}] Z_1$. Note that the first two rows of Table 2 add up to the first row of Table 1 and the last two rows are the same as in Table 1. MB recommend that the statistic $F_2 = R(\alpha^*|\mu,\beta_1)/(a-2)MSE$ be used to test the hypothesis H_0^*: $\mu_i^* - \mu_1^* = 0$, $i=2,\ldots,a$. (i.e., No treatment effects after adjusting for the among class regression, β_1). Also, in the "special case," $\beta_1 = \beta$, they suggest to consider the pooled sums of squares $R^*(\beta_1 - \beta) + R(\alpha^*|\mu,\beta_1) = R(\alpha|\mu,\beta)$ to test the hypothesis that there are no treatment effects. (i.e., use the F_1 statistic considered in the traditional approach.)

Similar analysis, but with a different interpretation, is considered by Wishart and Sanders (1934). Smith (1957) best summarizes the interpretations of the different partitioned sums of squares in Table 2. Recall that models (2.1) and (2.4) are equivalent and that the model (2.1) is more commonly used. We therefore believe that it is useful to express the hypotheses being tested by different sums of squares in Table 2 in terms of the parameters in model (2.1).

It is easy to show that the noncentrality parameter (λ^*) of the distribution of $R^*(\beta_1-\beta)/\sigma^2$ is zero if and only if $(\beta_1 - \beta) + (Z_1'Z_1)^{-1}Z_1'\varphi\alpha^*$ is zero, where $\alpha^* = (\alpha_2^*,\ldots, \alpha_a^*)'$. In terms of the parameterization of (2.1), the noncentrality parameter λ^* is zero if and only if $(Z_1Z_1)^{-1}\, Z_1'\varphi\alpha$ is zero, where $\alpha = (\alpha_2, \ldots, \alpha_a)'$, i.e., if and only if $[\sum_{i=1}^a(\bar{x}_{i.} - \bar{x}_{..})^2]^{-1} \sum_{i=1}^a(\bar{x}_{i.} - \bar{x}_{..})\mu_i$ is zero. (i.e., the regression of μ_i on $\bar{x}_i$ is zero.) Therefore, the one degree of freedom sum of squares $R^*(\beta_1 - \beta)$ tests the hypothesis that the contrast $\sum_{i=1}^a(\bar{x}_{i.} - \bar{x}_{..})\,\mu_i$ is zero. It now follows that the noncentrality parameter of $R(\alpha^*|\mu,\beta_1)/\sigma^2$ is zero if and only if the <u>remaining</u> (a-2) linearly independent treatment contrasts (involving μ_i) are simultaneously zero. Also, since the noncentrality parameter of $R(\beta_1|\mu)/\sigma^2$ is zero if and only if $\beta_1 + (Z_1'Z_1)^{-1}\, Z_1'\varphi\alpha^*$ is zero, one may also use the one degree of freedom sum of squares $R(\beta_1|\mu)$ to test the hypothesis that $\beta + \sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\mu_i$ is zero. (MB interpret this as a test for $\beta_1 = 0$ when there are "no treatment effects." See the last paragraph of their Section 3.)

It is interesting to note that $R(\beta_1|\mu)$ and $R(\alpha^*|\mu,\beta_1)$ are n times the regression sum of squares and the residual sum of squares in the regression of $\bar{y}_{i.}$ on $\bar{x}_{i.}$, i=1,...,a. Therefore, a test based on the sum of squares $R(\alpha^*|\mu,\beta_1)$ may be interpreted as a test for the lack of a linear relationship between the treatment means $(\bar{y}_{i.})$ and the covariate treatment means $(\bar{x}_{i.})$; i.e., it tests the null hypothesis that the among class regression is linear. (This interpretation is given in Wishart and Sanders (1934).)

The analysis in Table 2 will, therefore, be useful whenever the contrast $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\mu_i$ is of practical importance. If this contrast is important then the hypothesis that $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\mu_i$ is zero can be tested using $R^*(\beta_1 - \beta)/MSE$. Recall, however, that one of the postulates underlying the application of covariance to control extraneous variation on estimated treatment responses is that the concomitant variable, x, is unaffected by the treatments, either by direct causation or through correlation with another variable. (See Smith (1957), Section 2.) Since the postulate affirmed that μ_i and $\bar{x}_{i.}$ are unrelated we expect on the average of many experiments the regression of μ_i on $\bar{x}_{i.}$ to be zero. (i.e., the contrast $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\mu_i$ is expected to be zero.) Smith (1957) therefore writes, "To test for a relationship which is known to exist is entirely pointless ... individual contrasts dictated by the $\bar{x}_{i.}$ which happened to be associated with $\bar{y}_{i.}$ in a given experiment would rarely be of interest." Smith (1957) also points out that when the postulate cannot be made, or if it may be in doubt, this contrast may acquire some individual interest. (See the examples given in Smith (1957) and the examples given in subsection 2.4 of this paper.) Also, if the postulate cannot be made, one has to be careful with the interpretation of the adjusted means and the hypothesis of no treatment effects after adjusting for the covariate.

In practice, sometimes the statistic

$$F_3 = \frac{A_{xx}/(a-1)}{E_{xx}/(an-a)} = \frac{\sum_{i=1}^{a} n(\bar{x}_{i.} - \bar{x}_{..})^2/(a-1)}{\sum_{i=1}^{a} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.})^2/(an-a)} \qquad (2.6)$$

is used to test the hypothesis whether the postulate is valid. The above F_3 statistic in fact tests whether the covariate varied over the different treatment groups. The covariate may significantly vary with the treatments without showing a linear relationship to μ_i such as when the x-values are <u>imposed</u> by the treatments instead of being <u>affected</u> by the treatments. (See Smith (1957) Section 3 for examples.) In such a case, the F_3 statistic may be significant even though $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\mu_i$ may be zero. If F_3 were significant, one may have to worry about extrapolation problems.

2.3. Among Class and Within Class Regressions:

The regression of $y_{ij} - \bar{y}_{i.}$ on $x_{ij} - \bar{x}_{i.}$ is called the within class (internal) regression whereas the regression of $\bar{y}_{i.}$ on $\bar{x}_{i.}$ is called the among class (external) regression. Let $\hat{\beta}$ and $\hat{\beta}_1$ denote the estimates of internal and external regression coefficients, respectively. Wishart and Sanders (1934), Quenouille (1952), Cochran (1957), Smith (1957) and Winer (1971), among others, were interested in comparing the estimates $\hat{\beta}$ and $\hat{\beta}_1$. (See Wishart and Sanders (1934) and the next subsection of this paper for a pictorial interpretation of among and within class regressions.)

Using the parameterization (2.1), it is easy to show that (a) $\hat{\beta}_1$ and $\hat{\beta}$ are uncorrelated (b) $E(\hat{\beta}) = \beta$, (c) $V(\hat{\beta}) = \sigma^2 E_{xx}^{-1}$, (d) $V(\hat{\beta}_1) = \sigma^2 A_{xx}^{-1} + [\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})^2]^{-1} [\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\mu_i]$, and (e) $E(\hat{\beta}_1) = \beta$. Therefore, $E(\hat{\beta} - \hat{\beta}_1) = 0$ if and only if the contrast $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\mu_i = 0$. It can also be shown that, with $\hat{\sigma}^2$ = MSE,

$$\frac{(\hat{\beta} - \hat{\beta}_1)^2}{\hat{\sigma}^2(E_{xx}^{-1} + A_{xx}^{-1})} = \frac{R^*(\beta_1 - \beta)}{\hat{\sigma}^2} \qquad (2.7)$$

which is the one degree of freedom test statistic given in Table 2. We, therefore, prefer that a comparison of $\hat{\beta}$ and $\hat{\beta}_1$ be used as a test for $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\mu_i = 0$, rather than as a test of β (within treatment slope) equals β_1 (among treatment slope). (Winer (1971) gives (2.7) as the test statistic for $\beta_{\text{within class}} = \beta_{\text{between class}}$.)

Smith (1957) suggests that (2.7) is not appropriate for comparing β and β_1 and suggests using instead the statistic

$$d^2 = \frac{(\hat{\beta} - \hat{\beta}_1)^2}{\hat{\sigma}^2 E_{xx}^{-1} + \hat{\sigma}_1^2 A_{xx}^{-1}} \qquad (2.8)$$

where $\hat{\sigma}_1^2 = R(\alpha^*|\mu,\beta_1)/(a-2)$ from Table 2. Recall that $\hat{\sigma}_1^2$ is also n times the residual mean square error in the regression of $\bar{y}_{i.}$ on $\bar{x}_{i.}$ and $\hat{\sigma}_1^2 A_{xx}^{-1}$ is the estimated variance of $\hat{\beta}_1$ from this regression. Since $\hat{\sigma}_1^2$ is not an estimate of σ^2, we consider (2.7) as the appropriate test.

The best explanation of the different test statistics considered above is given by Snedecor and Cochran (1972, Section 14.7). They consider the analysis of covariance for the case where the treatments can be considered as a <u>random</u> sample from some population of treatments. Consider the model

$$y_{ij} = \mu^* + \beta_1(\bar{x}_{i.} - \bar{x}_{..}) + \beta(x_{ij} - \bar{x}_{i.}) + \alpha_i^* + e_{ij} \qquad (2.9)$$

where $\alpha_i^* \sim NID(0,\sigma_\alpha^2)$; $e_{ij} \sim NID(0,\sigma^2)$ and $\{\alpha_i^*\}$ and $\{e_{ij}\}$ are independent. In model (2.9), β_1 and β are both estimable. In fact, $\hat{\beta}_1$ and $\hat{\beta}$ are unbiased for β_1 and β, respectively. Under the assumptions of model (2.9), it is easy to show that

(a) $\hat{\sigma}_1^2 = R(\alpha^*|\mu,\beta_1)/(a-2)$ is unbiased for $\sigma^2 + n\,\sigma_\alpha^2$;

(b) $F_2 = \hat{\sigma}_1^2/\hat{\sigma}^2$ is an appropriate F-statistic for testing H_0: $\sigma_\alpha^2 = 0$ (i.e., no variation among treatments).

(c) The test statistic (2.8), suggested by Smith (1957) is an appropriate test statistic for testing $\beta_1 = \beta$. However, the exact distribution of (2.8) is unknown and one needs to use Satterthwaite's approximation for degrees of freedom.

(d) If $\beta_1 = \beta$, an <u>estimated</u> generalized least squares estimator for the common value is

$$\hat{\beta}_{EGLS} = (A_{xx}\hat{\sigma}_1^{-2} + E_{xx}\hat{\sigma}^{-2})^{-1}\,(A_{xx}\hat{\sigma}_1^{-2}\,\hat{\beta}_1 + E_{xx}\hat{\sigma}^{-2}\hat{\beta}) \;. \qquad (2.10)$$

(e) A test statistic for testing $H_0:\beta_1 = 0$ is $R(\beta_1|\mu)/\hat{\sigma}_1^2$.

(f) If <u>$\sigma_\alpha^2 = 0$</u> , then the test statistic in (2.7) has a (noncentral) $F_{1,an-a}$ distribution and it can be used to test the hypothesis $\beta_1 = \beta$. If $\sigma_\alpha^2 = 0$, Snedecor and Cochran (1976) suggest using a more powerful test which is obtained by using a pooled estimator

$$\tilde{\sigma}^2 = \frac{(a-2)\hat{\sigma}_1^2 + (an-a)\hat{\sigma}^2}{an-2}$$

instead of $\hat{\sigma}^2$ in (2.7). That is, compare the one degree of freedom sums of squares $R^*(\beta_1 - \beta)$ with the pooled sums of squares obtained by adding the (a-2) degree of freedom sum of squares $R(\alpha^*|\mu,\beta_1)$ with the residual sum of squares.

2.4. Examples:

In this subsection we present one example where the covariate is not affected by the treatments and another example where the treatments may have affected the covariate. For both examples, the GLM procedure in SAS (1982) is used to compute the Type I and Type III sums of squares.

Example 1: A study was conducted at the Governor Morehead School in Raleigh, North Carolina, to evaluate some techniques intended to improve the "listening-reading" skill. The subjects were visually impaired. The listening-reading treatments were: (1) Instruction in listening techniques plus practice listening to selected readings; (2) the same as (1) but with copies of the selected readings in braille; and (3) the same as (1) but with copies of selected readings in ink print. The number of individuals per group was 4. The response data are measures of reading accuracy as measured by the Gilmore Oral Reading Test. Both pre- and post-test data were taken. The data and the analysis of covariance are summarized in Tables 3 and 4 respectively.

Table 3. Pre-test and Post-test Data

	T1				T2				T3			
Pre-test Score (X)	89	82	88	94	89	90	91	92	89	99	84	87
Post-test Score (Y)	87	86	94	96	84	94	97	93	96	97	100	98

Table 4. Analysis of Covariance for Pre-test and Post-test Data

Source	df	Type I SS	Type III SS	F
Treatments	2	111.5	105.7	2.57
Among Class Regression	(1)	(13.7)	(7.9)	(0.38)
Treatments after β_1	(1)	(97.8)	(97.8)	(4.76)
Within Class Regression	1	13.22	13.22	0.64
Error	8	164.3		

The within class ($\hat{\beta}$) and among class ($\hat{\beta}_1$) regression estimates are 0.25 (± 0.32) and 1.14 (± 1.40) respectively. From Table 4, we conclude that there is not enough evidence of a relationship between the pre-test and post-test scores after the removal of the treatment effects (F = 0.64). Also, there is not enough evidence to conclude that the different instructions produce different post-test scores, after adjusting for the pre-test scores (F = 2.57).

In this example, instructions could not have affected the pre-test scores and with randomization, there should be no association between $\bar{x}_{i.}$ and μ_i. Consequently, the contrast $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\mu_i$ is expected to be zero *a priori* and the partition of the treatment sums of squares is pointless (as effectively argued by Smith (1957)). Had the test of within class regression (β) been significant, we would proceed to obtain adjusted treatment means using $\hat{\beta}$ to make adjustments.

Example 2: Rawlings (1988, p. 219) presents an example for one-way analysis of covariance where the purpose of the covariance analysis is to aid in the interpretation of the treatment effects. To compare the ascorbic acid content in cabbage from two genetic cultivars planted on three different dates (2x3 = 6 treatments) a completely random design with 10 experimental units per treatment was used. It was anticipated that ascorbic acid content (y) might be dependent on the size of the cabbage head; hence, head weight (x) was recorded for possible use as a covariate. The analysis of covariance of section 2.2 is given in Table 5. (We have ignored the factorial nature of the treatments in our analysis.)

Table 5. Analysis of Covariance for the Cabbage Data

Source	df	Type I SS	Type III SS	F
Treatments	5	3549.8	1435.3	7.70
Among Class Regression	(1)	(2650.6)	(536.1)	(14.39)
Treatments after β_1	(4)	(899.2)	(899.2)	(6.03)
Within Class Regression	1	516.05	516.05	13.85
Error	53	1975.05		

Note that all of the F-statistics in Table 5 are significant. There is a clear evidence that the treatment means differ significantly even after adjusting for the covariate (F = 7.70). Also, the covariate has significant effect on the content after adjusting for the treatments (F = 13.85). Here, the head weight may vary with different cultivars and planting times; i.e., the covariate may be affected by the treatments. Since the F-statistic (14.39) for the among class regression is significant, we reject the hypothesis that $\sum_{i=1}^{6}(\bar{x}_{i.} - \bar{x}_{..})\mu_i = 0$. This suggests that the treatment may have affected the covariate. Also, the significance of the F-statistic (6.03) for treatments after β_1 indicates that not all variation in treatment means can be explained by a linear relationship between $\bar{x}_{i.}$ and μ_i. The within class ($\hat{\beta}$) and among class ($\hat{\beta}_1$) regression estimates are -11.4 (± 1.21) and -4.5 (± 1.35) respectively. See Figure 1 for an interpretation of $\hat{\beta}$ and $\hat{\beta}_1$. (Recall, $\hat{\beta}_1$ is the slope in the regression of $\bar{y}_{i.}$ on $\bar{x}_{i.}$ and $\hat{\beta}$ is the slope in the regression of y_{ij} on $(x_{ij} - \bar{x}_{i.})$.)

Since, in this example, the treatments may have affected both the covariate and the dependent variable, a multivariate analysis of variance may be more appropriate to test the joint effect of the treatments on both variables and on the internal relationships between the two variables. The multivariate analysis also avoids the commonly made, but inappropriate, inference from analysis of covariance in these situations of a directional causal effect of the covariate on the independent variable. In this example, the multivariate test statistics for testing no treatment effects are all observed to be significant, indicating that treatments have affected both variables.

3. SPLIT-PLOT ANALYSIS OF COVARIANCE

Monlezun and Blouin (1988) consider model (1.1) with five different slope parameters for an analysis of covariance of a split-plot experiment. When different number (n_j) of whole-plots are used for different whole-plot treatments, they consider model (1.1) with $\beta_B = \beta_{AB}$. In this paper, for the sake of simplicity, we consider only the balanced design where $n_j = n$ for j=1, ..., a. A more commonly used model for split-plot analysis of covariance is

$$y_{ijk} = \mu + \alpha_j + \beta_W(\bar{x}_{ij.} - \bar{x}_{...}) + \delta_{ij} + \theta_k + \gamma_{jk}$$

$$+ \beta_S(x_{ijk} - \bar{x}_{.jk}) + e_{ijk} , \qquad (3.1)$$

which is obtained from model (1.1) by setting $\beta_A = \tau_A$ $(=\beta_W)$ and $\beta_B = \beta_{AB} = \tau_B(=\beta_S)$. (See Kempthorne (1952).) Using the arguments in Section 2, it can be shown that the models (1.1) and (3.1) are equivalent in the sense one is a reparameterization of the other; i.e., the column spaces of the two model matrices are identical.

In Table 6, we present the analysis of covariance based on model (1.1) suggested by MB. We also give expressions for the expected mean squares using the parameterization of (3.1). (The sums of squares given in Table 6 can be obtained using any of the standard statistical packages. Interested readers may obtain from the authors a technical report that describes the computations for both balanced and unbalanced split-plot analyses of covariance, using SAS (1982).) The matrices φ_A, φ_B, φ_{AB}, $\varphi_{A,x}$, $\varphi_{B,x}$ and $\varphi_{AB,x}$, used in Table 6, are as defined in MB. For the balanced case, the matrices Γ_A, Γ_B and Γ_{AB} defined in their paper reduce to n^{-1} times φ_A, φ_B and φ_{AB}, respectively. Let $\sigma_A^2 = \sigma^2 + b\sigma_\delta^2$. In Table 6, the sums of squares with R in boldface are sequential (Type I) sums of squares; the sums of squares with R^* notation are obtained by suitable subtractions and the remaining sums of squares are partial (Type III) sums of squares from models (3.1) and (1.1).

Table 6. Split-plot Analysis of Covariance

Source	d.f.	Sums of Squares	E[Sums of Squares]
Whole-plot (A)	a-1	$R(\alpha\|\mu,\tau_A)$	$\alpha' P_{[\varphi_{A,w}]}\alpha + (a-1)\sigma_A^2$
(β_A)	(1)	$R^*(\beta_A - \tau_A)$	$\alpha'\varphi_A' P_{[Z_A^*]}\varphi_A\alpha + \sigma_A^2$
(A after β_A)	(a-2)	$R(\alpha\|\mu,\beta_A)$	$\alpha'\varphi_A' P_{[\varphi_{A,X}]}\varphi_A\alpha + (a-2)\sigma_A^2$
Covariate (τ_A)	1	$R(\tau_A\|\mu,\alpha)$	$\beta_W^2(Z_2'Z_2) + \sigma_A^2$
Error(a) after τ_A	an-a-1	Error(a)-$R(\tau_A\|\mu,\alpha)$	σ_A^2 (an-a-1)
Split-plot (B)	b-1	$R(\theta\|\mu,\tau_B,\gamma)$	$\theta' P_{[\varphi_{B,u}]}\theta + (b-1)\sigma^2$
(β_B)	(1)	$R^*(\beta_B - \tau_B)$	$\theta'\varphi_B' P_{[Z_B^*]}\varphi_B\theta + \sigma^2$
(B after β_B)	(b-2)	$R(\theta\|\mu,\beta_B)$	$\theta'\varphi_B' P_{[\varphi_{B,X}]}\varphi_B\theta + (b-2)\sigma^2$
Interaction (AB)	ab-a-b+1	$R(\gamma\|\mu,\tau_B,\theta)$	$\gamma' P_{[\varphi_{AB,u}]}\gamma$ + (ab-a-b+1)σ^2
(β_{AB})	(1)	$R^*(\beta_{AB} - \tau_B)$	$\gamma'\varphi_{AB}' P_{[Z_{AB}^*]}\varphi_{AB}\gamma + \sigma^2$
(AB after β_{AB})	(ab-a-b)	$R(\gamma\|\mu,\beta_{AB})$	$\gamma'\varphi_{AB}' P_{[\varphi_{AB,X}]}\varphi_{AB}\gamma$ + (ab-a-b)σ^2
Covariate (τ_B)	1	$R(\tau_B\|\mu,\alpha,\delta,\theta,\gamma)$	$\beta_S^2(Z_3'Z_3) + \sigma^2$
Error(b) after τ_B	a(b-1)(n-1)	Error(b)-$R(\tau_B\|\mu,\alpha,\delta,\theta,\gamma)$	[a(b-1)(n-1)-1]σ^2

The sums of squares presented in Table 6 may be obtained by estimating models (3.1) and (1.1). The sum of squares for whole-plot (A) treatments, error (a) after τ_A, split-plot (B) treatments, whole-plot and split-plot treatment interactions (AB) and covariate (τ_B) at the split-plot level are the partial (Type III) sums of squares one usually computes for model (3.1). The sum of squares for error (b) after the covariate (τ_B) is the residual sums of squares for model (3.1) and the sum of squares for τ_A is the sequential (Type I) sum of squares in model (3.1). The analysis suggested by MB consists of partitioning the whole-plot treatment, split-plot treatment and treatment interaction sums of squares. The sums of squares $R(\alpha|\mu,\beta_A)$, $R(\theta|\mu,\beta_B)$ and $R(\gamma|\mu,\beta_{AB})$ are the partial (Type III) sums of squares for A, B and AB effects in model (1.1), respectively. The one degree of freedom sums of squares $R^*(\beta_A-\tau_A)$, $R^*(\beta_B-\tau_B)$ and $R^*(\beta_{AB}-\tau_B)$ are obtained by taking the differences of corresponding partial (Type III) sums of squares from models (3.1) and (1.1).

The one degree of freedom sum of squares corresponding to the among class regression (β_A) at the whole-plot level is given by

$$R^*(\beta_A - \tau_A) = \mathbf{y}'P_{[Z_A^*]}\mathbf{y}$$

where $\mathbf{Z}_A^* = [I - P_{[w]}]Z_1$, Z_1 is an nabx1 column vector with $Z_{1ijk} = \bar{x}_{.j.} - \bar{x}_{...}$ and $\mathbf{w}$ is an nabx1 vector with $w_{ijk} = \bar{x}_{ij.} - \bar{x}_{...}$. It is easy to see that the noncentrality parameter of the distribution of $R^*(\beta_A-\tau_A)/\sigma_A^2$ is zero if and only if $\Sigma_{j=1}^{a} (\bar{x}_{.j.} - \bar{x}_{...})\alpha_j$ is zero. The remaining (a-2) degrees of freedom whole-plot sums of squares

$$R(\alpha|\mu,\beta_A) = \mathbf{y}'P_{[\theta_{A,X}]}\mathbf{y}$$

may be used to test the hypothesis that the remaining (a-2) whole-plot contrasts involving α_j are simultaneously zero. It can be shown that $R(\alpha|\mu,\beta_A)$ is also nb times the residual sums of squares in the regression of $\bar{y}_{.j.}$ on $\bar{x}_{.j.}$. Also, recall $R(\alpha|\mu,\beta_A)$ is the sequential (Type I) sums of squares for the whole-plot treatments in model (1.1). The "usual" (a-1) degrees of freedom sum of squares (Type III sums of squares in model (3.1))

$$R(\alpha|\mu,\tau_A) = \mathbf{y}'P_{[\varphi_{A,w}]}\mathbf{y}$$

where $\varphi_{A,w} = [I - P_{[w]}]\varphi_A$, is $R^*(\beta_A-\tau_A) + R(\alpha|\mu,\beta_A)$ and it is used to test the hypothesis that there is no whole-plot treatment effect after adjusting for the covariate (at the whole-plot level). (Note that $R^*(\beta_A-\tau_A)$ therefore can be obtained as $R(\alpha|\mu,\tau_A) - R(\alpha|\mu,\beta_A)$.)

Similarly, the one degree of freedom sum of squares corresponding to the among class regression (β_B) at the split-plot level is

$$R^*(\beta_B - \tau_B) = \mathbf{y}'P_{[\mathbf{Z}_B^*]}\mathbf{y}$$

where $\mathbf{Z}_B^* = [I - P_{[\mathbf{u}]}]Z_B$, $\mathbf{Z}_B$ is an nabx1 vector with $Z_{Bijk} = \bar{x}_{..k} - \bar{x}_{...}$ and $\mathbf{u}$ is an nabx1 vector with $u_{ijk} = x_{ijk} - \bar{x}_{ij.}$. It can be shown that the noncentrality parameter of the distribution of $R^*(\beta_B - \tau_B)/\sigma^2$ is zero if and only if $\sum_{k=1}^{b} (\bar{x}_{..k} - \bar{x}_{...})\ \theta_k$ is zero. Also, the one degree of freedom sum of squares for the external regression of the interaction (β_{AB}) is

$$R^*(\beta_{AB} - \tau_B) = \mathbf{y}'P_{[\mathbf{Z}_{AB}^*]}\mathbf{y} \ .$$

where $Z_{AB}^{*} = [I - P_{[u]}]Z_{AB}$ and Z_{AB} is an nabx1 vector with $Z_{ABijk} = \bar{x}_{\cdot jk} - \bar{x}_{\cdot j\cdot} - \bar{x}_{\cdot k\cdot} + \bar{x}_{\cdot\cdot\cdot}$. The noncentrality parameter of the distribution of $R^{*}(\beta_{AB} - \tau_B)/\sigma^2$ is zero if and only if $\sum_{j=1}^{a} \sum_{k=1}^{b}(\bar{x}_{\cdot jk} - \bar{x}_{\cdot j\cdot} - \bar{x}_{\cdot k\cdot} + \bar{x}_{\cdot\cdot\cdot})\ \gamma_{jk}$ is zero. It can be shown that the (Type I = Type III in model (1.1)) sums of squares $R(\theta|\mu,\beta_B)$ and $R(\gamma|\mu,\beta_{AB})$ are na times the residual sum of squares in the regression of $\bar{y}_{\cdot\cdot k}$ on $\bar{x}_{\cdot\cdot k} - \bar{x}_{\cdot\cdot\cdot}$ and n times the residual sum of squares in the regression of $\bar{y}_{\cdot jk}$ on $\bar{x}_{\cdot jk} - \bar{x}_{\cdot j\cdot} - \bar{x}_{\cdot k\cdot} + \bar{x}_{\cdot\cdot\cdot}$, respectively. Also, the (Type III in model (3.1)) sums of squares $R(\theta|\mu,\tau_B,\gamma) = \mathbf{y}'P_{[\varphi_{B,u}]}\mathbf{y}$ and $R(\gamma|\mu,\tau_B,\theta) = \mathbf{y}'P_{[\varphi_{AB,u}]}\mathbf{y}$ where $\varphi_{B,u} = [I - P_{[u]}]\varphi_B$ and $\varphi_{AB,u} = [I - P_{[u]}]\varphi_{AB}$ are $R^{*}(\beta_B - \tau_B) + R(\theta|\mu,\beta_B)$ and $R^{*}(\beta_{AB} - \tau_B) + R(\gamma|\mu,\beta_{AB})$, respectively. They can be used to test the hypotheses H_B: no split-plot treatment effect and H_{AB}: no interaction between whole-plot and split-plot treatments, respectively. (Note therefore that the sums of squares $R^{*}(\beta_B - \tau_B)$ and $R^{*}(\beta_{AB} - \tau_B)$ may be obtained as $R(\theta|\mu,\tau_B,\gamma) - R(\theta|\mu,\beta_B)$ and $R(\gamma|\mu,\tau_B,\theta) - R(\gamma|\mu,\beta_{AB})$, respectively.)

The one degree of freedom sums of squares (Type I in model (3.1)) for the whole-plot (τ_A) and split-plot (τ_B) within class regressions are respectively

$$R(\tau_A|\mu,\alpha) = \mathbf{y}'P_{[Z_2]}\mathbf{y}$$

and

$$R(\tau_B|\mu,\alpha,\delta,\theta,\gamma) = \mathbf{y}'P_{[Z_3]}\mathbf{y}$$

where $\mathbf{Z}_2$ is an nab x 1 vector with $Z_{2ijk} = \bar{x}_{ij\cdot} - \bar{x}_{\cdot j\cdot}$ and $\mathbf{Z}_3$ is an nab x 1 vector with $Z_{3ijk} = (x_{ijk} - \bar{x}_{\cdot jk} - \bar{x}_{ij\cdot} + \bar{x}_{\cdot j\cdot})$. These sums of squares are used to test $\beta_W = 0$ and $\beta_S = 0$ in (3.1), respectively.

Let us now consider the special case $\beta_W = \beta_S$ in model (3.1), i.e., the whole-plot within class regression is the same as the split-plot within class regression. For this special case, the analysis presented in Table 6 is still valid. The appropriate test statistics for testing H_A, H_B and H_{AB} are

$$F_A = R(\alpha|\mu,\tau_A)/(a-1)\hat{\sigma}_A^2$$

$$F_B = R(\theta|\mu,\tau_B,\gamma)/(b-1)\hat{\sigma}^2$$

and

$$F_{AB} = R(\gamma|\mu,\tau_B,\theta)/(a-1)(b-1)\hat{\sigma}^2$$

where $\hat{\sigma}_A^2$ is the whole-plot mean square error (a) after adjusting for the covariate (τ_A) and $\hat{\sigma}^2$ is the split-plot mean square error (b) after adjusting for the covariate (τ_B). When $\beta_W = \beta_S$ the criteria based on the above F-statistics, however, are not optimal. Asymptotically (as $n \to \infty$) optimal tests may be obtained using an estimated generalized least squares estimator suggested by Fuller and Battese (1973). The test criteria based on estimated generalized least squares and the criteria considered by Kirk (1982) and Winer (1971) are only approximate in finite samples. Even though the criteria based on the F-statistics F_A, F_B and F_{AB} are not optimal when $\beta_W = \beta_S$, we recommend these criteria since the exact distributions of the F-statistics suggested by Kirk (1982) and Winer (1971) are not known.

When unequal number (n_j) of units are used for different whole-plot treatments in a completely randomized design, the matrices φ_B and φ_{AB} corresponding to split-plot treatment effects and the interactions between whole-plot and split-plot treatments are no longer orthogonal to each other.

The expectations for the sums of squares in the analysis recommended by MB can be similarly obtained using the matrices Γ_B and Γ_{AB}. Since φ_B and φ_{AB} are not orthogonal to each other, MB consider the partitions $[\Gamma_B\ \varphi_{AB}]$ and $[\varphi_B\ \Gamma_{AB}]$ of the column space $[\varphi_B\ \varphi_{AB}]$ to compute the split-plot treatments and interaction sums of squares, respectively. To define a mean model with different slopes for the covariate, that is consistent with both partitions, they had to assume $\beta_B = \beta_{AB}$. This assumption, however, is not required since model (1.1) is equivalent to model (3.1) with only two slopes and the expectation of the different sums of squares in Table 6 can similarly be computed using the parameterization of model (3.1).

4. CONCLUDING REMARKS

The traditional one way analysis of covariance includes only one regression coefficient - the within class regression. Among class regression, considered by Wishart and Sanders (1934), is not usually included. Some textbooks (e.g, Winer (1971)) present a test for the equality of within and among class regression coefficients. More recently, Monlezun and Blouin (1988) extended the test statistics for the equality of among and within class regressions to the split-plot analysis of covariance. They recommend that the hypothesis of no treatment effects be tested using the treatment sums of squares obtained after adjusting for both among and within class regressions. Smith (1957) indicated that the test for the equality of internal (within class) and external (among class) regressions is actually a test for the regression of covariate means $\bar{x}_{i.}$ on α_i where $\alpha_i = \bar{\mu}_{i.} - \bar{\mu}_{..}$ and $\mu_{ij} = E(y_{ij}) - \beta(x_{ij} - \bar{x}_{..})$. If the postulate, that the covariate is not affected by the treatment, underlying the analysis of covariance is valid, with randomization, there should be no association between $\bar{x}_{i.}$ and $\bar{\mu}_{i.}$. We reiterate the argument of Smith (1957) that the regression of $\bar{x}_{i.}$ on $\bar{\mu}_{i.}$ is expected to be zero *a priori* and hence it is pointless to test the hypothesis that $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\bar{\mu}_{i.} = 0$. Therefore, we do not recommend that the one degree of freedom sums of squares for the contrast $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\bar{\mu}_{i.}$ be routinely separated from the adjusted (for within class regression) treatment sums of squares (as recommended by the analysis of MB).

If values of the covariate are imposed and/or affected by the treatments (that is, if the postulate is not valid) then the contrast $\sum_{i=1}^{a}(\bar{x}_{i.} - \bar{x}_{..})\bar{\mu}_{i.}$ may be of some interest and the analysis suggested by MB may be appropriate. However, in this case, we believe that care must be used in interpreting the adjusted means and a multivariate analysis of variance may be more appropriate.

As pointed out in section 2.3, the different test statistics considered by Smith (1957), Winer (1971) and the statistics included in the analysis suggested by MB are appropriate when the treatments are considered as a random sample from a population of treatments. In this case, the (a-2) degrees of freedom test statistic F_2 tests the hypothesis $\sigma_\alpha^2 = 0$ in model (2.9). Also, assuming $\sigma_\alpha^2 = 0$, the one degree of freedom test statistic in (2.7), comparing the within and among class regressions tests $\beta_1 = \beta_2$. So, if the treatments are random, we recommend that the analysis of covariance presented in Table 2 be used.

For the split-plot analysis of covariance, MB use the model (1.1) with five different slopes. We recommend that the analysis of covariance associated with model (3.1) (which is a reparameterization of model (1.1), with only two different slopes) be used and as in the case of the one way analysis of covariance, the partition of the different one degree of freedom sums of squares from the adjusted treatment sums of squares be used only when they are appropriate. In the case $\beta_W = \beta_S$, the F-statistics based on model (3.1) are not optimal. The optimal tests, in this case, depend on the unknown variance components and one may obtain asymptotically optimal tests using an estimated generalized least squares procedure. However, we recommend that the analysis based on (3.1), without the assumption $\beta_W = \beta_S$, be used since the test statistics are simple and the exact distributions of the test statistics are known.

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