ON PITMAN CLOSENESS OF PITMAN ESTIMATORS

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Institute of Mimeo Series No. 1881
July 1990

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ABSTRACT. Pitman estimators of location are unbiased, translation-equivariant and possess some optimality properties under quadratic loss. Similar optimality properties of Pitman estimators are studied with respect to the measure of Pitman closeness of estimators.

1. INTRODUCTION

Let $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ be n independent and identically distributed (i.i.d.) random vectors (r.v.) with a probability density function (pdf) $f(\underset{\sim}{x}; \underset{\sim}{\theta})$. In a (multivariate) location model, we set

$$f(\underset{\sim}{x}; \underset{\sim}{\theta}) = f(\underset{\sim}{x} - \underset{\sim}{\theta}),\ \underset{\sim}{x} \in R^p,\ \underset{\sim}{\theta} \in \Theta \subset R^p, \tag{1.1}$$

for some $p \geq 1$, where the form of $f(\cdot)$ is assumed to be known. The joint density (i.e., likelihood) function of $\underset{\sim}{X}, \ldots, \underset{\sim}{X}_n$ is given by

$$\ell_n(\underset{\sim}{\theta}) = \prod_{i=1}^{n} f(\underset{\sim}{X}_i; \underset{\sim}{\theta}) = \prod_{i=1}^{n} f(\underset{\sim}{X}_i - \underset{\sim}{\theta}) \,. \tag{1.2}$$

Then the Pitman estimator [Pitman (1939)] of $\underset{\sim}{\theta}$ is defined as

$$\hat{\underset{\sim}{\theta}}_{P,n} = \{\int \ldots \int \ell_n(\underset{\sim}{\theta}) d\underset{\sim}{\theta}\}^{-1} \int \ldots \int \underset{\sim}{\theta}\, \ell_n(\underset{\sim}{\theta}) d\underset{\sim}{\theta} \,. \tag{1.3}$$

AMS (1980) Subject Classifications: 62 C15, 62 H12

Keywords: Asymptotic representations; Bayes estimator; equivariance; median unbiasedness; MLE; optimality; qudratic loss; shrinkage estimator.

*Work supported partially by NSERC Grant No. A3088.

For the location model in (1.1), the Pitman estimator (PE) $\hat{\theta}_{P,n}$ is translation-equivariant and is unbiased for θ. Also, within the class of equivariant estimators, under a quadratic loss, $\hat{\theta}_{P,n}$ has minimal risk. Moreover, for p=1 (i.e., $\Theta \subset R$), $\hat{\theta}_{P,n}$ is minimax with respect to quadratic loss, and it is admissible under additional conditions. In fact, if we define the posterior density of θ (with respect to the uniform weight function) by

$$g(\theta) = \ell_n(\theta)/\{\int \dots \int \ell_n(\theta)d\theta\} \tag{1.4}$$

then by (1.3) and (1.4), we have

$$\hat{\theta}_{P,n} = \int \dots \int \theta g(\theta)d\theta = E\{\theta|X_1, \dots, X_n\}\ , \tag{1.5}$$

so that $\hat{\theta}_{P,n}$ is the Bayes estimator of θ with respect to the uniform weight function. A quadratic loss $L(T, \theta) = \|T - \theta\|^2$(or a more general form $\|T - \theta\|_W^2 = (T - \theta)'W(T - \theta)$, for some positive definite (p.d.) W) dominates the scenario in this respect.

Another brilliant idea of comparing two rival estimators (say, T_1 and T_2) of a common parameter θ, due to Pitman (1937), is the Pitman closeness criterion (PCC): T_1 is closer to θ than T_2, in the Pitman sense, if

$$P_\theta\{\|T_1 - \theta\| \le \|T_2 - \theta\|\} \ge 1/2, \quad \forall\ \theta \in \Theta, \tag{1.6}$$

with strict inequality holding for some θ. The Euclidean norm $\|\cdot\|$ may be replaced by a more general norm $L(\cdot,\cdot)$, and the corresponding criterion is then termed a generalized Pitman closeness criterion (GPCC). Since, in general, $T_1 - T_2$ may have a positive probability mass at 0, a somewhat different conclusion may evolve if in(1.6), $[\|T_1 - \theta\| \le \|T_2 - \theta\|]$ is replaced by $[\|T_1 - \theta\| < \|T_2 - \theta\|]$. This drawback may readily be eliminated by replacing (1.6) by the following:

T_1 is closer to θ than T_2 in the sense of the GPCC, if

$$P_\theta\{L(T_1, \theta) < L(T_2, \theta)\} \ge P_\theta\{L(T_1, \theta) > L(T_2, \theta)\}, \quad \forall\ \theta \in \Theta, \tag{1.7}$$

with strict inequality holding for some θ. If (1.7) holds for all T_2 belonging to a class $\mathcal{C}$, then T_1 is the Pitman closest estimator of θ, with respect to the loss function $L(\cdot,\cdot)$. Note that while comparing estimators with respect to a squared error (or quadratic loss) criterion, we need to confine ourselves to a class of estimators for which $E_{\theta}L(\underset{\sim}{T}, \underset{\sim}{\theta})$ exists, for all $\underset{\sim}{\theta} \in \Theta$. In this respect, the GPCC entails less restrictive conditions. However, verification of (1.7) for every $\underset{\sim}{T}_2 \in \mathfrak{C}$ may be, in general, quite difficult, and that is the main reason, why inspite of having some advantages (over the classical quadratic loss), the GPCC did not gain due popularity over the years. Another drawback of the GPCC is the possible lack of transitivity which generally holds for quadratic error losses; we may refer to Blyth (1972) for some nice discussion. Nevertheless, the recent developments in the general area of GPCC have opened up a broad avenue of idea tracks, and it is quite natural to inquire about the performance of various estimators in the light of the GPCC. Recent works of Ghosh and Sen (1989) and Nayak (1990) play a very important role in this context. We may also refer to Sen (1990) for a broad review of GPCC in various contexts.

Since the estimator in (1.3) and the GPCC in (1.6) – (1.7) were both sparked by Pitman (1937, 1939), it is quite natural to study how the Pitman estimator performs in light of GPCC. The present study solely relates to this issue. In Section 2, we consider the one-parameter location model and discuss the Pitman closest characterizations of Pitman estimators. In this study, the recent results of Ghosh and Sen (1989) and Nayak (1990) are incorporated in charaterizing suitable conditions under which Pitman estimators are Pitman closest ones. Section 3 deals with the multi-parameter case. Since Pitman estimators are not, in general, admissible (under quadratic loss) and are deominated by Stein-rule or shrinkage estimators, incorporating the recent results of Sen, Kubokawa and Saleh (1989), further characterizations of GPC of Pitman estimators are investigated. The concluding section deals with the asymptotic case. In this setup, the location model in (1.1) is extended to a more general setup, and some non-regular cases are also treated along with.

2. ONE PARAMETER LOCATION MODEL: GPC OF PE.

For the location model in (1.1), we have already remarked that the PE $\hat{\theta}_{P,n}$ in (1.3) is a translation-equivariant, unbiased estimator of θ. Moreover, if an uniformly minimum variance unbiased (UMVU) estimator of θ exists, then it is identical with the Pitman estimator of θ (although such UMVU estimators may not always exist). However, within the class of equivariant estimators, $\hat{\theta}_{P,n}$ has the minimal risk under squared error loss. Further, $\hat{\theta}_{P,n}$ has the minimax (risk) property (under quadratic loss), in the sense that

$$\sup_{\theta} E_{\theta}(T_n - \theta)^2 \geq \sup_{\theta} E_{\theta}(\hat{\theta}_{P,n} - \theta)^2, \quad \forall\ T_n . \tag{2.1}$$

Moreover, if $f(\cdot)$ is a normal pdf, $\hat{\theta}_{P,n}$ coincides with the sample mean $\overline{X}_n$ $\left(= \frac{1}{n} \sum_{i=1}^{n} X_i\right)$ (which is the maximumn likelihood estimator (MLE) of the normal mean), and hence, $\hat{\theta}_{P,n}$ is admissible (under quadratic error loss); see Blyth (1951). However, for a non-normal $f(\cdot)$, $\hat{\theta}_{P,n}$ is generally different from $\overline{X}_n$ or the corresponding MLE of θ, and is, often, computationally very cumbersome. Nevertheless, the Bayes interpretation in (1.5) remains in tact. In all of these characterizations, quadratic loss plays a vital role.

It was observed by Ghosh and Sen (1989) that in the characterization of Pitman closeness of estimators of θ, sufficiency and median unbiasedness play a vital role. Recall that an estimator T is median unbiased for θ if

$$P_\theta\{T \le \theta\} = P_\theta\{T \ge \theta\}, \ \ \forall\ \theta \in \Theta\,. \tag{2.2}$$

Also, note that if T is a sufficient statistic for the estimation of θ, then defining $\ell_n(\theta)$ as in (1.2), we have, by the factorization theroem,

$$\ell_n(\theta) = h_n(T, \theta)\ \ell_n^*(X_1, ..., X_n), \quad \theta \in \Theta\,, \tag{2.3}$$

where $\ell_n^*(\cdot)$ does not depend on θ and the pdf $h_n(T, \theta)$ depends on $X_1, ..., X_n$ only through T. As such, by (1.4) and (2.3), we obtain that $g(\theta) = h_n(T, \theta)/\int h_n(T, \theta)d\theta$, $\theta \in \Theta$, so that by (1.5),

$$\begin{aligned}\hat{\theta}_{P,n} &= \int \theta h_n(T, \theta)d\theta/\int h_n(T, \theta)d\theta \\ &= \psi_n(T) = \text{a function of the sufficient statsitic T} \\ &= \text{a sufficient statistic itself.}\end{aligned} \tag{2.4}$$

This feature of $\hat{\theta}_{P,n}$ along with its unbiasedness make it possible to use the classical Rao-Blackwell theorem to clinch its minimal risk property (under a quadratic error or a suitable covex loss). The sufficiency and unbiasedness of $\hat{\theta}_{P,n}$ may not, however, suffice for its Pitman closest characterization. In this context, we may refer to Theorem 1 of Ghosh and Sen (1989) which provides the desired characterization. Let $\mathfrak{C}$ be the class of all estimators of the form $U_n = \hat{\theta}_{P,n} + Z_n$ where (i) $\hat{\theta}_{P,n}$ and Z_n are independently distributed, and (ii) $\hat{\theta}_{P,n}$ is median unbiased for θ. Then, note that

$$P_\theta\{|\hat\theta_{P,n} - \theta| < |U_n - \theta|\} - P_\theta\{|\hat\theta_{P,n} - \theta| > |U_n - \theta|\}$$
$$= P_\theta\{2Z_n(\hat\theta_{P,n} - \theta) + Z_n^2 > 0\} - P_\theta\{2Z_n(\hat\theta_{P,n} - \theta) + Z_n^2 < 0\}$$
$$= P_\theta\{\hat\theta_{P,n} - \theta > -\tfrac{1}{2}Z_n, Z_n > 0\} + P_\theta\{\hat\theta_{P,n} - \theta < -\tfrac{1}{2}Z_n, Z_n < 0\}$$
$$- P_\theta\{\hat\theta_{P,n} - \theta < -\tfrac{1}{2}Z_n, Z_n > 0\} - P_\theta\{\hat\theta_{P,n} - \theta > -\tfrac{1}{2}Z_n, Z_n < 0\}$$
$$\geq 0, \text{ by (i) and (ii).} \tag{2.5}$$

Note that (i) holds whenever Z_n is ancillary, and equivariance considerations often lead to the estimators of the type U_n. However, (i) and (ii) are sufficient, not necessary. Looking at the penultimate step in (2.5), we may observe that if the conditional distribution of $\hat\theta_{P,n}$, given Z_n, has median θ (a.e. Z_n), then (2.5) is nonnegative. This property, we may term the uniform conditional median unbiasedness. Since

$$P_\theta\{\hat\theta_{P,n} \leq \theta\} = E_\theta[P\{\hat\theta_{P,n} \leq \theta | Z_n\}], \quad \forall\, \theta \in \Theta, \tag{2.6}$$

this uniform conditional median unbiasedness implies the usual median unbiasedness, but the converse may not be true. Under (i), however, (ii) ensures uniform conditional median unbiasedness. Considerations of minimal sufficiency and maximal invariants often lead us to consider the class of estimators $\mathfrak{C}$, where Z_n depends only on the maximal invariants. In this setup, the distributional independence of $\hat\theta_{P,n}$ and Z_n holds. So, we need to verify that $\hat\theta_{P,n}$ is median unbiased for θ. In the location model, given that $\hat\theta_{P,n}$ is unbiased for θ, we need to verify that the mean and median for $\hat\theta_{P,n}$ are the same, and a sufficient condition for this is that the distribution of $\hat\theta_{P,n}$ is symmetric about θ. As an illustrative example, consider the normal location model where $f(x; \theta)$ is the normal density with mean θ. For this Gaussian shift model, $\hat\theta_{P,n}$ coincides with the sample mean $\overline{X}_n$ which has a pdf symmetric about θ, and hence, we conclude that within the class $\mathfrak{C}$ of equivariant estimators, $\hat\theta_{P,n}$ is the closest one in the Pitman sense. For this normal mean model, $\hat\theta_{P,n}$ is known to be admissible and minimax (with respect to quadratic loss) within the entire class of estimators of θ. Thus, it is quite natural to inquire whether the Pitman Closest character of $\hat\theta_{P,n}$ remains in tact within the entire class of estimators of θ (which are not necessarily translation-equivariant). The answer is in the negative and may be verified easily with the following example due to Efron (1975). Let $f(x; \theta)$ be the normal density with mean θ and unit variance. Let then

$$\hat{\theta}_{E,n} = \overline{X}_n - \Delta(\overline{X}_n) = \hat{\theta}_{P,n} - \Delta(\hat{\theta}_{P,n}) , \tag{2.7}$$

where $\Delta(\cdot)$ is skew-symmetric (i.e., $\Delta(-x) = -\Delta(x)$, $x \in R$) and

$$\Delta(y) = \tfrac{1}{2} \min\left\{y,\ n^{1/2}\Phi(-yn^{1/2})\right\}, \quad y \geq 0 , \tag{2.8}$$

$\Phi(x)$ being the standard normal distribution function (d.f.). Since $P_\theta\{\hat{\theta}_{E,n} = \hat{\theta}_{P,n}\} = P_\theta\{\Delta(\overline{X}_n) = 0\} = 0$, we may directly use the proof of Efron (1975) and show that in the light of (1.7), $\hat{\theta}_{E,n}$ dominates $\hat{\theta}_{P,n}$. Recall that $\hat{\theta}_{E,n}$ is not a translation-equivariant estimator, and hence, it does not belong to the class $\mathfrak{C}$ of equivariant estimators. Thus it is possible to choose an estimator outside the class $\mathfrak{C}$ which may dominate $\hat{\theta}_{P,n}$ in the sense of Pitman closeness. Nevertheless, within the class $\mathfrak{C}$ of equivariant estimators, the PE $\hat{\theta}_{P,n} = \overline{X}_n$ is optimal in the light of the PCC as well.

As a second example, let us consider the simple exponential (location) model where

$$f(x; \theta) = \exp\{(\theta - x)\}\, I(x \geq 0), \ \ x \in R, \ \ \theta \in \Theta \subset R . \tag{2.9}$$

Let $X_{(1)} = \min\{X_1, ..., X_n\}$. Then $X_{(1)}$ is a sufficient statistic (for θ) and its pdf is given by

$$h_n(X_{(1)}, \theta) = n \exp\{-n(X_{(1)} - \theta)\} I(X_{(1)} \geq \theta) . \tag{2.10}$$

Hence, using (2.4), we obtain that

$$\hat{\theta}_{P,n} = X_{(1)} - 1/n \quad (\Rightarrow E(\hat{\theta}_{P,n}) = \theta) . \tag{2.11}$$

On the other hand, from Theorem 1 of ghosh and Sen (1989), we obtain that the Pitman closest estimator (within the class of translation-equivariant estimators) is given by $X_{(1)} - n^{-1}\log 2 = \hat{\theta}_{P,n} + n^{-1}(1-\log 2)$. Thus, here the PE $\hat{\theta}_{P,n}$ is not the Pitman closest one, but can be made so by a small shift $(-n^{-1}(1-\log 2))$.

As a third example, consider the uniform distribution on $[\theta - \frac{1}{2}\delta, \theta + \frac{1}{2}\delta]$, $\theta \in \Theta \subset R$, $\delta \in R^+$, for which

$$f(x, \theta) = \delta^{-1} I(\theta - \tfrac{1}{2}\delta \leq x \leq \theta + \tfrac{1}{2}\delta) . \tag{2.12}$$

Let $X_{(1)} = \min\{X_1, ..., X_n\}$ and $X_{(n)} = \max\{X_1, ..., X_n\}$. then $(X_{(1)}, X_{(n)})$ is (jointly) sufficient for (θ, δ), and an unbiased estimator of θ based on this sufficient statistic is $M_n = (X_{(1)} + X_{(n)})/2$. Also, note that $R_n = (X_{(n)} - X_{(1)})$ estimates δ (and is translation-invariant). In this model, $P_{\theta,\delta}\left\{[X_{(1)} < \theta - \frac{1}{2}\delta] \cup [X_{(n)} > \theta + \frac{1}{2}\delta]\right\} = 0$, and the joint pdf of $(X_{(1)}, X_{(n)})$ is given by

$$n(n-1)\,\delta^{-n}\,R_n^{n-2}\,I\left(\theta - \tfrac{1}{2}\delta \le X_{(1)} \le X_{(n)} \le \theta + \tfrac{1}{2}\delta\right). \tag{2.13}$$

As such, using (2.4), we obtain that $\hat{\theta}_{P,n} = M_n$. Note that M_n is translation-equivariant while R_n is translation-invariant. Thus, if we consider the group G_2 of affine transformations $g_{a,b}(X_1, ..., X_n) = (a + bX_1, ..., a + bX_n)$, $a \in R$, $b \in R^+$, and a loss function $L(T, \theta) = \rho((T - \theta)/\delta)$ with an arbitrary nonnegative ρ, then any equivariant estimator based on $(X_{(1)}, X_{(n)})$ is of the form $M_n + v(R_n)$. Moreover, the conditional pdf of M_n, given R_n, is

$$g(m|R) = (\delta - R)^{-1}\,I\left(\theta - \tfrac{1}{2}(\delta - R) \le m \le \theta + \tfrac{1}{2}(\delta - R)\right), \tag{2.14}$$

which is symmetric about θ (a.e. R_n). Hence the uniform conditional median unbiasedness property holds for M_n. Thus, within the class of equivariant estimators of the form $M_n + v(R_n)$, $\hat{\theta}_{P,n} = M_n$ is the Pitman closest estimator of θ. The question may arise whether this PC characterization of M_n holds for a larger class of estimators which may not be translation-equivariant. The answer is again in the negative. Towards this consider the one-parameter model where δ is known (and take $\delta = 1$ without any loss of generality). Then, the conditional distribution of M_n, given R_n, is uniform on $[\theta - \frac{1}{2}(1 = R_n), \theta + \frac{1}{2}(1 - R_n)]$, and $M_n - \theta$ has a (conditional or unconditional) law independent of θ. Hence, we may construct an Efron-type estimator (viz. (2.7)) : $\hat{\theta}_{E,n} = M_n - \Delta(M_n, R_n)$, where in the formulation of $\Delta(\cdot,\cdot)$, we need to use the conditional d.f. of $M_n - \theta$, given R_n, instead of the normal d.f. Φ in (2.8). For a non-Gaussian shift, however, even under quadratic loss, the PE may not be admissible within the broader class of estimators which may not be equivariant, and hence, in this sense, the picture is not too different for the PC criterion.

Let us consider another important example. Let $X_1, ..., X_k$ $(k \ge 1)$ be k independent r.v.'s, each having tyhe Poisson (θ) distribution. Then $T = \sum_{i=1}^{k} X_i$ is sufficient for θ and $\overline{X} = k^{-1}T$ is unbiased for θ. Hence, the MLE $\overline{X}$ is an optimal estimator of θ (with respect to quadratic loss or a convex loss function, in general). By (1.3), we have here

$$\hat{\theta}_{P,n} = \left(\int_0^\infty \theta e^{-k\theta}\theta^T d\theta\right) \Big/ \left(\int_0^\infty e^{-k\theta}\theta^T d\theta\right)$$

$$= k^{-1}\left(\int_0^\infty y^{T=1}e^{-y}dy\right) \Big/ \left(\int_0^\infty y^T e^{-y}dy\right)$$

$$= k^{-1}\Gamma(T+2)/\Gamma(T+1) = (T+1)/k$$

$$= \overline{X} + k^{-1}. \tag{2.15}$$

Thus, the MLE ($\overline{X}$) and the PE ($\overline{X} + k^{-1}$) are not the same in this simple example. Moreover, the Poisson distribution does not belong to the location-scale family, although $\theta = EX$. Hence, translation-equivariance is not important for the Poisson distribution. Note that

$$E_\theta(\hat{\theta}_{P,n}) = \theta + k^{-1}, \tag{2.16}$$

while using the identity that $P_\theta\{T \le r\} = P\{\chi^2_{2(r+1)} > 2k\theta\}$, for every $r = 0, 1, 2, \ldots$, we obtain that

$$\begin{aligned} P_\theta\{\hat{\theta}_{P,n} \le \theta\} - P_\theta\{\hat{\theta}_{P,n} \ge \theta\} &\le 2P_\theta\{\hat{\theta}_{P,n} \le \theta\} - 1 \\ &= 2P_\theta\{T \le k\theta - 1\} - 1 \\ &= 2P_\theta\{\chi^2_{2[k\theta]} > 2k\theta\} - 1 \qquad ([s] = \text{integer part of } s) \\ &\le 2P_\theta\{\chi^2_{2[k\theta]} > [2k\theta]\} - 1 \\ &< 0, \text{ as } P\{\chi^2_m > m\} < \tfrac{1}{2}, \quad \forall\, m \ge 1. \end{aligned} \tag{2.17}$$

Hence, for every $k(\ge 1)$, the PE $\hat{\theta}_{P,n} = \overline{X} + k^{-1}$ is not median unbiased for θ. Hence, even if we would have confined ourselves to translation-equivariant estimators of θ, the PE $\hat{\theta}_{P,n}$ is not the Pitman closest one within this class. As the median of $T - k\theta$ depends on the unknown θ, a simple shift as in after (2.11) is not possible here.

In all the examples cited above, sufficiency plays a vital role. Sans sufficiency, the picture may be quite different. As an illustration, consider the double-exponential pdf

$$f(x;\theta) = \tfrac{1}{2}\exp\{-\textstyle\sum_{i=1}^n |X - \theta|\}, \quad x \in R, \quad \theta \in \Theta \subset R, \tag{2.18}$$

so that $\ell_n(\theta) = 2^{-n} \exp\{-\sum_{i=1}^{n}|X_i - \theta|\}$. The MLE of θ is the sample median ($\tilde{X}_n$, say), but the computation of the PE $\hat{\theta}_{P,n}$ poses serious problems, particularly when n is not small. This inherent drawback of PE has been largely eliminated by taking recourse to asymptotic methods resting on some representations of PE in terms of independent summands, and we shall consider them in the concluding section. Nevertheless, the PE, when computable, retains the minimal risk property within the class of equivariant estimators (with respect to quadratic loss). But in the absence of ancillarity, it may be difficult to claim that for U_n, $\hat{\theta}_{P,n}$ and Z_n are independently distributed, and hence, (2.5) may not work out when $\hat{\theta}_{P,n}$ is not a sufficient statistic. Thus, a different method of characterizing possible PC characters of PE is needed in this setup. In an asymptotic setup, we shall see in Section 4 that the BAN characterization of PE takes care of the situation in a very convenient manner.

3. GPC OF PE IN THE MULTIVARIATE CASE

The characterization of translation-equivariance and unbiasedness of PE of location parameters remains in tact for the multivariate models as well. Moreover, the Bayes characterization in (1.5) and sufficiency-characterizations in (2.4) extend to the multivariate case. As such, we may consider the following PC characeterization of PE in the multivariate case:

Let $\hat{\underset{\sim}{\theta}}_{P,n}$ be the PE of $\underset{\sim}{\theta}$ and consider the class $\mathfrak{C}$ of all estimators of the form $\underset{\sim}{U}_n = \hat{\underset{\sim}{\theta}}_{P,n} + \underset{\sim}{Z}_n$, where (i) $\hat{\underset{\sim}{\theta}}$ is multivariate median unbiased for $\underset{\sim}{\theta}$ and (ii) $\hat{\underset{\sim}{\theta}}_{P,n}$ and $\underset{\sim}{Z}_n$ are independently distributed. Then, within the class $\mathfrak{C}$, the PE $\hat{\theta}_{P,n}$ is the Pitman-closest estimator.

It may be recalled that a vector-estimator $\underset{\sim}{T}$ is said to be multivariate median unbiased (MMU) for $\underset{\sim}{\theta}$ if for every $\underset{\sim}{\ell}$, $\underset{\sim}{\ell}'(\underset{\sim}{T} - \underset{\sim}{\theta})$ is median unbiased for 0 [see, Sen (1989)]. If the distribuiton of $\underset{\sim}{T}$ is diagonally symmetric about $\underset{\sim}{\theta}$, then the multivariate median unbiasedness holds, although the converse may not be true. Here also, if the conditional distribution of $\hat{\underset{\sim}{\theta}}_{P,n}$, given $\underset{\sim}{Z}_n$, is diagonally symmetric about $\underset{\sim}{\theta}$, then the PC characterization of $\hat{\underset{\sim}{\theta}}_{P,n}$ holds without requiring $\hat{\underset{\sim}{\theta}}_{P,n}$ and $\underset{\sim}{Z}_n$ to be independently distributed. The proofs of these results are very similar to the ones considered by Ghosh and Sen (1989) and Sen (1989), and hence, are not reproduced here. In this multivariate case, in (1.7), $L(\underset{\sim}{T}, \underset{\sim}{\theta})$ is taken as $\|\underset{\sim}{T} - \underset{\sim}{\theta}\|^2_{\underset{\sim}{W}}$ for a suitable p.d.$\underset{\sim}{W}$., and the interesting point is that the definition of the PE in (1.3) is independent of the choice of this $\underset{\sim}{W}$, and the PC characterization of $\hat{\underset{\sim}{\theta}}_{P,n}$ (within suitable class of equivariant estimators) holds for all $\underset{\sim}{W}$. However, the MMU property cited above may not generally hold for a multivariate distribution. [For the multivariate location model, it holds for the entire class of elliptically symmetric distributions.] Towards this, we consider the following simple example. Let $\underset{\sim}{X} = (X_1, ..., X_p)'$ have the p($\geq$ 1)-variate Poisson distribution $P(\underset{\sim}{\theta})$, $\theta = (\theta_1, ..., \theta_p)'$ where

$$P\{\underset{\sim}{X} = \underset{\sim}{x}\} = e^{-(\theta_1+...+\theta_p)}\, \theta_1^{x_1} \ldots \theta_p^{x_p}, \text{ for } x_j \geq 0,\ 1 \leq j \leq p\ . \tag{3.1}$$

In this case, using (1.3), we have

$$\hat{\underset{\sim}{\theta}}_{P,n} = (X_1 + 1, ..., X_p + 1)', \tag{3.2}$$

[see (2.15) for k = 1], so that $E\hat{\underset{\sim}{\theta}}_{P,n} = \underset{\sim}{\theta} + \underset{\sim}{1}$; $\underset{\sim}{1} = (1, ..., 1)'$. Thus, $\hat{\underset{\sim}{\theta}}_{P,n}$ is not unbiased for $\underset{\sim}{\theta}$, and hence, the usual unbiasedness property of $\hat{\underset{\sim}{\theta}}_{P,n}$ does not hold here. Further, we may proceed as in (2.17) (with k = 1) and conclude that none of the p marginal distributions of $1 + X_1, ..., 1 + X_p$ has the median equal to the mean (θ_j, $1 \leq j \leq p$), so that the MMU property for $\hat{\underset{\sim}{\theta}}_{P,n}$ can not be true. Thus, even here, $\underset{\sim}{X}$ is sufficient for $\underset{\sim}{\theta}$, the Pitman estimator $\hat{\underset{\sim}{\theta}}_{P,n}$ does not possess the optimality property either with respect to quadratic loss or the Pitman closeness criterion. Another example is the multivariate negative exponential pdf

$$f(\underset{\sim}{x}; \underset{\sim}{\theta}) = \exp\{-(\underset{\sim}{x} - \underset{\sim}{\theta})'\underset{\sim}{1}\}\, I(\underset{\sim}{x} \geq \underset{\sim}{\theta}), \quad \underset{\sim}{x} \in R^p, \underset{\sim}{\theta} \in R^p. \tag{3.3}$$

Let $\underset{\sim}{X}_i = (X_{i1}, ..., X_{in})$ be n iidrv's with the pdf (3.3). For each j (= 1, ..., p), we let $X_{(1)j} = \min_{i \leq n} X_{ij}$, and then using (2.4), we have

$$\hat{\underset{\sim}{\theta}}_{P,n} = (X_{(1)1} - 1/n, ..., X_{(1)p} - 1/n)'. \tag{3.4}$$

In this case, although marginally, $X_{(1)j} - n^{-1}\log 2$ is the Pitman closest estimator of θ_j, $1 \leq j \leq p$, the MMU property does not hold, and hence, even with the adjustment $n^{-1}(1-\log 2)\underset{\sim}{1}$, $\hat{\underset{\sim}{\theta}}_{P,n}$ may not be the Pitman closest estimator of $\underset{\sim}{\theta}$(within the class of translation-equivariant estimators). This example relates to a 'nonregular' case as the range of $\underset{\sim}{X}$ depends on $\underset{\sim}{\theta}$. Even in a regular case, with respect to a quadratic loss, admissibility and minimaxity results for the classical MLE require delicate treatments in the multivariate case. For estimating the mean vector ($\underset{\sim}{\theta}$) of a p-variate normal distribution, the MLE ($\overline{\underset{\sim}{X}}_n$) is known to be admissible for p = 1 or 2. For $p \geq 3$, Stein (1956) showed that $\overline{\underset{\sim}{X}}_n$ is not admissible, and there exists some other (generally, non-equivariant) estimators which dominate $\overline{\underset{\sim}{X}}_n$ in quadratic loss. Such estimators are known as <u>shrinkage</u> or <u>Stein-rule</u> estimators. The past three decades have witnessed a phenomenal growth of the literature of research on such shrinkage estimators. In the normal case, the MLE and PE are the same, and, in general, in the presence of sufficient statistics, one is a function of the other. In view of this intricate relationship between the PE and MLE, we may wonder whether such Stein-rule versions exist for the PE, and, if they do so, whether they dominate their classical forms in the light of the PCC as well. The PC dominance of Stein-rule estimators has only been studied in the recent past. Sen, Kubokawa and Saleh (1989) have shown that for the multivariate normal mean estimation problem, the usual shrinkage versions of the MLE dominates the classical MLE under PCC as well. Moreover, such a dominance holds for $p \geq 2$ (compared to $p \geq 3$, under quadratic loss). Even for $p = 1$, the Efron (1975) estimator in (2.7) has a similar dominance property (over $\overline{X}_n$). Such shrinkage versions are not translation-equivariant, and for the multivariate normal mean problem, the MLE and PE are the same (and are translation-eqivariant). Thus, it is clear from the above discussion that though within the class of translation-equivariant estimators, the PE may be PC, it may not be, in general, PC within a large class of estimators where translation-equivariance may not hold. Moreover, the PC dominance of a shrinkage version of $\overline{\underset{\sim}{X}}_n$ (over the classical PE $\overline{\underset{\sim}{X}}_n$), in the multivariate normal mean case, depends on some intricate properties of noncentral chi-square distributions, and, at the present time, it is not precisely known how, for a possibly nonnormal distribution, such PC dominance results can be extended for the finite sample size case, even when there exists sufficient statistics. Characterization of such PC dominance results (in the exact sense) constitutes an important research topic. As a first step towards this, we shall show in the next section that in an asymptotic setup, because of the affinity of PE and MLE, the distribution theory of shrinkage MLE (studied, for example, by Sen (1986b)) provides a clear answer to this query; in some cases, the finite sample analogues may be worked out on individual basis.

4. GPC OF PE: THE ASYMPTOTIC CASE

The formulation in (1.3) through (1.5) enables us to define a PE for a general model (where θ need not be a location parameter). In an asymptotic setup, it is not necessary to be confined only to the location model, and the results to follow pertain to a much more general setup. Much of these developments rests on the theor of Hájek-LeCam regular estimators [viz., Hájek (1970) and Inagaki (1973)] for which certain (broad) conditions on the underlying pdf $f(\underset{\sim}{x}' \ \underset{\sim}{\theta})$ suffice. These conditions have been very elaborately formulated in Inagaki (1973), and hence, we omit these details by cross reference to his paper. There is, however, a basic difference between the LeCam-Hájek-Inagaki setup and ours, and this will add some extra generality to our formulation.

Let $\hat{\underset{\sim}{\theta}}_n$ be the MLE of $\underset{\sim}{\theta}$, so that

$$\ell_n(\hat{\underset{\sim}{\theta}}_n) = \sup\{\ell_n(\underset{\sim}{\theta}) : \underset{\sim}{\theta} \in \Theta\} \ . \tag{4.1}$$

We write

$$\ell_n(\underset{\sim}{\theta}) = \ell_n(\hat{\underset{\sim}{\theta}}_n) \cdot \exp\{-[\log \ell_n(\hat{\underset{\sim}{\theta}}_n) - \log\ell_n(\theta)]\}, \underset{\sim}{\theta} \in \Theta . \tag{4.2}$$

By (1.3) and (4.2), we have then

$$\hat{\underset{\sim}{\theta}}_{P,n} = \hat{\underset{\sim}{\theta}}_n + \frac{\int \dots \int (\underset{\sim}{\theta} - \theta) \exp\{-[\log\ell_n(\hat{\underset{\sim}{\theta}}_n) - \log\ell_n(\underset{\sim}{\theta})]\}d\underset{\sim}{\theta}}{\int \dots \int \exp\{-[\log\ell_n(\hat{\underset{\sim}{\theta}}_n) - \log\ell_n(\underset{\sim}{\theta})]\}d\underset{\sim}{\theta}} . \tag{4.3}$$

Invoking the LeCam-Hájek-Inagaki representation for the MLE $\hat{\underset{\sim}{\theta}}_n$, we obtain that with $P_{\underset{\sim}{\theta}}$-probability 1 uniformly in $\underset{\sim}{\theta}$ belonging to a compact set K, as $n \to \infty$,

$$n^{1/2} \mathfrak{I}(\underset{\sim}{\theta})(\hat{\underset{\sim}{\theta}}_{n}, - \underset{\sim}{\theta}) = n^{-1/2} \frac{\partial}{\partial \underset{\sim}{\theta}} \log\ell_n(\underset{\sim}{\theta}) + \underset{\sim}{o}(1) , \tag{4.4}$$

where

$$n^{-1/2} \frac{\partial}{\partial \underset{\sim}{\theta}} \log\ell_n(\underset{\sim}{\theta}) \sim N(\underset{\sim}{0}, \mathfrak{I}(\underset{\sim}{\theta})), \text{ under } P_{\underset{\sim}{\theta}} , \tag{4.5}$$

and $\mathfrak{I}(\underset{\sim}{\theta})$ denotes the information matrix:

$$\mathfrak{I}(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}\left\{n^{-1} \frac{\partial}{\partial \underset{\sim}{\theta}} \log\ell_n(\underset{\sim}{\theta}) \frac{\partial}{\partial \underset{\sim}{\theta}'} \log\ell_n(\underset{\sim}{\theta})\right\} . \tag{4.6}$$

[See, for example, Theorem 8.1 of Ibragimov and Has'minskii (1981)]. The smoothness conditions on $f(\underset{\sim}{x}; \underset{\sim}{\theta})$ implicit in (4.4) ensure that with $P_{\underset{\sim}{\theta}}$-probability 1, uniformly in $\underset{\sim}{\theta} \in K$, as $n \to \infty$,

$$\log\ell_n(\hat{\underset{\sim}{\theta}}_n) - \log\ell_n(\underset{\sim}{\theta}) = \frac{n}{2}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta})' \mathfrak{I}(\underset{\sim}{\theta})(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})\{1 + o(1)\} , \tag{4.7}$$

while by (4.1) and an assumed convexity condition on K [c.f. Ibragimov and Has'minskii (1981, p. 83)], outside K, (4.7) can be made $0(n)\|\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}\|$, so that by (4.3),

$$\hat{\underset{\sim}{\theta}}_{P,n} = \hat{\underset{\sim}{\theta}}_n + \underset{\sim}{o}_p(n^{-1/2}), \text{ as } n \to \infty . \tag{4.8}$$

At this point, we may remark that in the context of (asymptotic) minimal risk property of the PE, one needs a stronger version of (4.8) where $o_p(n^{-1/2})$ is replaced by a stronger mode. For example, under a quadratic error loss criterion, we need to show that

$$nE_{\theta}\left\{\|\hat{\theta}_{P,n} - \hat{\theta}_n\|_W^2\right\} \to 0 \text{ as } n\to\infty, \text{ uniformly in } \theta \in K, \tag{4.9}$$

and this stronger mode of convergence may call for additional regularity conditions. These have been studied in detail in the literature; the books by Ibragimov and Has'minskii (1981) and LeCam (1986) are especially noteworthy in this context. For our purpose, however, the weaker relation in (4.8) suffices.

Consider now the class $\mathcal{C}$ of AN (asymptotically normal) estimators $\{T_n\}$ of θ, such that

$$\begin{pmatrix} n^{1/2}(T_n - \theta) \\ n^{-1/2}\frac{\partial}{\partial\theta}\log\ell_n(\theta) \end{pmatrix} \sim \mathcal{N}_{2p}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\nu & I \\ I & \mathcal{I}(\theta)\end{pmatrix}\right), \tag{4.10}$$

where

$$\nu - \mathcal{I}^{-1}(\theta) \text{ is positive semi-definite.} \tag{4.11}$$

The MLE $\hat{\theta}_n$ belongs to this class and is a BAN estimator in the sense that the corresponding $\nu = \mathcal{I}^{-1}(\theta)$. By virtue of (4.8), the PE $\hat{\theta}_{P,n}$ is also BAN. Hence, by an appeal to Theorem 3.1 of Sen (1986a), we conclude that within the class $\mathcal{C}$, the PE is asymptotically Pitman closest. For this characterization, we do not need (4.9), and hence, in general, we may need less restrictive regularity conditions than those pertaining to the minimal risk property of the PE.

In the multi-parameter case, the class $\mathcal{C}$ does not include shrinkage or Stein-rule estimators, which are not (even) asymptotically normal! Sen (1986b) has shown that the shrinkage estimation theory applies neatly in an asymptotic setup to general MLE, even in more complex situations. By virtue of (4.8), we are in a position to adapt the same (shrinkage estimation theory) for the PE. In this context, we may define the unrestricted PE (UPE) as in (1.3), and this can be done in a general setup (including the location model in (1.1) as a particular case). As contrasted to the parameter space Θ, we may conceive of a restricted parameter space Θ^* ($\subset\Theta$). Such a restricted parameter space may be formulated by linear restraints on θ or even by suitable nonlinear ones. Thus, in (1.3), (1.4) and (1.5), replacing the domain Θ by Θ^*, we may define the restricted PE (RPE) $\hat{\theta}^*_{P,n}$. As such, we may consider a version of the (log-) likelihood ratio test statistic:

$$\mathcal{L}_n = -2\log\left\{\ell_n(\hat{\theta}^*_{P,n})/\ell_n(\hat{\theta}_{P,n})\right\}. \tag{4.12}$$

With respect to the usual quadratic loss, one may then consider a shrinkage version of the PE as

$$\hat{\theta}^S_{P,n} = \hat{\theta}_{P,n} + (p-2)\mathcal{L}_n^{-1}(\hat{\theta}^*_{P,n} - \hat{\theta}_{P,n}), \tag{4.13}$$

where, we need to confine ourselves to $p \geq 3$. Following the line of attack of Sen (1986b) and using (4.8), it follows that in terms of <u>asymptotic distributional risks (ADR)</u>, $\hat{\theta}^S_{P,n}$ dominates $\hat{\theta}_{P,n}$ whenever θ is "close to" Θ^*, while for any (fixed) θ not belonging to Θ^*, $\hat{\theta}^S_{P,n}$ and $\hat{\theta}_{P,n}$ are asymptotically equivalent (with respect to their ADR). In this context, the "closeness" of θ to Θ^* is defined by a Pitman-type alternative:

$$\theta \in \mathcal{K}_n = \left\{\theta: d(\theta, \Theta^*) \leq \mathcal{K}n^{-1/2}\right\}, \tag{4.14}$$

where $\mathcal{K}$ ($< \infty$) is arbitrary and $d(\theta, \Theta^*)$ refers to the distance between θ, defined in the usual fashion. Outside the domain $\mathcal{K}_n$, $\mathcal{L}_n$ becomes asymptotically large (in probability), and hence, $\hat{\theta}^S_{P,n}$ and $\hat{\theta}_{P,n}$ becomes asymptotically equivalent, in probability. The ADR of $\hat{\theta}^S_{P,n}$, under (4.14), is the same as the ADR of the shrinkage MLE, treated in detail in Sen (1986b), and hence, we do not repeat this here. It may be noted that (4.15) can be rewritten as

$$\hat{\theta}^S_{P,n} = \hat{\theta}^*_{P,n} + \left\{1 - (p-2)\mathcal{L}_n^{-1}(\hat{\theta}_{P,n} - \hat{\theta}^*_{P,n})\right\}, \tag{4.15}$$

which is more analogous to the usual Stein-rule versions of the classical MLE. Also, instead of the shrinkage factor $(p-2)$, some other nonnegative factor c: $0 < c < 2\ (p-2)$, may be used. the choice of $c = p - 2$ is governed by the minimization of the ADR of $\hat{\theta}^S_{P,n}$ on Θ^*. Moreover, for the location model, when Θ^* refers to a given pivot θ_o, $\hat{\theta}^*_{P,n}$ is to be replaced by θ_o, and instead of $\mathcal{L}_n$ one may use $n\|\hat{\theta}_{P,n} - \theta_o\|^2_{\hat{I}_n}$ where $\hat{I}_n$ is the estimated $\mathcal{I}(\theta)$, from the given sample. In the above development, we have $p \geq 3$, and, as has been pointed out in Sen, Kubokawa and Saleh (1989), for the Pitman closeness study, it may not be necessary to limit oneself to $p \geq 3$. With this in mind, we consider the following <u>shrinkage PE</u> (SPE) of θ:

$$\hat{\theta}^{*S}_{P,n} = \hat{\theta}^*_{P,n} + (1 - c\mathcal{L}_n^{-1})(\hat{\theta}_{P,n} - \hat{\theta}^*_{P,n}), \tag{4.16}$$

where

$$0 < c < (p - 1)(3p + 1)/2p ; \quad p \geq 2 . \tag{4.17}$$

In view of (4.4), (4.8) and the main result of Sen, Kubokawa and Saleh (1989), we can show that under (4.14), $\hat{\underset{\sim}{\theta}}^{*S}_{P,n}$ asymptotically dominates $\hat{\underset{\sim}{\theta}}_{P,n}$ in the sense of Pitman closeness. For any fixed $\underset{\sim}{\theta}$ (outside Θ^*), $\hat{\underset{\sim}{\theta}}^{*S}_{P,n}$ and $\hat{\underset{\sim}{\theta}}_{P,n}$ are asymptotically equivalent in the PC sense.

It is well known that the usual Stein-rule estimator (for the multivariate normal mean) can be improved (with respect to quadratic loss) by its positive-rule version. Sen, Kubokawa and Saleh (1989) showed that a similar conclusion holds with respect to the Pitman closeness criterion. As such, parallel to (4.16), we consider the following positive-rule version of the SPE:

$$\begin{aligned} \hat{\underset{\sim}{\theta}}^{*S+}_{P,n} &= \hat{\underset{\sim}{\theta}}^{*}_{P,n} + (1 - c\mathcal{L}_n^{-1})^{+}(\hat{\underset{\sim}{\theta}}_{P,n} - \hat{\underset{\sim}{\theta}}^{*}_{P,n}) \\ &= \begin{cases} \hat{\underset{\sim}{\theta}}^{*}_{P,n} , & \text{if } \mathcal{L}_n \leq c , \\ \hat{\underset{\sim}{\theta}}^{*S}_{P,n} , & \text{if } \mathcal{L}_n > c , \end{cases} \end{aligned} \tag{4.18}$$

(where $y^{+} = \max\{0, y\}$). By virtue of (4.4), (4.5), (4.7) and (4.8), in the asymptotic case, we are in a position to incorporate (2.4) through (2.7) of Sen, Kubokawa and Saleh (1989) and conclude that in the sense of PC too, $\hat{\underset{\sim}{\theta}}^{*S+}_{P,n}$ asymptotically dominates $\hat{\underset{\sim}{\theta}}^{*S}_{P,n}$ (as well as $\underset{\sim}{\theta}_{P,n}$). This dominance is also perceptible for $\underset{\sim}{\theta} \in \mathcal{K}_n$ (in (4.14)), and outside $\mathcal{K}_n$, they are equivalent (asymptotically) in the PC sense.

It may be remarked that in this asymptotic setup, the assumed regularity conditions pertain to the asymptotic (joint) normality of the UPE $\hat{\underset{\sim}{\theta}}_{P,n}$ and the (noncentral) chi square distribution of the likelihood ratio statistic $\mathcal{L}_n$. These results may not apply when either of these distributional assumptions may not hold. We illustrate this with the following examples.

Example 1. Multivariate Cauchy distribution with the pdf:

$$f(\underset{\sim}{x}; \underset{\sim}{\theta}) = \pi^{-(p+1)/2}\Gamma\left(\frac{p+1}{2}\right)\left\{1 + (\underset{\sim}{x} - \underset{\sim}{\theta})'(\underset{\sim}{x} - \underset{\sim}{\theta})\right\}^{-(p+1)/2} , \tag{4.19}$$

where $\underset{\sim}{\theta} \in \Theta \subset R^p$ and $\underset{\sim}{x} \in R^p$. For $p = 1$, (4.19) reduces to the standard Cauchy pdf. A scale factor can easily be accommodated in (4.19). Although in this case the PE $\hat{\underset{\sim}{\theta}}_{P,n}$ of $\underset{\sim}{\theta}$ is computationally very cumbersome (and has to be obtained by iterative procedures), the regularity conditions pertaining to (4.4) and (4.8) all hold. Hence, the shrinkage and positive-rule versions of the PE discussed earlier work out well, and their asymptotic dominance in the PC sense holds.

Example 2. Multivariate logistic distribution with the pdf

$$f(\underset{\sim}{x}; \underset{\sim}{\theta}) = p! \exp\left\{-(\underset{\sim}{x} - \underset{\sim}{\theta})'\underset{\sim}{1}\right\}\left\{1 + \sum_{j=1}^{p} e^{-(x_j - \theta_j)}\right\}^{-(p+1)} \tag{4.20}$$

$\underset{\sim}{\theta} \in \underset{\sim}{\Theta} \subset R^p$ and $\underset{\sim}{x} \in R^p$. Here also $\underset{\sim}{1}$ may be replaced by a vector $\underset{\sim}{\gamma} = (\gamma_1, ..., \gamma_p)'$ where the γ_j represent the scale factors. The exact computation of the PE $\hat{\underset{\sim}{\theta}}_{P,n}$ of $\underset{\sim}{\theta}$ is a formidable task, especially for large n. However, the regularity conditions pertaining to (4.4) and (4.8) all hold, and hence, the asymptotic theory of the shrinkage and postive-rule versions of $\hat{\underset{\sim}{\theta}}_{P,n}$ works out well. In this case, to obtain the PE by iteration, it may be convenient to start with the Wilcoxon score estimators of the θ_j $(1 \leq j \leq p)$ (which are the median of the midranges for each coordinate).

Example 3. Multivariate negative exponential pdf in (3.3). In this case, the PE $\hat{\underset{\sim}{\theta}}_{P,n}$ of $\underset{\sim}{\theta}$ is given by (3.4), and is easy to compute. However, here

$$n(\hat{\underset{\sim}{\theta}}_{P,n} - \underset{\sim}{\theta}) = O_p(1), \quad n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}) = O_p(1), \tag{4.21}$$
$$n(\hat{\underset{\sim}{\theta}}_{P,n} - \hat{\underset{\sim}{\theta}}_n) = 1$$

and the MLE $\hat{\underset{\sim}{\theta}}_n$ does not pertain to (4.4) – (4.5); it pertains to a multivariate exponential law (with the scaling factor n instead of the conventional $n^{1/2}$). Similarly, for the likelihood ratio test statistic, we would have a different asymptotic distribution. Hence, the proposed shrinkage and positive-rule versions of the PE may not have the desired asymptotic dominance property, and a modified approach is needed.

Example 4. Multivariate Poisson distribution in (3.1). The PE $\hat{\underset{\sim}{\theta}}_{P,n}$ is given by (3.2). Recall that $(X_j - \theta_j)/\sqrt{\theta_j}$, j = 1, ..., p, are all asymptotically (as $\theta_j \rightarrow \infty$) normally distributed with zero mean and unit variance, and they are independent too, Hence, in this case the proposed asymptotics work out well, when $\theta_j \rightarrow \infty$, for each j = 1, ..., p. For small values of θ_j, a different approach may be more appropriate.

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