LEAST SQUARES ESTIMATION FOR A MULTIVARIATE WEIBULL MODEL OF HOUGAARD BASED ON ACCELERATED LIFE TEST OF COMPONENT AND SYSTEM

Jye-Chyi Lu

Department of Statistics
North Carolina State University
Raleigh, NC 27695

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ABSTRACT

Accelerated life testing of products and material under severe conditions quickly yields information on life. In this article, we present a simple method to incorporate the information collected from the accelerated life test on component and (series) system levels. The underlying distribution of the lifetimes of the components is assumed to be a multivariate Weibull due to Hougaard. The least squares estimators of the model parameters are proposed along with the derivation of their asymptotic distribution.

1. INTRODUCTION

Accelerated Life Test (ALT) is commonly employed when product or material reliability is high and testing under normal use condition would make test time prohibitively long. With ALT, test units are subjected to stress conditions that are more severe than those encountered in normal use so that more failures are apt to take place in a limited time. Data of failure times under such over-stress conditions are drawn with regard to life length or reliability of the product under its normal use condition. Nelson (1974a) provided a bibliography of applications. Bhattacharyya (1986) reviewed the principal methodological approaches to ALT analysis in regard to plausibility of the model, flexibility of empirical fit and usefulness in practical application.

Before a system is developed, ALT on component levels are usually conducted (see examples in Mann, Schafer and Singpurwalla, 1974). When the components are assembled into a system, life-test data may also be obtained from system testing. Nelson (1973, 1974b), Klein and Basu (1981a, 1981b, 1982), analyzed the ALT data from series system testing. When life test data are available for a system as well as the components, it is desirable to utilize all available data to improve the component reliability estimation and system design, particularly in situations where the available data are limited. With regular life test on normal use condition, Miyamura (1982) analyzed life-test results of the electromagnetic valve and of the air conditioner, which is a series system of the electromagnetic valve and other components. Easterling and Prairie (1971) gave an example on testing of parallel system: for a certain thermal battery with two bridgewires, life test data may be obtained at the component level (bridgewires) as well as the system level (battery). With the assumption of the exponential lifetimes, Easterling and Prairie and Miyamura used the method of maximum likelihood, and Mastran (1975) presented a Bayesian procedure to estimate the parameters of the distribution of component lifetimes. In the context of ALT, the estimation method based on component and system data have not been adequately developed.

In all these aforementioned studies, the component lifetimes were assumed to be *independent* for the sake of simplicity of mathematical treatment. In a two-component system, the failure rate of one component might be increased upon the failure of the other component. Common cause failure or similar environmental factor (stress) might lead to the dependence of the components. Modeling the lifetimes of the components as Gumbel's (1960) bivariate exponential BVE, Lu and Bhattacharyya (1988b) developed several exact inference procedures based on the data obtained from the regular life test.

In this article, we consider a system of m identical components whose lifetimes may be dependent due to the effect of common environmental factors. We develop estimation methods for the life length of the system based on ALT data from series system and component testing. This problem is also motivated from the following consideration: sometimes, due to the limitation of time or cost, the experimental stress were set far from the normal use condition. Consequently, an extrapolation to the use condition may be quite unreliable. To narrow the gap between experimental and use condition or to check the model for extrapolation, one would like to observe some data at stress closer to the use-condition. Since the failure time of a series system is the minimum of the failure times of its components, testing of a series system might be conducted under lower stress. Hence, besides analyzing the result from component testing, utilizing the information from life test of a series system allows us to check acceleration model in lower stress.

In Section 2 we introduce the set-up of the experiment. We apply a flexible and physically motivated distribution to the lifetimes of both system and components (under each stress). Between different stress levels, a stress acceleration function is formulated. In view of the computational complexity and the lack of closed-form solution for maximum likelihood (ML) estimation, we propose a set of two-stage least squares (LS) estimators in Section 3. Section 4 is devoted to the derivation of the asymptotic distribution of the LS estimator. We consider both the cases of the number of replications tends to infinity and the number of stress settings tends to infinity.

2. LIFE DISTRIBUTION OF COMPONENTS AND SYSTEM, AND STRESS ACCELERATION FUNCTION

Let us denote the lifetimes of m components in a system as $Z_1, \ldots, Z_m$ and suppose that they have the same distribution individually, regardless if the components are assembled into a system. We apply the Hougaard (1986) multivariate Weibull distribution MVW to model lifetimes of the components. The survival function (SF) of the MVW is of the form

$$\bar{F}(z_1, z_2, \ldots, z_m) = \exp\left[-\sum_{k=1}^{m} (z_k/\theta_k)^{\beta_k/\delta}\right]^{\delta},$$

$$\delta \in (0, 1],\ \beta_i, \theta_i > 0,\ i = 1, \cdots, k,\ z_1, \cdots, z_m \geq 0. \tag{2.1}$$

This distribution enjoys several important properties such as a physical motivation (cf. Hougaard, 1986), existence of absolutely continuous probability density function, Weibull marginals and minimum for equal shape parameters case (or stability relation phased in Tawn, 1988). Its bivariate case can be obtained by using a power transformation of Gumbel's BVE which has been studied extensively by Lu and Bhattacharyya (1988a, b) for paired and system-component data collected from regular life testing procedures.

We consider the following set-up of the accelerated life test. On the component level, we suppose that an experiment is conducted under k_1 stress levels $\underset{\sim}{x}_i$, $i = 1, \dots, k_1$. At stress $\underset{\sim}{x}_i$, n_i components are put on test simultaneously and the experiment stops as soon as the first r_i failures are observed (type II censored sample). The lifetimes of these n_i components are modeled as independent random variables (rv) $Z_{i1}, \dots, Z_{in_i}$. The type II censored sample refers to a specified subset of the order statistics $Z_{i(1)} < \cdots < Z_{i(r_i)}$ of $Z_{i1}, \dots, Z_{in_i}$. For system testing, we suppose that an experiment is conducted under $k_2 = k - k_1$ stress levels $\underset{\sim}{x}_i$, $i = k_1 + 1, \dots, k$. At stress $\underset{\sim}{x}_i$, we observe the type II censored sample $Z_{i(1)s} < \dots < Z_{i(r_i)s}$, which is a subset of the independently identical rv's $Z_{i1s}, \dots, Z_{in_is}$, where Z_{ijs}, $j = 1, \dots, n_i$ are the lifetimes of series systems.

Under a stress $\underset{\sim}{x}_i$, we assume that the random variables Z_{ij}, $j = 1, \cdots, n_i$ are independently identically distributed (iid) as Weibull distribution with the scale and shape parameters θ and β, respectively. Since the series system consists of m identical components which have the MVW (2.1) as joint distribution, the system life Z_{ijs} then has the Weibull distribution with the scale and shape parameters $m^{-\delta/\beta}\theta$ and β, respectively.

For the stress acceleration function between different stresses, we assume that the parameter θ depends on a p-vector stress according to a log-linear relation $\log \theta_i = \underset{\sim}{x}_i' \underset{\sim}{\alpha}$, where

$$\underset{\sim}{x}_i = (x_{i1}, \cdots, x_{ip})', \quad \underset{\sim}{\alpha} = (\alpha_1, \cdots, \alpha_p)', \quad i = 1, \cdots, k,$$

while the dependence (δ) and shape (β) parameters are independent of stress. The assumption of a log-linear relation to stress is not only simple and flexible but is also motivated in many practical contexts. The Arrhenius reaction rate model, Inverse power law and Eyring model are some of the widely used engineering models which fit into the log-linear relation.

3. LEAST SQUARES ESTIMATION

The method of maximum likelihood estimation involves considerable computational complexity and, lacks a closed-form solution. An analytical treatment of exact properties of the ML estimators does not appear to be feasible. In the context of ALT, for the analysis of results from component testing, some interesting procedures have been developed for the life distributions in location-scale family. A simple estimation procedure with type II censored data, proposed by Nelson and Hahn (1972, 1973), is based on an application of the least squares method in two stages. This method leads to unbiased estimators of the mean and any percentile log-life as well as their exact variances as opposed to only asymptotic results obtainable for the MLE's.

For a given stress of component testing, the log-life Y ($= \log Z$) is written as

$$Y = \log \theta + \eta\ W,$$

where W has the standard extreme-value distribution with probability density function (pdf)

$$\exp\left[W - \exp(W)\right], \quad -\infty < W < \infty.$$

Similarly, for a given stress of system testing, the log-life $T(= \log Z_S)$ is written as

$$T = \log \theta - \eta\ \delta \log m + \eta\ W.$$

Since these are linear regression models, least squares estimation based on order statistics can be used. To combine these two linear models together, we define

$$\gamma = \delta\eta, \quad \lambda = \log \theta, \qquad \zeta_i = \begin{cases} 0 & \text{for} \quad i = 1, \ldots, k_1, \\ -\log m & \text{for} \quad i = k_1 + 1, \ldots, k. \end{cases}$$

$$\alpha_j = E(W_{i(j)}),$$

Note that α_j's and ζ_i's are known constants while γ, λ are unknown parameters, λ depends on $\underset{\sim}{x}$ by linear relation and γ does not depend on $\underset{\sim}{x}$. The observed log-life from component and system testing can be presented by a general linear model of the form

$$y_{i(j)} = \lambda + \eta\alpha_j + \gamma\zeta_i + e_{ij}, \quad j = 1, \ldots, r_i, \quad i = 1, \ldots, k, \tag{3.1}$$

where e_{ij}'s have mean 0 and covariance matrix

$$\eta^2(\sigma_{jj'}) = \eta^2\, Cov\left[W_{i(j)}, W_{i(j')}\right].$$

The means and covariances of $W_{i(j)}$ are known constants (tables available, e.g., White, 1964).

The linear unbiased least squares estimation is obtained in two stages. In the first stage, we ignore the regression structure and estimate the parameters

$(\lambda_i, \eta_i, \gamma_i)$ from the *ith* data set, $y_{i(1)} \leq \cdots \leq y_{i(r_i)}$ through the least square estimation method. However, for $i = k_1 + 1, \cdots, k$ (system testing) the estimable parameters are $\tau_i = \lambda_i + \gamma\zeta_i$, and η due to the deficiency in rank. Hence, we should rewrite the linear model (3.1) as

$$y_{i(j)} = \tau_i + \eta\alpha_j + e_{ij}, \quad j = 1, \cdots, r_i, \quad i = 1, \cdots, k.$$

Let us denote

$$\Delta_{1i} = \sum_{j=1}^{r_i} \sum_{j'=1}^{r_i} \sigma^{jj'}, \qquad \Delta_{2i} = \sum_{j=1}^{r_i} \sum_{j'=1}^{r_i} \alpha_j \sigma^{jj'},$$

$$\Delta_{3i} = \sum_{j=1}^{r_i} \sum_{j'=1}^{r_i} \alpha_j \alpha_{j'} \sigma^{jj'}, \qquad \Delta_i = \Delta_{1i}\Delta_{3i} - \Delta_{2i}^2,$$

$$a_{ij} = \frac{\sum_{j'=1}^{r_i} (\Delta_{3i} - \Delta_{2i}\alpha_{j'})\sigma^{jj'}}{\Delta_i}, \qquad b_{ij} = \frac{\sum_{j'=1}^{r_i} (\Delta_{1i}\alpha_{j'} - \Delta_{2i})\sigma^{jj'}}{\Delta_i},$$

where $\sigma^{jj'}$ is the (j, j') element of the inverse of the covariance matrix $(\sigma_{jj'})$. We thus have the stage-1 ordered linear unbiased estimators (O-BLUE) (cf. Lloyd, 1952) of the form

$$\tau_i^* = \sum_{j=1}^{r_i} a_{ij} y_{i(j)}, \quad \eta_i^* = \sum_{j=1}^{r_i} b_{ij}\, y_{i(j)}, \; i = 1, \cdots, k, \tag{3.2}$$

as well as their exact covariance matrix

$$\eta^2 \begin{bmatrix} d_{1i} & d_{2i} \\ d_{2i} & d_{3i} \end{bmatrix}, \tag{3.3}$$

where

$$d_{1i} = \frac{\Delta_{3i}}{\Delta_i}, \quad d_{3i} = \frac{\Delta_{1i}}{\Delta_i}, \quad d_{2i} = \frac{-\Delta_{2i}}{\Delta_i}.$$

Hence, from the known values of α_j and $\sigma_{jj'}$, the coefficients a_{ij} and b_{ij} as well as d_{1i}, d_{2i} and d_{3i} can be evaluated (e.g., White, 1964).

Remark: In system testing if $\delta = 1$, *i.e.*, independence case, we can simplify τ_i^* to:

$$\tau_i^* = \lambda_i^* + \eta_i^* (-\log m) = \sum_{j=1}^{r_i} a_{ij}\, y_{i(j)}.$$

Then we can obtain the LS estimators of λ_i^* as follows:

$$\lambda_i = \sum_{j=1}^{r_i} \left[a_{ij} + b_{ij} \log m\right] y_{i(j)} = \sum_{j=1}^{r_i} c_{ij}\, y_{i(j)},\ i = k_1 + 1, \cdots, k,$$

where

$$c_{ij} = \frac{1}{\Delta_i} \sum_{j'=1}^{r_i} \left[(\Delta_{3i} - \Delta_{2i} \log m) + \alpha_{j'} (\Delta_{1i} - \Delta_{2i} \log m)\right] \sigma^{jj'}.$$

In the second stage, we take account of the regression structure $\lambda = \log \theta = \mathrm{X}\, \underset{\sim}{\alpha}$. Let us define

$$\underset{\sim}{\lambda}^* = (\lambda_1^*, \cdots, \lambda_k^*)', \quad \underset{\sim}{\eta}^* = (\eta_1^*, \cdots, \eta_k^*)',$$

$$\underset{\sim}{\tau} = (\tau_1, \cdots, \tau_k)', \quad \underset{\sim}{\tau}^* = (\tau_1^*, \cdots, \tau_k^*)',$$

$$\underset{\sim}{e}_1 = (e_{11}, \cdots, e_{1k})', \quad \underset{\sim}{e}_2 = (e_{21}, \cdots, e_{2k})'$$

$$\underset{\sim}{\zeta} = (\zeta_1, \cdots, \zeta_{k_1}, \zeta_{k_1+1}, \cdots, \zeta_k)' = (0, \cdots, 0, -\log m, \cdots, -\log m)',$$

$$\mathrm{X} = (\underset{\sim}{X}_1, \cdots, \underset{\sim}{X}_k)', \quad \underset{\sim}{\alpha} = (\alpha_1, \cdots, \alpha_p)',$$

$$\mathrm{D}_j = \mathrm{diag}\,(d_{j1}, \cdots, d_{jk}), \quad j = 1, 2, 3.$$

We recall that $\tau_i = \lambda_i + \gamma\zeta_i = \underset{\sim}{x}'\, \underset{\sim}{\alpha} + \zeta_i\, \gamma$. Using the estimators obtained in the first stage, we form linear models of $\underset{\sim}{\tau}^*$ and $\underset{\sim}{\eta}^*$ separately as

$$\underset{\sim}{\tau}^* = \mathrm{X}^*\, \underset{\sim}{\alpha}^* + \underset{\sim}{e}_1, \quad \underset{\sim}{\eta}^* = \underset{\sim}{1}\, \eta + \underset{\sim}{e}_2, \tag{3.4}$$

where $\mathrm{X}^* = (\mathrm{X}, \underset{\sim}{\zeta})$, $\underset{\sim}{\alpha}^* = (\underset{\sim}{\alpha}', \gamma)'$, and their pair $(\underset{\sim}{e}_1, \underset{\sim}{e}_2)$ has mean $(\underset{\sim}{0}, \underset{\sim}{0})$, is independent across rows and has the covariance structure (3.3) across columns. Based on these linear models, the weighted least squares unbiased estimator (WLUE) are obtained as

$$\tilde{\underset{\sim}{\alpha}}^* = (\mathrm{X}^{*\prime}\, \mathrm{D}_1^{-1}\, \mathrm{X}^*)^{-1} \mathrm{X}^{*\prime}\, \mathrm{D}_1^{-1}\, \underset{\sim}{\tau}^*,$$

$$\tilde{\eta} = (\underset{\sim}{1}'\, \mathrm{D}_3^{-1}\, \underset{\sim}{1})^{-1}\, \underset{\sim}{1}'\, \mathrm{D}_3^{-1}\, \underset{\sim}{\eta}^* = \sum_{i=1}^{k} d_{3i}^{-1}\, \eta_i^* \Big/ \sum_{i=1}^{k} d_{3i}^{-1}.$$

Since X^* and $\underset{\sim}{\alpha}^*$ matrices can be decomposed into two parts, we further separate the WLUE's, $\tilde{\underset{\sim}{\alpha}}$ and $\tilde{\gamma}$, as follows:

$$\begin{bmatrix} \tilde{\underset{\sim}{\alpha}} \\ \tilde{\gamma} \end{bmatrix} = \begin{bmatrix} \mathrm{X}'\, \mathrm{D}_1^{-1}\, \mathrm{X} & \mathrm{X}'\, \mathrm{D}_1^{-1}\, \underset{\sim}{\zeta} \\ \underset{\sim}{\zeta}'\, \mathrm{D}_1^{-1}\, \mathrm{X} & \underset{\sim}{\zeta}'\, \mathrm{D}_1^{-1}\, \underset{\sim}{\zeta} \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{X}'\, \mathrm{D}_1^{-1}\, \underset{\sim}{\tau}^* \\ \underset{\sim}{\zeta}'\, \mathrm{D}_1^{-1}\, \underset{\sim}{\tau}^* \end{bmatrix}. \tag{3.5}$$

Let

$$A_1 = X' D_1^{-1} X, \quad A_2 = \underset{\sim}{\zeta}' D_1^{-1} \underset{\sim}{\zeta}, \quad A_3 = X' D_1^{-1} \underset{\sim}{\zeta},$$

$$P = (X' D_1^{-1} X)^{-1} X' D_1^{-1}, \quad Q = I - X P, \quad \sigma_* = \underset{\sim}{\zeta}' D_1^{-1} Q \underset{\sim}{\zeta} .$$

Inverting the matrix in (3.5), we get

$$\begin{bmatrix} \tilde{\underset{\sim}{\alpha}} \\ \tilde{\gamma} \end{bmatrix} = \begin{bmatrix} A_1^{-1} + A_1^{-1} A_3 \sigma_*^{-1} A_3' A_1^{-1} & - A_1^{-1} A_3 \sigma_*^{-1} \\ - \sigma_*^{-1} A_3' A_1^{-1} & \sigma_*^{-1} \end{bmatrix} \begin{bmatrix} X' D_1^{-1} \underset{\sim}{\tau}^* \\ \underset{\sim}{\zeta}' D_1^{-1} \underset{\sim}{\tau}^* \end{bmatrix} .$$

These yield the WLUE of $\underset{\sim}{\alpha}$ and γ as

$$\tilde{\underset{\sim}{\alpha}} = P (I - \underset{\sim}{\zeta} \sigma_*^{-1} \underset{\sim}{\zeta}' D_1^{-1} Q) \underset{\sim}{\tau}^*, \quad \tilde{\gamma} = \sigma_*^{-1} \underset{\sim}{\zeta}' D_1^{-1} Q \underset{\sim}{\tau}^*, \tag{3.6}$$

and their variances and covariances are given by

$$Cov(\tilde{\underset{\sim}{\alpha}}) = \eta^2 \left[X' D_1^{-1} X - X' D_1^{-1} \underset{\sim}{\zeta} (\underset{\sim}{\zeta}' D_1^{-1} \underset{\sim}{\zeta})^{-1} \underset{\sim}{\zeta}' D_1^{-1} X\right]^{-1},$$

$$Var(\tilde{\gamma}) = \eta^2 \left[\underset{\sim}{\zeta}' D_1^{-1} \underset{\sim}{\zeta} - \underset{\sim}{\zeta}' D_1^{-1} X (X' D_1^{-1} X)^{-1} X' D_1^{-1} \underset{\sim}{\zeta}\right]^{-1},$$

$$Cov(\tilde{\underset{\sim}{\alpha}}, \tilde{\gamma}) = - (X' D_1^{-1} X)^{-1} X' D_1^{-1} \underset{\sim}{\zeta} \; Var(\tilde{\gamma}).$$

Based on these results, the simple estimators of shape and dependence parameters can be constructed as

$$\tilde{\beta} = 1/\tilde{\eta}, \quad \tilde{\delta} = \tilde{\gamma}/\tilde{\eta}.$$

Remark: Instead of formulating the linear models (3.4) of $\underset{\sim}{\tau}^*$ and $\underset{\sim}{\eta}^*$ separately, we can put them together into a single linear model and then apply the least squares method to obtain the best linear unbiased estimators (BLUE) in one stage (cf. Lloyd, 1952; Nelson and Hahn, 1973). However, this procedure involves inverses of larger matrices than the ones in two-stage least squares approach. Moreover, the WLUE's are widely advocated in engineering; they are highly efficient with respect to the BLUE's; they also provide information for checking the correctness of the model (cf. Escobar, 1986).

Next, we discuss some special cases of $\tilde{\underset{\sim}{\alpha}}$ and $\tilde{\gamma}$. To simplify the notations, we will drop all the subscripts of summations. For instance,

$$\sum d^{-1}\, x\, \tau^{*} = \sum_{j=1}^{k} d_{Ij}^{-1}\, x_j\, \tau_j^{*} \quad \text{and} \quad \sum\nolimits_s d_I^{-1}\, x^2 = \sum_{j=k_1+1}^{k} d_{Ij}^{-1}\, x_j^2 .$$

Example 1: In a simple linear regression case, with $p = 2$ and

$$\underset{\sim}{\alpha} = (\alpha_0, \alpha_1)', \quad X' = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_k \end{bmatrix},$$

the weighted WLUE's $\underset{\sim}{\tilde{\alpha}}$ and $\tilde{\gamma}$ can be simplified as follows:

$$\underset{\sim}{\tilde{\alpha}} = \begin{bmatrix} \tilde{\alpha}_0 \\ \tilde{\alpha}_1 \end{bmatrix} = \mathrm{P}\, \underset{\sim}{\tau}^{*} - \mathrm{P}\, \underset{\sim}{\zeta}\, \sigma_*^{-1}\, \underset{\sim}{\zeta}'\, \mathrm{D}_I^{-1}\, \mathrm{Q}\, \underset{\sim}{\tau}^{*}$$

$$\tilde{\alpha}_0 = \Delta^{-1}\Big\{(\sum d_I^{-1}\, \tau^{*})\big[\sum d_I^{-1}\, x(x+1)\big] - (\sum d_I^{-1}\, x\, \tau^{*}) \times \big[\sum d_I^{-1}\,(x+1)\big]\Big\},$$

$$\tilde{\alpha}_1 = (\log m)^2\, \Delta^{-2}\, \sigma_*^{-1}\Big\{(b_1 - b_2)\,\big[\Delta \sum\nolimits_s \tau^{*} - \sum a\, \tau^{*}\big]\Big\},$$

$$\tilde{\gamma} = \frac{-\log m}{\Delta\, \sigma_*}\,(\Delta\, \Sigma_s\, \tau^{*} - \Sigma\, a\, \tau^{*}),$$

where

$$\Delta = (\sum d_I^{-1})(\sum d_I^{-1}\, x^2) - (\sum d_I^{-1}\, x)^2,$$

$$\sigma_* = (\log m)^2\, \Delta^{-1}\Big\{\sum\nolimits_s d_I^{-1} - (\sum\nolimits_s d_I^{-1})^2\,(\sum d_I^{-1}\, x^2) + 2(\sum\nolimits_s d_I^{-1})(\sum\nolimits_s d_I^{-1}\, x)(\sum d_I^{-1}\, x) - (\sum d_I^{-1})(\sum\nolimits_s d_I^{-1}\, x^2)^2\Big\},$$

$$a_i = d_{Ii}^{-1}\Big\{(\sum d_I^{-1}\, x^2)(\sum\nolimits_s d_I^{-1}) + x_i(\sum d_I^{-1})(\sum\nolimits_s d_I^{-1}\, x) - (x_i \sum\nolimits_s d_I^{-1} + \sum\nolimits_s d_I^{-1}\, x)\Big\},$$

$$b_1 = (\sum_s d_1^{-1})(\sum d_1^{-1} x^2) - (\sum d_1^{-1} x)(\sum_s d_1^{-1} x),$$

$$b_2 = (\sum d_1^{-1})(\sum_s d_1^{-1} x) - (\sum_s d_1^{-1})(\sum d_1^{-1} x).$$

Example 2: Following Example 1, if the replications and censorings are equal in system or component testing, that is $n_i = n_c$, $r_i = r_c$ for $i = 1, \ldots, k$, and $n_i = n_s$, $r_i = r_s$ for $i = k_1 + 1, \ldots, k$, we have $d_{ij} = d_{1c}$ for $i = 1, \ldots, k_1$, and $d_{ij} = d_{1s}$ for $i = k_1 + 1, \ldots k$. Hence, Δ, a_i and b_j, $j = 1, 2$ in Example 1 can be furthur simplified as follows:

$$\Delta = (k_1 d_{1c}^{-1} + k_2 d_{1s}^{-1})(d_{1c}^{-1} \sum_c x^2 + d_{1s}^{-1} \sum_s x^2) - (d_{1c}^{-1} \sum_c x + d_{1s}^{-1} \sum_s x)^2,$$

$$a_{it} = d_{1t}^{-1}(d_{1c}^{-1} \sum_c x^2 + d_{1s}^{-1} \sum_s x^2)(k_2 d_{1s}^{-1}) + d_{1t}^{-1}(x_i d_{1s}^{-1} \sum_s x) \times (k_1 d_{1c}^{-1} + k_2 d_{1s}^{-1}) - d_{1t}^{-1}\left[d_{1s}^{-1}(k_2 x_i + \sum_s x)\right],$$

where $t = c$ for $i = 1, \ldots, k_1$, $t = s$ for $i = k_1 + 1, \ldots, k$. And,

$$b_1 = (k_2 d_{1s}^{-1} \sum_c x^2 + d_{1s}^{-1} \sum_s x^2) - (d_{1s}^{-1} \sum_s x)(d_{1c}^{-1} \sum_c x + d_{1s}^{-1} \sum_s x),$$

$$b_2 = (k_1 d_{1c}^{-1} + k_2 d_{1s}^{-1})(d_{1s}^{-1} \sum_s x) - (k_2 d_{1s}^{-1})(d_{1c}^{-1} \sum_c x + d_{1s}^{-1} \sum_s x).$$

Example 3: If the replications and censorings are all equal at all stress levels, i.e., $n_i = n$, $r_i = r$, then $d_{1i} = d_1$ for $i = 1, \cdots, k$ and the weighted WLUE's become unweighted WLUE's. That is,

$$\tilde{\underset{\sim}{\alpha}}^* = (\mathrm{X}^{*\prime}\,\mathrm{X}^*)^{-1}\,\mathrm{X}^{*\prime}\,\underset{\sim}{\tau}^*, \quad \tilde{\eta} = \frac{1}{k}\sum_{i=1}^{k} \eta_i^*,$$

we can decompose the X^* and $\underset{\sim}{\alpha}^*$ matrices to get the linear unbiased estimators of $\underset{\sim}{\alpha}$ and γ as

$$\tilde{\underset{\sim}{\alpha}} = \mathrm{P}\,(\mathrm{I} - \underset{\sim}{\zeta}\,\sigma_*^{-1}\,\underset{\sim}{\zeta}'\,\mathrm{Q})\,\underset{\sim}{\tau}^*, \quad \tilde{\gamma} = \sigma_*^{-1}\,\underset{\sim}{\zeta}'\,\mathrm{Q}\,\underset{\sim}{\tau}^*,$$

with variances and covariances

$$Cov(\tilde{\underset{\sim}{\alpha}}) = \eta^2\left[\mathrm{X}'\,\mathrm{X} - \mathrm{X}'\,\underset{\sim}{\zeta}\,(\underset{\sim}{\zeta}'\,\underset{\sim}{\zeta})^{-1}\,\underset{\sim}{\zeta}'\,\mathrm{X}\right]^{-1},$$

$$Var(\tilde{\gamma}) = \eta^2\left[\underset{\sim}{\zeta}'\,\underset{\sim}{\zeta} - \underset{\sim}{\zeta}'\,\mathrm{X}\,(\mathrm{X}'\,\mathrm{X})^{-1}\,\mathrm{X}'\,\underset{\sim}{\zeta}\right]^{-1},$$

$$Cov(\tilde{\underset{\sim}{\alpha}}, \tilde{\gamma}) = -\,(\mathrm{X}'\,\mathrm{X})^{-1}\,\mathrm{X}'\,\underset{\sim}{\zeta}\;Var(\tilde{\gamma}),$$

where

$$P = (X' X)^{-1} X', \quad Q = I - X P, \quad \sigma_* = \underset{\sim}{\zeta}' Q \underset{\sim}{\zeta}.$$

In linear regression case, we have

$$\underset{\sim}{\tilde{\alpha}}_0 = \Delta^{-1}\Big\{(\sum x^2)(\sum \tau^*) + (\sum x)(\sum \tau^*) - (\sum x)(\sum x\, \tau^*) - k \sum x\, \tau^*\Big\},$$

$$\underset{\sim}{\tilde{\alpha}}_1 = (\log m)^2\, \Delta^{-2}\, \sigma_*^{-1}\Big\{(b_1 - b_2)\,(\Delta \textstyle\sum_s \tau^* - \sum a\, \tau^*)\Big\},$$

$$\tilde{\gamma} = -(\log m)\, \Delta^{-1}\, \sigma_*^{-1}\, (\Delta \textstyle\sum_s \tau^* - \sum a\, \tau^*),$$

$$\Delta = k \sum x^2 - (\sum x)^2, \quad a_i = (k_2 \sum x^2) + k\, x_i \textstyle\sum_s x - (k_2\, x_i + \sum_s x),$$

$$b_1 = k_2 \sum x^2 - (\textstyle\sum_s x)(\sum x), \quad b_2 = k \textstyle\sum_c x - k_2 \sum x,$$

$$\sigma_* = (\log m)^2\, \Delta^{-1}\Big\{k_2 - k_2^{\,2}\,(\sum x^2) + 2\, k_2 (\textstyle\sum_s x)(\sum x) - k\, (\sum_s x^2)^2\Big\}.$$

4. ASYMPTOTIC DISTRIBUTION OF THE WEIGHTED LEAST SQUARES UNBIASED ESTIMATOR

Since the simple estimator $(\tilde{\theta}_u, \tilde{\beta}, \tilde{\delta})$ is a function of the WLUE $\underset{\sim}{\tilde{\xi}} = (\underset{\sim}{\tilde{\alpha}}, \tilde{\eta}, \tilde{\gamma})'$, we only need to derive the asymptotic distribution of $\underset{\sim}{\tilde{\xi}}$. In the study of the asymptotic distribution of WLUE, we consider two cases: (i) the number of replications $N = \sum_{i=1}^{k} n_i$ tends to infinity and, (ii) the number of stress settings k tends to infinity.

In the first case, the number of stress settings k is fixed. We assume that

$$n_i/N \to \pi_i, \quad i = 1, \cdots, k \text{ as } N \to \infty.$$

Since $\underset{\sim}{\tilde{\xi}}$ is a function of $\underset{\sim}{\tau}^*$ and $\underset{\sim}{\eta}^*$, we first obtain the asymptotic distribution of $(\underset{\sim}{\tau}^*, \underset{\sim}{\eta}^*)$. In stage-1, the O-BLUE τ_i^* and η_i^* are asymptotically efficient estimators of τ_i and η_i, respectively (cf. Bennett, 1952; Chernoff *et al.*, 1967). Hence, the joint asymptotic distribution of τ_i^* and η_i^* is as follows:

$$\sqrt{n_i}\,\big[(\tau_i^* - \tau_i), (\eta_i^* - \eta_i)\big] \xrightarrow{d} N_2\,(\underset{\sim}{0}, \Sigma_0), \quad i = 1, \cdots, k,$$

where Σ_0 is the Cramer-Rao lower bound of the form

$$\Sigma_0 = \eta^2 \begin{bmatrix} c_{1i} & -c_{2i} \\ -c_{2i} & c_{3i} \end{bmatrix}, \tag{4.1}$$

and c_{ji}, $j = 1, 2, 3$ are tabulated in Bain (1978). In view of the independence of the observations in different stress levels, we obtain the asymptotic distribution of $\underset{\sim}{\tau}^*$ and $\underset{\sim}{\eta}^*$ as follows

$$\sqrt{N}\left[(\underset{\sim}{\tau}^* - \underset{\sim}{\tau}), (\underset{\sim}{\eta}^* - \underset{\sim}{\eta})\right] \xrightarrow{d} N_{2k}(\underset{\sim}{0}, \Sigma),$$

where

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_3 \end{bmatrix}, \quad \Sigma_j = (-1)^{j+1} \times \eta^2 \times \mathrm{diag}(c_{ji}/\pi_1, \cdots, c_{jk}/\pi_k).$$

We introduce the following notations to relate the WLUE $\underset{\sim}{\tilde{\xi}}$ to $(\underset{\sim}{e}_1, \underset{\sim}{e}_2)$.

$$\mathrm{C}_j = \mathrm{diag}(c_{ji}, \cdots, c_{jk}), \quad \mathrm{P}_c = (\mathrm{X}'\,\mathrm{C}_1^{-1}\,\mathrm{X})^{-1}\,\mathrm{X}'\,\mathrm{C}_1^{-1},$$

$$\mathrm{Q}_c = \mathrm{I} - \mathrm{X}\,\mathrm{P}_c, \quad \sigma_c = \underset{\sim}{\zeta}'\,\mathrm{C}_1^{-1}\,\mathrm{Q}_c\,\underset{\sim}{\zeta},$$

To simplify the expressions, let us denote

$$\mathrm{B}_1 = \mathrm{P}_c\left[\mathrm{I} - \underset{\sim}{\zeta}\,\sigma_c\,\underset{\sim}{\zeta}'\,\mathrm{C}_1^{-1}\,\mathrm{Q}_c\right],$$

$$\mathrm{B}_2 = \sigma_c^{-1}\,\underset{\sim}{\zeta}'\,\mathrm{C}_1^{-1}\,\mathrm{Q}_c, \quad \mathrm{B}_3 = (\underset{\sim}{1}'\,\mathrm{C}_2^{-1}\,\underset{\sim}{1})^{-1}\,\mathrm{C}_2^{-1}.$$

Since the elements of the covariance matrix (3.3) of O-BLUE (τ_i^*, η_i^*) converge to the corresponding elements of the covariance matrix (4.1), we have $\mathrm{D}_j^{-1} \rightarrow \mathrm{C}_j^{-1}$, $j = 1, 2, 3$. Then, we have the following asymptotics:

$$\mathrm{P}\left[\mathrm{I} - \underset{\sim}{\zeta}\,\sigma_*\,\underset{\sim}{\zeta}'\,\mathrm{D}_1^{-1}\,\mathrm{Q}\right] \rightarrow \mathrm{B}_1, \quad \sigma_*^{-1}\,\underset{\sim}{\zeta}'\,\mathrm{D}_1^{-1}\,\mathrm{Q} \rightarrow \mathrm{B}_2,$$

$$(\underset{\sim}{1}'\,\mathrm{D}_2^{-1}\,\underset{\sim}{1})^{-1}\,\underset{\sim}{1}'\,\mathrm{D}_2^{-1} \rightarrow \mathrm{B}_3, \text{ as } N \rightarrow \infty.$$

Using the equality $\mathrm{P}(\mathrm{I} - \underset{\sim}{\zeta}\,\sigma_*^{-1}\,\underset{\sim}{\zeta}'\,\mathrm{D}_2^{-1}\,\mathrm{Q})\,\mathrm{X}\,\underset{\sim}{\alpha} = \underset{\sim}{\alpha}$, we can replace $\underset{\sim}{\tau}^*$ by $\mathrm{X}^*\,\underset{\sim}{\alpha}^* + \underset{\sim}{e}_1$ in the expression of $\underset{\sim}{\tilde{\alpha}}$ (3.6) to conclude that

$$\sqrt{N}\,(\underset{\sim}{\tilde{\alpha}} - \underset{\sim}{\alpha}) - \mathrm{B}_1'\left(\sqrt{N}\,\underset{\sim}{e}_1\right) = o_p(1).$$

Similarly, we have $\sqrt{N}\,(\tilde{\gamma} - \gamma) - \mathrm{B}_2'\left(\sqrt{N}\,\underset{\sim}{e}_1\right) = o_p(1)$ and

$$\sqrt{N}\,(\tilde{\eta} - \eta) - \mathrm{B}_3'\left(\sqrt{N}\,\underset{\sim}{e}_2\right) = o_p(1).$$

Since the joint asymptotic distribution $\sqrt{N}\,(\underset{\sim}{e}_1, \underset{\sim}{e}_2)$ is normal with mean $\underset{\sim}{0}$ and covariance matrix Σ we thus establish the asymptotic distribution of the WLUE $\tilde{\underset{\sim}{\xi}}$.

Theorem 1: The asymptotic distribution of

$$\sqrt{N}\left[(\tilde{\underset{\sim}{\alpha}} - \underset{\sim}{\alpha}), (\tilde{\eta} - \eta), (\tilde{\gamma} - \gamma)\right]$$

is normal with mean $\underset{\sim}{0}$ and covariance matrix Σ_s, where

$$\Sigma_s = \begin{bmatrix} B_1' \Sigma_1 B_1 & B_1' \Sigma_1 B_2 & B_1' \Sigma_2 B_3 \\ & B_2' \Sigma_1 B_2 & B_2' \Sigma_2 B_3 \\ \text{Symmetric} & & B_3' \Sigma_3 B_3 \end{bmatrix} .$$

In the case of the number of stress setting k tends to infinity, we establish the asymptotic distribution of the WLUE $\tilde{\underset{\sim}{\xi}} = (\tilde{\underset{\sim}{\alpha}}, \tilde{\eta}, \tilde{\gamma})$ by using a series of lemmas. Let $\underset{\sim}{a}_j' = (a_{j,1}, \cdots, a_{j,k})'$ and $\underset{\sim}{b} = (b_1, \cdots, b_k)'$ be vectors of constants and denote $A = (\underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_p)$. For $k = 1, 2, \ldots$ let us define $T_j = \sum_{i=1}^{k} a_{j,i} X_i$, $j = 1, \cdots, p$, and $T_{p+1} = \sum_{i=1}^{k} b_i Y_i$. We need the following assumptions for a general result.

ASSUMPTIONS:

A1. The sequences $\underset{\sim}{a}_j'$, $j = 1, \cdots, p$ and $\underset{\sim}{b}'$ satisfy

(i) $\sum_{i=1}^{k} a_{j,i}^2 = 1, \quad \sum_{i=1}^{k} b_i^2 = 1, \quad k = 1, 2, \ldots$

(ii) $\max_{i \le k} a_{j,i}^2 \to 0, \quad \max_{i \le k} b_i^2 \to 0, \quad$ as $k \to \infty$.

A2. The linear functions T_j, $j = 1, \cdots, p$, are orthogonal, that is for all $j \neq j'$ and k, $\underset{\sim}{a}_j' \underset{\sim}{a}_{j'} = 0$.

A3. $\underset{\sim}{a}_j' D \underset{\sim}{b} = \sum_{i=1}^{k} a_{j,i} b_i \sigma_i \to d_j$ as $k \to \infty$, $j = 1, \cdots, p$, where

$D = \text{diag}(\sigma_1, \cdots, \sigma_k)$, and σ_i is the variance between X_i and Y_i.

An exercise of Lindeberg's Theorem gives the following lemma which is a modification of Lemma 1 of Bhattacharyya and Soejoeti (1981).

Lemma 1: Let (X_i, Y_i), $i = 1, 2, \cdots$ be independent random vectors with $\underset{\sim}{0}$ mean, unit variance, covariance σ_i, and $\lim(B_k/r_i^{1/2}) = 0$ as $k \to \infty$, where $B_k = \left(\sum_{i=1}^{k} q_i\right)^{1/3}$ and $q_i = E|X_i|^3$ (and $E|Y_i|^3$) exist for each i, and σ_i is a known constant. If Assumptions A1–A3 are satisfied, then

$$T = (T_1, \cdots, T_p, T_{p+1})' \xrightarrow{d} N_{p+1}(\underset{\sim}{0}, \Sigma_T),$$

where

$$\Sigma_T = \begin{bmatrix} I & \underset{\sim}{d} \\ \underset{\sim}{d}' & 1 \end{bmatrix}, \quad \underset{\sim}{d} = (d_1, \cdots, d_p)'.$$

Remark: Because that the numbers of replications in any two stress levels might be different, the distributions of τ_i^* and η_i^*, $i = 1, \cdots, k$ (as well as e_{1i} and e_{2i}) given in (3.2) generally are not the same. Hence, we need to impose Liapunov's condition on the moments of X_i and Y_i (or e_{1i} and e_{2i} in next lemma). This assumption holds for the extreme-value distribution.

Let $S_1 = X' D_1^{-1} X$, and C_1 be the symmetric matrix such that $C_1^2 = S_1$. Similarly, we also define

$$S_2 = \underset{\sim}{\zeta}' D_1^{-1} \underset{\sim}{\zeta}, \quad S_3 = \underset{\sim}{1}' D_2^{-1} \underset{\sim}{1}, \quad C_2^2 = S_2, \quad C_3^2 = S_3,$$

$$D_i^{-1/2} = \text{diag}(d_{i1}^{-1/2}, \cdots, d_{ik}^{-1/2}), \; D_i^{1/2} = \text{diag}(d_{i1}^{1/2}, \cdots, d_{ik}^{1/2}), \; i = 1, 2, 3.$$

We denote $\underset{\sim}{Q}_k = (\underset{\sim}{Q}^*, t_k, s_k)'$, where

$$\underset{\sim}{Q}_k^* = \frac{1}{\sqrt{k}} C_1^{-1} X' D_1^{-1} \underset{\sim}{e}_1, \quad t_k = \frac{1}{\sqrt{k}} \underset{\sim}{\zeta}' D_1^{-1} \underset{\sim}{e}_1, \quad s_k = \frac{1}{\sqrt{k}} \underset{\sim}{1}' D_2^{-1} \underset{\sim}{e}_2.$$

The assumptions of the asymptotics of design matrices are as follows:

ASSUMPTIONS:

B1. The limit of $k^{-1} S_1$ exists and is a nonsingular matrix denoted by B. Similarly, we assume the existence of $k^{-1} S_2 \to w_1$, $k^{-1} S_3 \to w_2$ as $k \to \infty$, where both w_1 and w_2 are constants.

B2. The limit of $k^{-1} X' D_1^{-1} \underset{\sim}{\zeta}$ exists and is denoted by $\underset{\sim}{c}$. We also have $k^{-1} X' D_1^{-1} D_3 D_2^{-1} \underset{\sim}{1} \to \underset{\sim}{d}$ and $k^{-1} \underset{\sim}{1}' D_2^{-1} D_3 D_1^{-1} \underset{\sim}{\zeta} \to s$, as $k \to \infty$.

Lemma 2: If Assumptions B1 and B2 are satisfied, then

$$\sqrt{k}\, \underset{\sim}{Q}_k \xrightarrow{d} N_{(p+2)\times(p+2)}(\underset{\sim}{0}, \Sigma_1),$$

$$\Sigma_1 = \begin{bmatrix} I_{p\times p} & B^{-1/2} \underset{\sim}{c}\, w_1^{-1/2} & B^{-1/2} \underset{\sim}{d}\, w_2^{-1/2} \\ (B^{-1/2} \underset{\sim}{c}\, w_1^{-1/2})' & 1 & w_1^{-1/2} w_2^{-1/2} s \\ (B^{-1/2} \underset{\sim}{d}\, w_2^{-1/2})' & w_1^{-1/2} w_2^{-1/2} s & 1 \end{bmatrix},$$

where the limits B, $\underset{\sim}{c}$, $\underset{\sim}{d}$, w_1, w_2, and s are defined in Assumptions B1 and B2.

Proof: To prove this lemma, we shall make use of Lemma 5.1. We consider the independent random vectors (Z_{1i}, Z_{2i}, Z_{3i}), $i = 1, 2, \ldots$, where $Z_{1i} = Z_{2i} = d_{1i}^{-1/2}\, e_{1i}$ and $Z_{3i} = d_{3i}^{-1/2}\, e_{2i}$. Thus (Z_{1i}, Z_{2i}, Z_{3i}) has mean $\underset{\sim}{0}$ and unit variance. The covariance of (Z_{1i}, Z_{2i}) and (Z_{ti}, Z_{3i}), $t = 1, 2$, are 1 and d_{3i}^*, respectively, where $d_{3i}^* = d_{2i}/(d_{1i} d_{3i})^{1/2}$. Let us define

$$\underset{\sim}{Z}_1 = (Z_{11}, \cdots, Z_{1k})', \quad \underset{\sim}{Z}_2 = (Z_{21}, \cdots, Z_{3k})', \quad \underset{\sim}{Z}_3 = (\underset{\sim}{Z}_{31}, \cdots, Z_{3k})'.$$

$$D_3^* = \text{diag}\,(d_{31}^*, \cdots, d_{3k}^*),$$

$$A = D_1^{-1/2}\, X\, C_1^{-1} = (\underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_p), \qquad \underset{\sim}{b}_1' = D_1^{-1/2}\, \underset{\sim}{\zeta}\, C_2^{-1},$$

$$\underset{\sim}{b}_2' = D_2^{-1/2}\, \underset{\sim}{1}\, C_3^{-1},$$

where $\underset{\sim}{a}_j$ denote the jth column of the matrix A. Then, $\sqrt{k}\, \underset{\sim}{Q}_k^*$ and $\sqrt{k}\, t_k$ and $\sqrt{k}\, s_k$ can be written in terms of $\underset{\sim}{Z}_1$, $\underset{\sim}{Z}_2$ and Z_3 as $\sqrt{k}\, \underset{\sim}{Q}_k^* = A'\, X$, $\sqrt{k}\, t_k = \underset{\sim}{b}_1' Z_2$, and $\sqrt{k}\, s_k = \underset{\sim}{b}_2' \underset{\sim}{Z}_3$, respectively.

Next, we show that under Assumptions B1 and B2, the Assumption A1–A3 hold. By Assumption B1 $A'A = C_1^{-1}\, X'\, D_1^{-1}\, X\, C_1 = \underset{\sim}{I}_p$, $\underset{\sim}{b}_2' \underset{\sim}{b}_2 = 1$, the conditions A1(i) and A2 hold. By Assumption B2 $A\, \underset{\sim}{b}_1' \to \underset{\sim}{c}$. $A\, \underset{\sim}{b}_2' \to \underset{\sim}{d}$ and $\underset{\sim}{b}_1' \underset{\sim}{b}_2 \to s$ as $k \to \infty$, so the Assumption A3 holds. Denoting the ith diagonal element of $AA' = D_1^{-1/2}\, X\, S_1^{-1}\, X'\, D_1^{-1/2}$ by q_i, we have

$$\max_{i \le k} |\, q_i\,| \le \sum_{j=1}^{p} \sum_{\ell=1}^{p} \left| k\, S^{j\ell} \right| \max_{i \le k} \frac{\left\{ x_{ji}\, x_{\ell i} \right\}}{d_{1i}\, k}\,.$$

By Assumption B1, $k\, S^{j\ell}$ and $\sum_{i=1}^{p} x_{ji}\, x_{\ell i}/(d_{1i}\, k)$ are convergent, and the latter implies that $\max_{i \le k} x_{ji}\, x_{\ell i}/(d_{ij}\, k) \to 0$, so $\max_{i \le k} |q_i| \to 0$. Since $q_i = \sum_{j=1}^{p} a_{j,i}^2$, we have $\max_{i \le k} a_{j,i}^2 \to 0$ for every $j = 1, \cdots, p$. This result along with the convergences of S_2/k and S_3/k lead to the Assumption A1 (ii). Finally, a direct appliation of Lemma 1 leads to the desired asymptotic distribution of $\sqrt{k}\, \underset{\sim}{Q}_k$. □

Let us define $t_k^* = \frac{1}{\sqrt{k}}\, C_2 t_k$ and $s_k^* = \frac{1}{\sqrt{k}}\, C_3\, s_k$. The following lemma gives the asymptotic distribution concerning (P, t_k^*, s_k^*).

Lemma 3: Let P be equal to $(X'\, D_1^{-1}\, X) X'\, D_1^{-1}$ and $t_k^* = \frac{1}{\sqrt{k}}\, C_2 t_k$ and $s_k^* = \frac{1}{\sqrt{k}}\, C_3\, s_k$. If Assumptions B1 and B2 are satisfied, then

$$\sqrt{k} \left[P\, \underset{\sim}{e}_1,\, t_k^*,\, s_k^* \right] \xrightarrow{d} N_{p+2}(\underset{\sim}{0}, \Sigma_2),$$

where

$$\Sigma_2 = \begin{bmatrix} B^{-1} & B^{-1}\, \underset{\sim}{c} & B^{-1}\, \underset{\sim}{d} \\ (B^{-1}\, \underset{\sim}{c})' & w_1 & s \\ (B^{-1}\, \underset{\sim}{d})' & s & w_2 \end{bmatrix}.$$

Proof: By $C_1^2 = S_1 = X'\, D_1\, X$ we have

$$\sqrt{k}\, \underset{\sim}{Q}_k^* = (C_1^{-1}\, X'\, D_{1s}^{-1})(D_{1s}^{-1}\, \underset{\sim}{e}_1) = C_1\, S_1^{-1}\, X'\, D_1^{-1} \underset{\sim}{e}_1 = C_1\, P\, \underset{\sim}{e}_1.$$

Therefore, from Lemma 2 $\sqrt{k}\left[\frac{1}{\sqrt{k}}\, C_1\, P\, \underset{\sim}{e}_1,\ t_k^*,\ s_k^*\right]$ is asymptotically normal with mean $\underset{\sim}{0}$ and covariance matrix Σ_1. By an application of *Delta* method along with the asymptotic $\frac{1}{\sqrt{k}}\, C_1 \to B^{1/2}$, we complete the proof. □

Lemma 4: If Assumptions B1 and B2 are satisfied, then

$$\sqrt{k}\, (P\, \underset{\sim}{e}_1,\ \tfrac{1}{k}\, \underset{\sim}{\zeta}'\, D_1^{-1}\, Q\, \underset{\sim}{e}_1,\ s_k^*) \xrightarrow{d} N_{p+2}\, (\underset{\sim}{0},\ \Sigma_3),$$

where

$$\Sigma_3 = \begin{bmatrix} B^{-1} & \underset{\sim}{0} & B^{-1/2}\, \underset{\sim}{d} \\ \underset{\sim}{0}' & \sigma_a & \sigma_b \\ (B^{-1/2}\, \underset{\sim}{d})' & \sigma_b & w_2 \end{bmatrix},$$

and $\sigma_a = w_1 - \underset{\sim}{c}'\, B^{-1}\, \underset{\sim}{c}$ and $\sigma_b = s - \underset{\sim}{c}'\, B^{-1}\, \underset{\sim}{d}$.

Proof: By $Q = I - X\, P$ we have

$$\frac{1}{\sqrt{k}}\, \underset{\sim}{\zeta}'\, D_1^{-1}\, Q\, \underset{\sim}{e}_1 = \sqrt{k}\, t_k^* - \left(\tfrac{1}{k}\, \underset{\sim}{\zeta}'\, D_1^{-1}\, X\right)\left(\sqrt{k}\, P\, \underset{\sim}{e}_1\right).$$

From Assumption B2, we know that $\frac{1}{k}\, \underset{\sim}{\zeta}'\, D_1^{-1}\, X \to \underset{\sim}{c}'$ as $k \to \infty$. Hence, the asymptotic distribution of $\frac{1}{\sqrt{k}}\, \underset{\sim}{\zeta}'\, D_1^{-1}\, Q\, \underset{\sim}{e}_1$ is the same as the asymptotic distribution of $\sqrt{k}\, t_k^* - \underset{\sim}{c}'\left(\sqrt{k}\, P\, \underset{\sim}{e}_1\right)$. Using the joint asymptotic distribution of $\sqrt{k}\, (P\, \underset{\sim}{e}_1,\ t_k^*,\ s_k^*)$ in Lemma 3, we complete the proof. □

The following asymptotics are readily to be obtained by Assumptions B1 and B2. (i) $P\, \underset{\sim}{\zeta} \to B^{-1}\, \underset{\sim}{c}$ and (ii) $k^{-1}\, \sigma_* \to w_1 - \underset{\sim}{c}'\, B^{-1}\, \underset{\sim}{c}$, where

$\sigma_* = \underset{\sim}{\zeta}' \mathrm{D}_1^{-1} \mathrm{Q} \underset{\sim}{\zeta}$. The asymptotic distribution of the WLUE $\underset{\sim}{\tilde{\xi}}$ is given in the following theorem.

Theorem 2: If Assumptions B1 and B2 are satisfied, then the joint asymptotic distribution of $\sqrt{k}\left[\left(\underset{\sim}{\tilde{\alpha}} - \underset{\sim}{\alpha}\right), (\tilde{\gamma} - \gamma), (\tilde{\eta} - \eta)\right]$ is normal with mean $\underset{\sim}{0}$ and covariance matrix Σ, where

$$\Sigma = \begin{bmatrix} \Sigma_\beta & \Sigma_{\beta\gamma} & \Sigma_{\beta\eta} \\ \Sigma'_{\beta\gamma} & \sigma_a^{-1} & \sigma_a^{-1}\sigma_b \\ \Sigma'_{\beta\eta} & \sigma_a{}^{-1}\sigma_b & w_2^{-1} \end{bmatrix},$$

and

$$\Sigma_\beta = \mathrm{B}^{-1}\left[\mathrm{I} + \underset{\sim}{c}\,(w_1 - \underset{\sim}{c}'\,\mathrm{B}^{-1}\underset{\sim}{c})^{-1}\underset{\sim}{c}'\,\mathrm{B}^{-1}\right],$$

$$\Sigma_{\beta\gamma} = -\,\mathrm{B}^{-1}\underset{\sim}{c}(w_1 - \underset{\sim}{c}'\,\mathrm{B}^{-1}\underset{\sim}{c}), \qquad \Sigma_{\beta\eta} = \mathrm{B}^{-1}\left[\underset{\sim}{d} - \underset{\sim}{c}\,\sigma_a{}^{-1}\sigma_b\right] z^{-1},$$

$$\sigma_a = w_1 - \underset{\sim}{c}'\,\mathrm{B}^{-1}\underset{\sim}{c}, \qquad \sigma_b = s - \underset{\sim}{c}'\,\mathrm{B}^{-1}\underset{\sim}{d}.$$

Proof: We first recall that

$$\sqrt{k}\,(\underset{\sim}{\hat{\alpha}} - \underset{\sim}{\tilde{\alpha}}) = \sqrt{k}\,\mathrm{P}\left[\mathrm{I} - \underset{\sim}{\zeta}\,\sigma_*^{-1}\underset{\sim}{\zeta}'\,\mathrm{D}_1^{-1}\,\mathrm{Q}\right]\underset{\sim}{e}_1,$$

$$\sqrt{k}\,(\tilde{\gamma} - \gamma) = \sqrt{k}\,\sigma_*^{-1}\underset{\sim}{\zeta}'\,\mathrm{D}_1^{-1}\,\mathrm{Q}\,\underset{\sim}{e}_1,$$

$$\sqrt{k}\,(\tilde{\eta} - \eta) = \sqrt{k}\,(\underset{\sim}{1}'\,\mathrm{D}_2^{-1}\underset{\sim}{1})^{-1}\underset{\sim}{1}'\,\mathrm{D}_2^{-1}\underset{\sim}{e}_2.$$

As $k \to \infty$ we have

$$\sqrt{k}\,(\underset{\sim}{\hat{\alpha}} - \underset{\sim}{\tilde{\alpha}}) - \left(\sqrt{k}\,\mathrm{P}\,\underset{\sim}{e}_1\right) - (\mathrm{B}^{-1}\underset{\sim}{c}\,\sigma_*^{-1})\left(\frac{1}{\sqrt{k}}\underset{\sim}{\zeta}'\,\mathrm{D}_1^{-1}\,\mathrm{Q}\,\underset{\sim}{e}_1\right) = o_p(1),$$

$$\sqrt{k}\,(\tilde{\gamma} - \gamma) - \sigma_a^{-1}\sqrt{k}\left(\frac{1}{k}\underset{\sim}{\zeta}'\,\mathrm{D}_1^{-1}\,\mathrm{Q}\,\underset{\sim}{e}_1\right) = o_p(1),$$

$$\sqrt{k}\,(\tilde{\eta} - \eta) - w_2^{-1}\sqrt{k}\left(\frac{1}{k}\underset{\sim}{1}'\,\mathrm{D}_2^{-1}\underset{\sim}{e}_2\right) = \sqrt{k}\,(\tilde{\eta} - \eta) - w_2\sqrt{k}\,s_k^* = o_p(1).$$

Since the joint asymptotic normality of $\sqrt{k}\left(\mathrm{P}\,\underset{\sim}{e}_1, \frac{1}{k}\underset{\sim}{\zeta}'\,\mathrm{D}_1^{-1}\,\mathrm{Q}\,\underset{\sim}{e}_1, s_k^*\right)$ has been established in Lemma 4, the joint asymptotic distribution of

$\sqrt{k}\left[(\tilde{\underset{\sim}{\alpha}} - \underset{\sim}{\alpha}), (\tilde{\gamma} - \gamma), (\tilde{\eta} - \eta)\right]$ is readily established as stated in theorem. □

Remark: If the dependence parameter δ is in the interior of (0, 1], the asymptotic distribution of $(\tilde{\theta}_u, \tilde{\beta}, \tilde{\delta})$ follows from *Delta* method. For the case of δ equal to 1, the correct limit distribution is obtained by an application of Eq. (2.2) given in Self and Liang (1987).

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