NORMAL SCALE MIXTURE APPROXIMATIONS TO THE LOGISTIC DISTRIBUTION WITH APPLICATIONS

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ABSTRACT

We present least maximum approximants to the logistic distribution function from the family of discrete scale mixtures of the standard normal distribtuion. The approximations are useful in statistical models involving convolutions of the logistic and normal distributions, such as logistic regression models with random effects or with predictors measured with error. The approximations are obtained using an adaptation of Remes' Second Algorithm and their application to measurement error models is discussed in detail.

1. INTRODUCTION

In this paper we develop mixture approximations to the logistic distribution, $F(t) = 1/(1+e^{-t})$, having the form

$$F_k(t) = \sum_{i=1}^{k} p_{k,i}\Phi(ts_{k,i}), \qquad (k = 1, 2, \ldots), \tag{1.1}$$

where Φ is the standard normal cumulative distribution function and $\{p_{k,i}, s_{k,i}\}_{i=1}^{k}$ are chosen to minimize

$$D_k^* = \sup_t \mid F(t) - F_k(t) \mid, \tag{1.2}$$

for $k = 1, 2, \ldots$. These approximations are tailor-made for statistical models involving convolutions of a normal distribution with F and/or its derivatives. Two important applications, logistic regression measurement error models and random effects logistic regression models, motivated our work and are discussed Section 2.

The quality of the approximation for k as small as 3 is remarkable and improves significantly for each successive k. Explanation for this behavior and mathematical justification for the class of appproximations (1.1) is provided by the following result which is proved in Section 3.

THEOREM. *Let F and Φ denote the standard logistic and normal cumulative distribution functions respectively. Then*

$$F(t) = \int_0^\infty \Phi(t/\sigma)q(\sigma)d\sigma, \tag{1.3}$$

where $q(\sigma) = (d/d\sigma)L(\sigma/2)$ and L is the Kolmogorov-Smirnov distribution,

$$L(\sigma) = 1 - 2\sum_{j=1}^{\infty}(-1)^{j+1}\exp(-2j^2\sigma^2). \tag{1.4}$$

Tables of $\{p_{k,i}, s_{k,i}\}_{i=1}^{k}$, for $k = 1, \ldots, 8$ are given in Section 4 along with technical details related to: (i) the method employed to minimize (1.2); (ii) the error functions

$D_k(t) = F(t) - F_k(t)$; and (iii) the error incurred in using F_k to approximate the logistic/normal integral arising in logistic regression measurement error models.

2. MODELS INVOLVING LOGISTIC-NORMAL INTEGRALS

In one form of a logistic regression measurement error model, the binary outcome Y is related to a p-dimensional predictor X, via $\text{pr}(Y = 1 \mid X = x) = F(\beta^T x)$. Observable data consist of observations on (Y, Z) where $X \mid (Z = z)$ is normally distributed with mean $\mu(z)$ and covariance matrix $\Omega(z)$. The conditional distribution of $Y \mid (Z = z)$ is given by

$$G(\eta, \tau) = \int_{-\infty}^{\infty} F(t)\tau^{-1}\phi\left(\frac{t-\eta}{\tau}\right) dt, \tag{2.1}$$

where $\eta = \beta^T \mu(z)$ and $\tau = \sqrt{\beta^T \Omega(z) \beta}$. This model and closely related models have been the topic of several recent papers: Carroll, Spiegelman, Baily, Lan and Abbott (1984), Armstrong (1985), Stefanski and Carroll (1985), Prentice (1986), Schafer (1987), Spiegelman (1988). With the exception of Spiegelman (1988), none of these authors propose direct numerical evaluation of (2.1).

We propose replacing F by F_k in (2.1) resulting in the approximate model

$$G_k(\eta, \tau) = \int_{-\infty}^{\infty} F_k(t)\tau^{-1}\phi\left(\frac{t-\eta}{\tau}\right) dt = \sum_{i=1}^{k} p_{k,i}\Phi\left(\frac{\eta s_{k,i}}{\sqrt{1+\tau^2 s_{k,i}^2}}\right). \tag{2.2}$$

The accuracy of the approximation (2.2) is discussed in Section 4. Its useful feature is its representation in terms of Φ and therefore its programmability in standard statistical software packages.

Now consider the random-effects logistic regression model

$$\text{pr}(Y_{i,j} = 1 \mid X_i = x_i,\ \epsilon_i) = F(\eta_i + \tau\epsilon_i), \qquad j = 1, \ldots, n_i, \quad i = 1, \ldots, N, \tag{2.3}$$

where $\eta_i = \beta^T x_i$ and $\{\epsilon_i\}_{i=1}^N$ is a sequence of independent standard normal variates. Models for correlated binary responses have been studied by several authors: Pierce and

Sands (1975), Williams (1982), Ochi and Prentice (1984), Prentice (1986), Zeger, Liang and Albert (1988), Prentice (1988), Stiratelli, Laird and Ware (1984) and McCullagh (1989). They arise frequently in animal experiments wherein ϵ_i in (2.3) represents a random inter-litter effect.

Quasilikelihood/variance-function estimation for this model requires the first two moments of $Y_{i,\cdot} \mid (X_i = x_i)$:

$$E(Y_{i,\cdot} \mid X_i = x_i) = n_i \mu_i = n_i \int_{-\infty}^{\infty} F(t) \tau^{-1} \phi \left(\frac{t - \eta_i}{\tau} \right) dt; \tag{2.4}$$

$$Var(Y_{i,\cdot} \mid X_i = x_i) = \Sigma_i = n_i^2 \mu_i (1 - \mu_i) - n_i (n_i - 1) \int_{-\infty}^{\infty} F^{(1)}(t) \tau^{-1} \phi \left(\frac{t - \eta_i}{\tau} \right) dt. \tag{2.5}$$

The identity $F^{(1)} = F(1 - F)$ is used to derive (2.5).

Again we propose substituting F_k for F in (2.4) leading to the approximate mean

$$\tilde{\mu}_i = \int_{-\infty}^{\infty} F_k(t) \tau^{-1} \phi \left(\frac{t - \eta}{\tau} \right) dt = \sum_{j=1}^{k} p_{k,j} \Phi \left(\frac{\eta_i s_{k,j}}{\sqrt{1 + \tau^2 s_{k,j}^2}} \right). \tag{2.6}$$

Substitution of $F_k^{(1)}$ for $F^{(1)}$ in (2.5) yields an approximate variance

$$\tilde{\Sigma}_i = n_i^2 \tilde{\mu}_i (1 - \tilde{\mu}_i) - n_i (n_i - 1) \int_{-\infty}^{\infty} F_k^{(1)}(t) \tau^{-1} \phi \left(\frac{t - \eta_i}{\tau} \right) dt$$

$$= n_i^2 \tilde{\mu}_i (1 - \tilde{\mu}_i) - n_i (n_i - 1) \sum_{j=1}^{k} \frac{p_{k,j} s_{k,j}}{\sqrt{1 + \tau^2 s_{k,j}^2}} \phi \left(\frac{\eta_i s_{k,j}}{\sqrt{1 + \tau^2 s_{k,j}^2}} \right). \tag{2.7}$$

The fact that neither the covariate nor the random effect varies with j in (2.3) is responsible for the simplicity of (2.7). More complex random effects logistic regression models have

$$\text{pr}(Y_i = 1 \mid X_i = x_i, \; \epsilon_i) = F(\eta_i + \epsilon_i), \quad i = 1, \ldots, n, \tag{2.8}$$

where $\eta_i = \beta^T x_i$ and $\epsilon^T = (\epsilon_1, \ldots, \epsilon_n)$ has a multivariate normal distribution with mean zero and covariance matrix $\Omega = \{\omega_{r,m}\}$ containing unknown parameters. The marginal expectation $E(Y_i \mid X_i = x_i)$ is given by μ_i in (2.4) with τ replaced by $\sqrt{\omega_{ii}}$ and can be approximated as in (2.6). Computation of the marginal covariance of Y_r and Y_m requires

$$\mu_{r,m} = E(Y_r Y_m \mid X_r = x_r,\ X_m = x_m) = E\{F(\eta_r + \epsilon_r)F(\eta_m + \epsilon_m)\}. \tag{2.9}$$

Substituting F_k for F in (2.9) yields the approximation

$$\tilde{\mu}_{r,m} = \sum_{i=1}^{k}\sum_{j=1}^{k} p_{k,i}p_{k,j}E\{\Phi(\eta_r s_{k,i} + \epsilon_r s_{k,i})\Phi(\eta_m s_{k,j} + \epsilon_m s_{k,j})\}$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{k} p_{k,i}p_{k,j}\Phi_2\left\{\frac{s_{k,i}\eta_r}{\sqrt{1+s_{k,i}^2\omega_{r,r}}}, \frac{s_{k,j}\eta_m}{\sqrt{1+s_{k,j}^2\omega_{m,m}}}, \frac{\omega_{r,m}s_{k,i}s_{k,j}}{\sqrt{(1+s_{k,i}^2\omega_{r,r})(1+s_{k,j}^2\omega_{m,m})}}\right\}, \tag{2.10}$$

where $\Phi_2(\cdot,\cdot,\rho)$ is the bivariate standard normal distribution function with correlation ρ.

In some statistical software packages $\Phi_2(\cdot,\cdot,\cdot)$ is an intrinsically defined function and with these programs quasilikelihood/variance-function estimation for (2.8) using (2.1) may be feasible when not too many of the $\omega_{r,m}$ are nonzero.

3. THE LOGISTIC DISTRIBUTION AS A GAUSSIAN SCALE MIXTURE

Consider the integral equation

$$f(t) = \int_0^\infty \sigma^{-1}\phi(t/\sigma)q(\sigma)d\sigma, \tag{3.1}$$

when $f(t) = e^{-t}/(1+e^{-t})^2$ and ϕ is the standard normal density.

Note that for $t > 0$, $F(t) = (1+e^{-t})^{-1} = \sum_{n=0}^{\infty}(-1)^n \exp(-nt)$. Upon differentiation and appeal to symmetry it follows that

$$f(t) = \sum_{n=1}^{\infty}(-1)^{(n+1)} n \exp(-n \mid t \mid). \tag{3.2}$$

Thus for $q(\sigma)$ given in (1.4)

$$\int_0^\infty \sigma^{-1}\phi(t/\sigma)q(\sigma)d\sigma = \sqrt{2/\pi}\sum_{n=1}^{\infty}(-1)^{(n+1)}n^2\int_0^\infty \exp(-t^2/2\sigma^2 - n^2\sigma^2/2)d\sigma$$
$$= \sqrt{2/\pi}\sum_{n=1}^{\infty}(-1)^{(n+1)}n^2\left\{\frac{\sqrt{2\pi}\exp(-n \mid t \mid)}{2n}\right\}$$
$$= \sum_{n=1}^{\infty}(-1)^{(n+1)}n\exp(-n \mid t \mid), \tag{3.3}$$

proving the theorem upon appeal to (3.2). The integral evaluation in (3.3) employs a standard change of variables; alternatively, see Gradshteyn and Ryzhik (1980, Formula 3.325, p. 307).

4. THE DISCRETE MIXTURE APPROXIMATION

The *least-maximum approximant*, F_k, minimizes $D_k^* = \sup_t \mid D_k(t) \mid$, where $D_k(t) = F(t) - F_k(t)$ subject to the parametric constraints on F_k. Since $D_k(t)$ is skew-symmetric, D_k^* is determined by the restriction of D_k to $[0, \infty)$.

Least-maximum approximations form the basis for computing all nonarithmetic functions wherein the approximant is usually a polynomial or rational function. In the case of a polynomial of degree m, the solution to the optimization problem is characterized by Chebyshev's Alternation Theorem (Hart, Cheney, Lawson, Maehley, Mesztenyi, Rice, Thatcher, and Witzgall, 1968, p. 45), which states that the difference function must have $m+1$ roots and $m+2$ local extrema, equal in magnitude and alternating in sign.

For our problem the approximant F_k is not polynomial nor is it linear in the $m = 2k-1$ parameters $\{(p_{k,i}, s_{k,i})_{i=1}^k : 0 \le s_{k,i};\ 0 \le p_{k,i} \le 1,\ \sum_{i=1}^k p_{k,i} = 1\}$. Chebyshev's Theorem does not apply but we conjecture that the least-maximum approximant has local extrema equal in magnitude and alternating in sign as in the polynomial case. Thus we solved for approximants F_k having difference functions D_k possessing m roots in addition to the constrained roots at 0 and ∞, and $m+1$ local sign-alternating extrema.

We adapted Remes' Second Algorithm (Hart, *et al.*, 1968, p.45) for finding polynomial least-maximum approximants in our problem. Given initial values $\{p_{k,i}^{(0)},\ s_{k,i}^{(0)}\}_{i=1}^{k}$ such that the initial difference function $D_k^{(0)}$ has $m+1$ sign-alternating extrema not necessarily of equal magnitude, m roots of $D_k^{(0)}$, $z_i^{(0)}$, $(i = 1, \ldots, m)$ were found. Then, letting $z_0^{(0)} = 0$ and $z_{m+1}^{(0)} = z^*$ for some large z^*, the $m+1$ extrema $x_i^{(0)}$ of $D_k^{(0)}$ in the intervals $(z_{i-1},\ z_i)$, $(i = 1, \ldots, m+1)$ were found. Updated parameters $\{p_{k,i}^{(1)},\ s_{k,i}^{(1)}\}_{i=1}^{k}$ were obtained by solving the $m+1$ nonlinear equations

$$D_k^{(0)}(x_i) = (-1)^{i+1} D, \qquad (i = 1, \ldots, m+1), \tag{4.1}$$

as functions of the $m+1$ arguments $\{p_{k,i},\ s_{k,i}\}_{i=1}^{k}$ and D. This three-step procedure was iterated until convergence. At the stationary point, D_k had the requisite number of sign-alternating, equal-magnitude extrema, giving us reason to believe that the least-maximum approximant on $(0,\ z^*)$ was found. Since for z^* sufficiently large, D_k is negative and monotonically increasing to zero, the restriction to $(0,\ z^*)$ is not binding and our error estimates apply to $(0,\ \infty)$.

The parameters of the approximations, and the maximum absolute differences D_k^* for $k = 1, \ldots, 8$ are given in Table 1. Note that $s_{1,1} = 0.58763$ is slightly smaller than the logistic/normal scale correction given by Cox (1970, p.28) and agrees closely with the factor $16\sqrt{3}/15\pi = 0.58808$ given by Johnson and Kotz (1970, p. 6).

Figure 1 displays a graph of the functions

$$J_k(t) = \frac{D_k(t)}{D_k^*} + 2k - 1, \quad (k = 1, \ldots, 8), \tag{4.2}$$

illustrating the oscillations in $D_k(\cdot)$, $(k = 1, \ldots, 8)$.

Two aspects of our problem complicated the implementation of Remes' Second Algorithm: i) finding initial parameter values; and ii) solving (4.1). For $k < 3$ suitable starting parameters were found by trial and error. For larger k starting parameters were found using a nonlinear least squares routine to minimize a discrete approximation to the L_{2r} norm of D_k. Acceptable starting values were obtained with $r = 1$; however, better starting values were found for $r = 2, \ldots, 4$, a consequence of the approximate equivalence of the sup norm and L_{2r} norm for r large.

In the usual applications of the algorithm, the approximant is either polynomial or a rational function in which cases the equations in (4.1) are linear or can be replaced with a linear system in all of the parameters except D (Cody, Fraser, and Hart, 1968). In our application the equations are nonlinear in all the parameters, thereby complicating this step of the algorithm.

5. LOGISTIC REGRESSION MEASUREMENT ERROR MODELS

5.1 Comparison with other approximations

Other approximations have been proposed for the integral in (2.1). Fuller (1987, p. 262) proposed the Taylor series approximation,

$$G_{TS}(\eta, \tau) = F(\eta) + (\tau^2/2)F^{(2)}(\eta); \tag{5.1}$$

and Carroll and Stefanski (1989) proposed a range-preserving modification to the above approximation,

$$G_{RPTS}(\eta, \tau) = F\left\{\eta + \frac{\tau^2 F^{(2)}(\eta)}{2F^{(1)}(\eta)}\right\}. \tag{5.2}$$

The approximations in (5.1) and (5.2) are compared to those in (2.2) in Figure 2. Here we have graphed the functions

$$D_A^{**}(\tau) = \log_{10}\left\{\sup_\eta \mid G(\eta, \tau) - G_A(\eta, \tau) \mid\right\}$$

for the various approximations over the range $0.01 < \tau < 5$. The eight nearly-linear graphs correspond to our approximations in (2.2), $k = 1, \ldots, 8$, top to bottom; the approximations in (5.1) and (5.2) are drawn with dashed lines; the remaining curve (dot-dash) corresponds to the approximation derived from using 20-point Gauss-Hermite quadrature to evaluate (2.1). The "exact" evaluation of (2.1) was obtained using the method proposed by Crouch and Spiegelman (1990) with accuracy set at 10^{-15} and also by an adaptive Simpson's rule with convergence criterion set at 10^{-15}. The latter two differed by less than 10^{-14} across the range of τ values. The two Taylor series approximations are exact at $\tau = 0$ and thus more accurate for very small τ; however, they quickly breakdown as τ increases from zero.

5.2 Example

We have fit, via maximum likelihood, the logistic regression measurement error model (2.1) to several data sets using the approximations in (5.1), (5.2) and those in (2.2), and compared these fits to those obtained from an "exact" fit of (2.1) using the Simpson's rule evaluation mentioned above. In these data sets, $\hat{\tau}$ was not large and the differences between all the estimated models was generally small. However, although the differences between the approximate fits (2.2) for $k = 4, \ldots, 8$ and the exact fit were numerically negligible, differences between the Taylor series approximations (5.1) and (5.2) and the exact fit were occasionally more noticeable, even though the differences were generally slight relative to standard errors. Table 2 contains results from two comparisons. In the first, data were generated such that, $X \mid (Z = z) \sim \mathcal{N}(z, 1/36)$, $Z \sim \mathcal{N}(0, 13/180)$, and $\mathrm{pr}(Y = 1 \mid X = x) = F(\beta_0 + \beta_1 x)$, $\beta_0 = -1.5$, $\beta_1 = 1.5$; and for Case 2, $X \mid (Z = z) \sim \mathcal{N}(z, .04)$, $Z \sim \mathcal{N}(0, .06)$, and $\mathrm{pr}(Y = 1 \mid X = x) = F(\beta_0 + \beta_1 x)$, $\beta_0 = -1.5$, $\beta_1 = 2.5$. The parameterizations were chosen so that the corresponding τ values, 0.25 and 0.50 respectively, span the range over which the Taylor series approximations breakdown; see Figure 2. Also the first matches closely a cohort study referenced in Stefanski and Carroll (1985). Sample size was $n = 300$ and standard errors were derived from the inverse of the conditional (on $\{Z_i\}$) expected information matrix evaluated at the corresponding estimate.

Finally we note that maximum likelihood estimation and large–sample inference in model (2.1) involves evaluation of not only the integral in (2.1) but also the first– and possibly second–order partial derivatives of (2.1) with respect to η and τ. The approximations in (2.2) have the advantage that all the required derivatives have closed–form expressions, thus making them computationally more attractive than other forms of numerical integration.

REFERENCES

Armstrong, B. (1985). Measurement error in the generalized linear model. *Communications in Statistics,* **B**, **14**, 529-544.

Carroll, R. J., Spiegelman, C., Bailey, K., Lan, K. K. G. and Abbott, R. D. (1984). On errors-in-variables for binary regression models. *Biometrika*, **71**, 19-26.

Carroll, R. J. and Stefanski, L. A. (1989). Approximate quasilikelihood estimation in measurement error models. *Journal of the American Statistical Association*, To appear.

Cody, W. J., Fraser, W. and Hart, J. F. (1968). Rational Chebyshev approximations using linear equations. *Numerische Mathematik*, **12**, 242-251.

Cox, D. R. (1970). *Binary Regression.* Chapman and Hall, London.

Crouch, A. and Spiegelman, D. (1990). The evaluation of integrals of the form $\int f(t)e^{-t^2}dt$. Application to logistic–normal models. *Journal of the American Statistical Association*, To appear.

Fuller, W. A. (1987). *Measurement Error Models.* Wiley, New York.

Gradshteyn, I. S. and Ryzhik, I. W. (1965). *Tables of Integrals, Series and Products.* Academic Press: New York

Hart, J. F., Cheney, E. W., Lawson, C. L., Maehley, H. J., Mesztenyi, C. K., Rice, J. R., Thatcher, H. G. and Witzgall, C. (1968). Computer Approximations. Wiley: New York.

Johnson, N. L. and Kotz, S. (1970). *Distributions in Statistics, Continuous Univariate Distributions, Vol.* 2. Boston: Houghton–Mifflin.

McCullagh, P. (1989). Binary data, dispersion effects and the sexual behaviour of Appalachian salamanders. Technical Report No. 227, Department of Statistics, University of Chicago.

Ochi, Y. and Prentice, R. L. (1984). Likelihood inference in a correlated probit regression model. *Biometrika*, **71**, 531-543.

Pierce, D. A. and Sands, B. R. (1975). Extra–binomial variation in binary data. Technical

Report No. 46, Department of Statistics, Oregon State University.

Prentice, R. L. (1986). Binary regression using an extended beta–binomial distribution, with discussion of correlation induced by covariate measurement errors. *Journal of the American Statistical Association*, **394**, 321-327.

Prentice, R. L. (1988). Correlated binary regression with covariates specific to each binary observation. *Biometrics*, **44**, 1033-1048.

Schafer, D. (1987). Covariate measurement error in generalized linear models. *Biometrika*, **74**, 385-391.

Spiegelman, D. (1988). Design and analysis strategies for epidemiologic research when the exposure variable is measured with error. Unpublished Ph.D. thesis, Department of Biostatistics, Harvard University.

Stefanski, L. A. and Carroll, R. J. (1985). Covariate measurement error in logistic regression. *Annals of Statistics*, **13**, 1335-1351.

Stiratelli, R., Laird, N. M. and Ware, J. H. (1984). Random–effects models for serial observations with binary response. *Biometrics*, **40**, 961-971.

Williams, D. A. (1982). Extra–binomial variation in logistic linear models. *Applied Statistics*, **31**, 144-148.

Zeger, S. L., Liang, K. Y., and Albert, P. S. (1988). Models for longitudinal data: A generalized estimating equation approach. *Biometrics*, **44**, 1049-1060.

Table 1

Least Maximum Approximants

k	D_k^*	$p_{k,i}$	$s_{k,i}$
1	9.5(−3)	1.00000 00000 00000	0.58763 21328 24711
2	5.1(−4)	0.56442 48434 56014	0.76862 11512 01617
		0.43557 51565 43986	0.43525 51800 69097
3	4.4(−5)	0.25220 15780 98282	0.90793 08374 49693
		0.58522 50592 35736	0.57778 72761 40136
		0.16257 33626 65982	0.36403 77294 79770
4	4.7(−6)	0.10649 89926 56952	1.02313 55805 00914
		0.45836 12270 14536	0.69877 43559 46609
		0.37418 90669 14829	0.47512 75246 40229
		0.06095 07134 13683	0.32106 46558 34542
5	6.0(−7)	0.04433 31519 39163	1.12271 60446 01626
		0.29497 33769 77114	0.80390 11123 64718
		0.42981 24819 00555	0.57619 76474 16307
		0.20758 95057 57111	0.41091 37884 75565
		0.02329 14834 26056	0.29147 98950 13730
6	8.4(−8)	0.01844 61051 35654	1.21126 19038 17665
		0.17268 13809 23308	0.89734 91017 45990
		0.37393 07960 25243	0.66833 39140 55091
		0.31696 99558 13251	0.49553 64080 73851
		0.10889 73000 53481	0.36668 89730 48750
		0.00907 44620 49063	0.26946 19401 81722
7	1.3(−8)	0.00771 18334 44756	1.29153 69623 80838
		0.09586 53530 40290	0.98192 29519 23815
		0.28194 83083 10964	0.75277 49130 79201
		0.34546 28488 09089	0.57547 48297 39348
		0.20984 86860 83383	0.43861 91442 39000
		0.05556 46179 00566	0.33410 42759 71241
		0.00359 83524 10953	0.25221 62211 45020
8	2.1(−9)	0.00324 63432 72134	1.36534 08062 96348
		0.05151 74770 33972	1.05952 39710 16916
		0.19507 79126 73858	0.83079 13137 65644
		0.31556 98236 32818	0.65073 21666 39391
		0.27414 95761 58423	0.50813 54253 66489
		0.13107 68806 95470	0.39631 33451 66341
		0.02791 24187 27972	0.30890 42522 67995
		0.00144 95678 05354	0.23821 26164 09306

Table 2

Comparison of Approximations

	Case 1			Case 2		
$\hat{G}$	$\hat{\beta}_1$	s.e.($\hat{\beta}_1$)	t–ratio	$\hat{\beta}_1$	s.e.($\hat{\beta}_1$)	t–ratio
G_{TS}	1.4639	0.5663	2.5850	2.8670	0.8286	3.4601
G_{RPTS}	1.4641	0.5667	2.5837	2.8743	0.8442	3.4046
G_1	1.4474	0.5504	2.6299	2.8109	0.8012	3.5085
G_2	1.4572	0.5663	2.5733	2.8450	0.8334	3.4138
G_3	1.4637	0.5667	2.5829	2.8509	0.8308	3.4313
G_4	1.4639	0.5665	2.5841	2.8509	0.8308	3.4316
G_5	1.4639	0.5665	2.5841	2.8509	0.8308	3.4316
G_6	1.4639	0.5665	2.5841	2.8509	0.8308	3.4316
G_7	1.4639	0.5665	2.5841	2.8509	0.8308	3.4316
G_8	1.4639	0.5665	2.5841	2.8509	0.8308	3.4316
G	1.4639	0.5665	2.5841	2.8509	0.8308	3.4316

Figure 1. Graph of the scaled and shifted difference functions in (4.2) showing the equal-magnitude sign-alternating behavior of the local extrema.

Figure 1

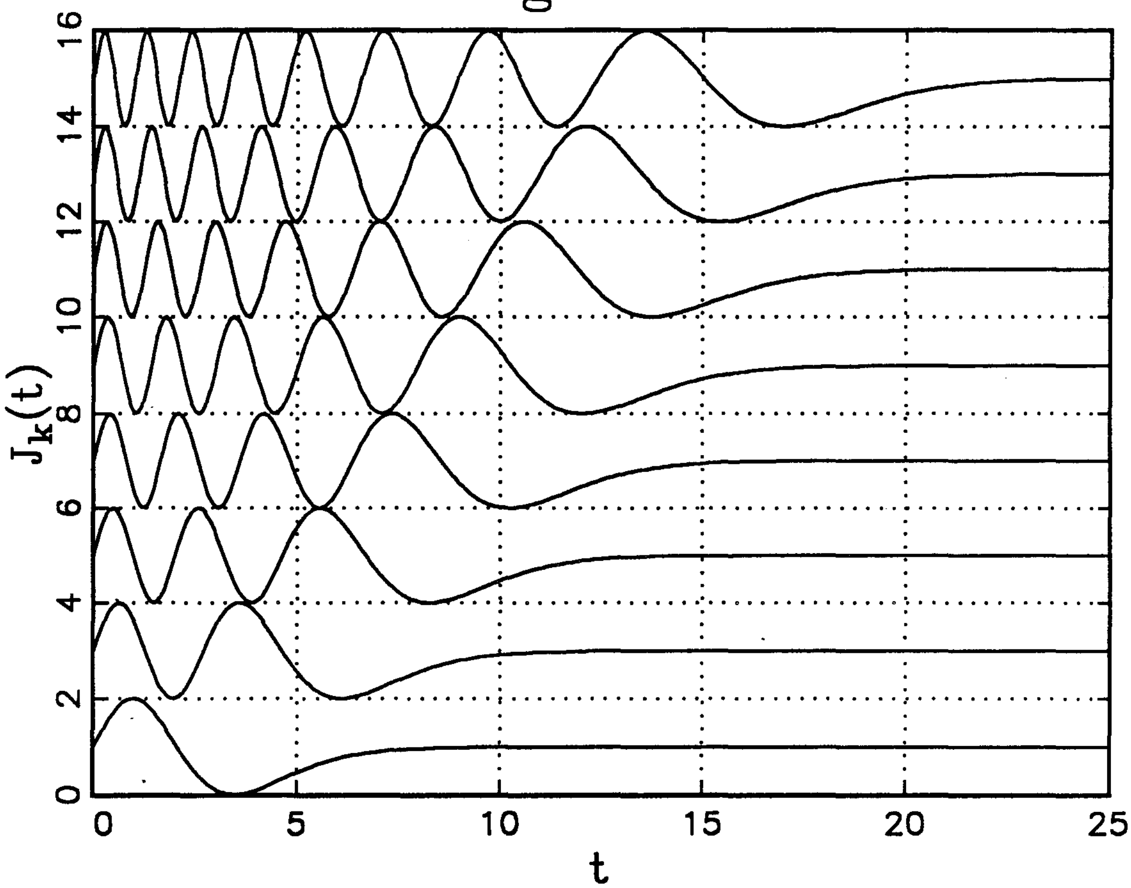

Figure 2. Graph of the $\log_{10}$ error functions $D_A^{**}(\tau)$. Solid lines, G_k, $k = 1, \ldots, 8$; dashed lines, Taylor series approximations; dot–dashed line, 20–point Gauss–Hermite quadrature.

Figure 2