THE INSTITUTE OF STATISTICS

UNIVERSITY OF NORTH CAROLINA SYSTEM

ANALYSIS OF MEANS FOR RELIABILITY CHARACTERISTICS

by

Jye-Chyi Lu and John Jones

Institution of Statistics Mimeograph Series No. 1989

November, 1990

NORTH CAROLINA STATE UNIVERSITY
Raleigh, North Carolina

Analysis of Means for Reliability Characteristics

Jye-Chyi Lu and John Jones

Department of Statistics

North Carolina State University, Raleigh, NC 27695-8203

KEY WORDS: Analysis of means; Interval Censored Data; Weibull and lognormal distributions; Percentile Lifetime; Pivotal Statistics; Normal Approximation.

ABSTRACT

Analysis of Means (ANOM) is a rational extension of the Shewhart control chart for comparing products from different design configurations in off-line quality improvement. ANOM is a powerful graphic-tool for engineers to present and interpret experimental results. We apply the ANOM idea to the analysis of lifetime data resulting from experiments planned for enhancing product reliability. The maximum likelihood method is utilized to estimate parameters of the Weibull and lognormal distributions for interval censored and accelerated life-tested data. Decision limits for the ANOM chart are constructed using quantiles of the distributions of the pivotal statistics. Asymptotic results are derived for comparison with the finite sample inferences obtained by Monte Carlo simulations.

1. INTRODUCTION

The article published in Technometrics (Hoadley and Kettenring, 1990) shows that statisticians have not done a good job communicating with engineers and physical scientists. Graphical analysis serves as a powerful tool for statisticians to exchange experiences with industrial practitioners. Nowadays, most engineers know how to use statistical process control charts to monitor the process parameters. Ott (1967) suggested the use of Analysis of Means (ANOM) to present and analyze experimental results. ANOM is a rational extension of the Shewhart control chart and is an useful supplement (or alternative) to the analysis of variance (ANOVA). Applications of ANOM to solve various industrial problems are provided in Ott (1975). Recently, the ANOM technique has been extended from the study of one-way layout (Raming, 1983) to testing interactions from multifactor experiments (Nelson, 1988), and the analysis of signal and noise (Ullman, 1989). Since ANOM is inherently a graphical procedure similar to control charts, it is apt to receive increasing interest from engineers and other industrial personnel for off-line quality improvement (see p. 170 of Hogg, *et. al.*, 1985).

For comparison of mean responses in a one-way classification the ANOM chart provides a central line, which is at the overall average, and the upper and lower decision limits denoted as UDL and LDL respectively. Plotting individual means on the chart, we can identify at a glance if one or more means are statistically different from the overall average. With the ANOVA procedure, however, the experimenters will test either that all of the means are equal, or that at least one of the means differs from the others. The ANOVA table provides the sum of squares and F statistics without revealing information about an individual sample average. In the analysis of mean effects from two-level fractional factorial experiments, these two procedures give identical results in detecting significance factors (see p. 410 of Ryan, 1989). The intrinsic differences between ANOM and ANOVA are addressed in Ryan (1989, p. 399) and Schilling (1973, p. 99-100). Nevertheless, one procedure needs not be used to the exclusion of the other but rather, should be used as a supplement to the other.

Traditionally, ANOM and ANOVA are used to monitor the quality characteristics, from which inferential procedures are derived based on a normal distribution (with equal scales) and complete samples. Moreover, most of the publications in ANOM are directed to the study of location effects. With the increasing interest in reducing the variability of products (Taguchi and Wu, 1979), we should apply the idea of ANOM to study dispersion effects (Ullman, 1989) as well as other parameters of interest such as percentiles in the analysis of lifetime data. For improving product reliability, engineers prefer to use simple graphical techniques to present and interpret their life-testing results. For example, analyzing the fatigue testing data given in Table 4.1, we compare product performances resulting from different design configurations in a steel hardening process. Since the steel specimens were quite durable, some experimental units still survived by the end of experiment which results in several censored data points. Moreover, since the strength testings for steel specimens were done at various stress levels, we utilize the regression technique to unify the observations and compare steel types at a fixed stress, 40 ksi.

The objective of this article is to show how simple graphical procedures such as ANOM can be used to compare the durability of products formulated from different production plans. Various data types and model assumptions are considered in this article. In Section 2, the router bit life data from Phadke (1986) (see Table 2.1) is used to construct an ANOM chart for the comparison of the mean lifetimes of four bit types. The Weibull distribution is used to model the interval censored bit lives. Monte Carlo simulation of size 5000 is employed to study the finite sampling distributions of the pivotal statistics used in the construction of the ANOM chart. Normal approximation are also provided for comparison with the exact results. In Section 3, the ANOM procedure is extended to study dispersion effects based on the fatigue measurements of rolling contact given in McCool (1983). Finite and large sample distributional results are investigated for the pivotal statistics of the scale parameters. In section 4, the aforementioned steel data are used to illustrate the procedure of comparing the 10th percentile lifetimes of steels in the case of possibly unequal scale parameters. The conclusion and future work of this study is given in Section 5.

2. ANALYSIS OF MEAN TIME TO FAILURE

In this section we utilize the router bit life data given in Table 2.1 to illustrate the procedure of analysis of mean time to failure. Note that the data points are reported in time intervals and, moreover, 12.5% to 60% of the data are right-censored. The choice of the underlying distribution for lifetime data analysis is based on the following criteria cited in Lawless (1982, p. 29-30 and p. 468-470): knowledge of the underlying aging process, empirical fit, availability of statistical method, etc. The SAS (1988) procedure Lifereg is employed to perform the maximum likelihood (ML) regression analysis for the most commonly used lifetime distributions such as the lognormal, the Weibull, the generalized gamma (Prentice, 1974) and the logistic.

[Please put Table 2.1 here]

In deciding the underlying distribution the two most commonly used models, the lognormal and the Weibull, are fit to the router bit data and the likelihood ratio test is conducted for equal scale parameters. For both models, we conclude that the scale parameters of the distributions of the four bit types are all equal. The corresponding p-values of the χ^2 distribution with five degrees of freedom are .1112 and .1827, respectively. Given the scale parameters test equal, we compare the fit of the lognormal and Weibull models to the data. Based on the ML estimates, the log-likelihood of the lognormal model is slightly higher, $-$ 52.117 compared to $-$ 52.953 for the Weibull model. However, the standard error for one of the parameter estimate, $\hat{u}_2$, is considerably larger in the case of lognormal model. The p-values of the χ^2 statistics for this particular estimate are .1825 and .0004 for fitting lognormal and Weibull distributions, respectively. Hence, the Weibull is selected as the underlying distribution to illustrate the application of ANOM to non-normal models. The SAS Weibull parameter estimates are provided in Table 2.2.

[Please put Table 2.2 here]

By taking the natural logarithm of the Weibull distribution, the Weibull is transformed to the extreme-value (EV) distribution, which belongs to the location and scale family. Let us denote the parameters of the EV distributions for router bit lives, as (u_i, b), $i = 1, 2, \ldots, k$, respectively, where $k = 4$ and the b_i's are all equal to b. The study of mean time to failure $u_i + \gamma b$ in EV distribution can be handled by studying the location parameters u_i's, where $\gamma = 0.5772\cdots$ is Euler's constant. In the ANOM procedure, the hypothesis H_0: $\delta_i = 0$, $i = 1, 2, 3, 4$ is tested against H_1: at least one of $\delta_i \neq 0$, where $\delta_i = u_i - \bar{u}$, and $\bar{u}$ is the arithmetic average of location parameters u_i's. We propose to study the distributions of the statistics $V_i = [(\hat{u}_i - \hat{\bar{u}}) - (u_i - \bar{u})]/\hat{b}$ in order to derive confidence intervals to test the null hypothesis, H_0. From Appendix G of Lawless (1982), we note that the statistics $(\hat{u}_i - u_i)/\hat{b}$, $i = 1, 2, 3, 4$, are pivotal quantities. This implies that the statistics V_i's are pivotal. Let us consider the marginal distribution of V_i in order to address the link between the decision limits of the ANOM chart and the test of the hypothesis H_0. Denoting the lower and upper quantiles of the distribution of V_i as L_{vi} and U_{vi}, we obtain the $100(1 - \alpha_i)\%$ confidence interval of the $u_i - \bar{u}$ as $(\hat{u}_i - \hat{\bar{u}} - \hat{b}\, U_{vi},\ \hat{u}_i - \hat{\bar{u}} - \hat{b}\, L_{vi})$. If the lower bound $\hat{u}_i - \hat{\bar{u}} - \hat{b}\, U_{vi}$ is greater than 0, *i.e.* $\hat{u}_i > \hat{\bar{u}} + \hat{b}\, U_{vi}$, we reject the null hypothesis H_0 and conclude that $u_i \neq \bar{u}$. This gives the upper decision limit in ANOM chart as UDL $= \hat{\bar{u}} + \hat{b}\, U_{vi}$. Similarly, we obtain the lower decision limit (LDL) as $\hat{\bar{u}} + \hat{b}\, L_{vi}$.

The histograms of the V_i statistics , based on results from 5000 simulations, are plotted in Figure 2.1. The summary statistics such as maximum, minimum, mean, standard deviation (s.d.), coefficient of skewness μ_3/σ^3, coefficient of kurtosis μ_4/σ^4 and various quantiles for V_i, $i = 1, 2, 3, 4$ are reported in Table 2.3, where $\mu_r = \mathrm{E}[(X - \mu)^r]$ is the rth central moment. Note that the means of V_i's are very close to zero and the s.d.'s are around 0.40. Except that histogram of V_1 is slightly skewed to the right, the other histograms are closed to normal with the kurtosis around 3.3. Denoting r_{ij} as the product-moment correlation coefficient between V_i and V_j, we obtain r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34} as -.2340, -.2612, -.4217, -.1800, -.4371, -.4249, respectively. We note that the correlations between V_4 to other V_i's are comparably larger than the others due to the data for bit type 4 being heavily (60%) censored. The pairwise scatter plots of these V_i statistics are given in Figure 2.2. All computations and random number generations are done in S (1989) programs on Sun Sparc station 1.

[Please put Figure 2.1, 2.2 and Table 2.3 here]

For computing decision limits in ANOM of non-censored normal samples, P. R. Nelson (1982) and L. S. Nelson (1983) reported exact critical values of the equicorrelated multivariate t-distribution for $k > 2$. The construction of a simultaneous interval based on the empirical joint distribution of the V_i's is rather complicated, especially in this case of unequal correlations. Moreover, this procedure is not feasible to give separate limits for individual sample in an ANOM setup. There are quite a few publications in the direction of simultaneous interval estimation (see Seber, 1977, p. 125-134 for reference). The intervals based on Bonferroni inequality and maximum modulus are the commonly used simultaneous interval estimation procedures. For a $100(1 - \alpha)\%$ overall coverage probability Bonferroni interval, one obtains the $100(1 - \alpha/k)\%$ confidence intervals for each of the k mean differences $\delta_i = u_i - \bar{u}$, $i = 1, 2, \ldots, k$. For example, if one tests the hypothesis H_0, at significance level $\alpha = 0.05$ with an ANOM chart, the decision limits (99.5% confidence interval) for u_1 of bit type 1 are calculated to be LDL = 3.5323 and UDL = 8.8915 with the estimates $\hat{\bar{u}} = 6.2660$, $\hat{b} = 2.6106$, $L_{v1} = -1.0472$, $U_{v1} = 1.0057$. Since $\hat{u}_1 = 6.4260$ is inside its limits, we do not reject $u_1 = \bar{u}$. The decision limits for the other parameters δ_i, $i = 2, 3, 4$ are (3.7039, 8.7163), (3.6675, 8.7659) and (3.1058, 9.6291), respectively. Since the parameter estimates $(\hat{u}_2, \hat{u}_3, \hat{u}_4) = (4.1128, 5.3626, 9.1726)$ are inside the decision limits, we conclude that $\delta_i = 0$, $i = 1, 2, 3, 4$ is not rejected at significant level 0.05.

For the interval based on maximum modulus, we need the $100(1 - \alpha)\%$ quantiles M_v (reported in Table 2.3) from the simulated distribution of the statistics, where $V_5 = \max(|V_1|, |V_2|, |V_3|, |V_4|)$. The decision limits then are computed as $\hat{\bar{u}} \pm \hat{b} M_v$ for all four samples. Since the 95% confidence interval for δ_i is (3.4059, 9.1261), we reject $u_i = \bar{u}$, $i = 1, 2, 3, 4$ at $\alpha = 0.05$.

The ANOM chart with 90% and 95% Bonferroni and maximum modulus intervals are plotted in Figure 2.3. We note that even though we have the same sample size for all bit types, the distributions and variations of the V_i's are different due to the censoring mechanism. Except for the case of Bit type 4, maximum modulus limits are wider than the limits constructed from the Bonferroni intervals.

[Please put Figure 2.3 here]

Instead of using the Monte Carlo simulation procedure to find the finite sampling distributions of the pivotal statistics V_i's, we can derive the asymptotic distribution of the V_i's to construct approximated Bonferroni intervals. We need the likelihood function of the data (y_{ijL}, y_{ijU}) for $i = 1, 2, 3, 4$ and $j = 1, 2, \ldots, 8$, where L and U

denote the lower and upper bounds of the interval data. The probability of the log-lifetime Y_{ij} being in the interval (y_{ijL}, y_{ijU}) is computed as $P(Y_{ij} > y_{ijL}) - P(Y_{ij} > y_{ijU}) = \overline{F}(y_{ijL}) - \overline{F}(y_{ijU})$, where $\overline{F}$ is the survival function. If the data is censored at time $t = \log(1700)$, the probability of Y_{ij} being larger than t is calculated as $\overline{F}(t)$. Using the survival function $\overline{F}(z) = exp[- exp(z)]$ of the standardized extreme-value distribution, we write the log-likelihood of the interval censored data as follows:

$$log\ L = \sum_{i=1}^{k}\sum_{j=1}^{n} \delta_{ij}\, log\{exp[- exp(z_{ijL})] - exp[- exp(z_{ijU})]\} \tag{3.1}$$
$$- \sum_{i=1}^{k}\sum_{j=1}^{n} (1 - \delta_{ij})\, exp(c_i),$$

where the indicator δ_{ij} is defined as $\delta_{ij} = 1$ if $y_{ij} < t$ and $\delta_{ij} = 0$ otherwise. The log-lifetimes y_{ij}'s given in Table 2.1 are standardized to $z_{ijm} = (y_{ijm} - u_i)/b$, $m = L$ and U, and the standardized censoring time is written as $c_i = (t - u_i)/b$, for $i = 1, 2, 3, 4$ and $j = 1, 2, ..., 8$. The asymptotic normal distribution for the MLE's of the model parameters is obtained in the usual way. Specifically, to obtain the Fisher information matrix $I(u_1, u_2, u_3, u_4, b)$ we require the expectations of the second derivatives of $- log L$. Since the calculation of $I(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{b})$ involves many numerical integrations, we consider the following simpler and equally valid procedure for large sample approximation (c.f. Lawless, 1985, p. 172):

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{b}) \sim N_5[(u_1, u_2, u_3, u_4, b), I_0^{-1}], \tag{3.2}$$

where I_0 is the observed information matrix, which is composed of the second derivatives of $- log L$ evaluated at the ML estimates of the parameters. The derivatives of the log-likelihood (3.1) are rather complicated and are therefore reported in the appendix. To get the asymptotic distribution of the statistics V_i's, we employ the *delta* method to the following transformations of the MLE's $(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{b})$:

$$f_i(x_1, x_2, x_3, x_4, x_5) = [(x_i - \bar{x}) - (u_i - \bar{u})]/x_5,\ i = 1, 2, ..., 5,$$

where $\bar{x} = (x_1 + x_2 + x_3 + x_4)/4$. Thus we concluded that the large sample distribution of V_i, $i = 1, 2, 3, 4$ is normal with zero means and standard deviations (.2721, .3466, .3498, .4514) with correlations (-.0647, -.2271, -.3771, -.2366, -.5456, -.4564) for $(r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34})$. Note that the standard deviations and correlations are quite comparable with the finite sample results.

The decision limits of the ANOM chart for the ith router bit life based on large sample Bonferroni interval are then given as $\text{LDL}_i = \hat{u} + \hat{b}\, L^*_{vi}$ and $\text{UDL}_i = \hat{u} + \hat{b}\, U^*_{vi}$, where L^*_{vi} and U^*_{vi} are the lower and upper $\alpha/(2k)$ quantiles of the marginal normal distributions of the V_i's. The large sample 90% confidence limits are plotted in Figure 2.3. We observe that the estimates $\hat{u}_2$ and $\hat{u}_4$ are outside the large sample 90% confidence limits, but are inside the finite sample Bonferroni intervals. The large sample decision limits with 95% overall coverage probability are (4.4917, 8.0403), (4.0055, 8.5265), (3.9848, 8.5471) and (3.224, 9.2096) for $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ and $\hat{u}_4$, respectively. These are shorter than the corresponding limits derived from the finite sample simulations. This implies that, in finite sample, the overall coverage probability based on the limits derived from large sample approximation will, in general, not reach the desired level.

To check the coverage probability of the decision limits constructed for the ANOM chart given in Figure 2.3, we simulate 1000 samples and obtain the MLE's of $(u_{im} - \bar{u}_m)$ and b_m, for $i = 1, 2, 3, 4$, four bit types, and $m = 1, 2, \ldots, 1000$. The empirical coverage probability is the proportion of the cases of all estimates from the original data $(\hat{u}_i - \hat{\bar{u}})$ that are inside their corresponding $100(1 - \alpha/k)\%$ confidence interval $(\hat{u}_{im} - \hat{\bar{u}}_m - \hat{b}_m U_{vi}, \hat{u}_{im} - \hat{\bar{u}}_m - \hat{b}_m L_{vi})$, $i = 1, 2, 3, 4$. For the overall 90% Bonferroni intervals, the decision limits constructed from the finite sampling distribution have a 88.6450% coverage probability (C.P.), which is slightly higher than the C.P. 87.6908% found for the limits derived from the maximum modulus, and is much higher than the C.P. 77.0038% obtained for the large sample intervals. For 95% overall coverage probability the C.P.'s for finite sample Bonferroni, maximum modulus and large sample Bonferroni intervals are 93.1298%, 92.8435% and 82.5382%, respectively. As expected, the large sample procedure does not provide an accurate approximation and should be used with caution, especially on the charts for comparing dispersion and percentile effects which will be discussed in next two sections.

3. ANALYSIS OF VARIATIONS OF FATIGUE FAILURE TIMES

Recently, there has been increasing interest in reducing product variability. Various case studies are given in Taguchi and Wu (1979) and publications cited in Ryan (1989, Chapter 14). Considering the fatigue failure times of rolling contact given in McCool (1983) and Table 3.1, we develop a graphical procedure to analyze the dispersion effects of lifetimes from the use of different testers. Following the notation given in Section 2, we utilize the ML method to estimate the location and scale parameters (u_i, b_i), $i = 1, 2, \ldots, 10$ of the extreme-value distributions for failure times obtained from tester 1 to 10 respectively. The estimation results are summarized in the bottom of Table 3.1. The estimates of the scale parameters $\hat{b}_i$ range from 0.0918 to

0.3540. The likelihood ratio (LR) test for the equality of scale parameters, H_0: $b_i = b$, $i = 1, 2, \ldots, 10$, gives the test statistic as 29.4590. With the p-value .000542 computed from χ^2_9 distribution, we conclude that the scale parameters b_i's are not equal for the ten testers.

[Please put Table 3.1 here]

Since the variances σ_i^2's of the EV distribution are equal to $\pi^2 b_i^2/6$, the study of dispersion effect can be handled by investigating the differences of the scale parameters b_i's. Since the statistics $\hat{b}_i/b_i$, $i = 1, 2, \ldots, 10$ are pivotal quantities (see Appendix G of Lawless, 1982), we note that the quantities Z_i's are pivotal, where

$$Z_i = [\, \log \hat{b}_i - (\Sigma \log \hat{b}_i)/k \,] - [\, \log b_i - (\Sigma \log b_i)/k \,],\ i = 1, 2, \ldots, 10,$$

where the summations range from $i = 1$ to k, $k = 10$. The choice of log $\hat{b}_i$ in Z_i is based upon experience in reliability studies (see Lawless, 1982, p. 173): normal approximation works better in log $\hat{b}_i$ than in $\hat{b}_i$. We study the finite sample distributions of these Z_i statistics via a simulation study of 5000 replications to construct the chart for the analysis of dispersion effects. The histograms of the Z_i statistics plotted in Figure 3.1 do not show any special pattern other than normal curves. The means of Z_i's are very close to zero and their standard deviations are around .2600. We present the needed quantiles in Table 3.2 for the Bonferroni and maximum modulus intervals for various overall coverage probabilities, 80%, 90%, 95%, 98% and 99%.

Based on the Bonferroni intervals, the decision limits of the ANOM chart are computed from the following formulas: central line $= (\Sigma \log \hat{b}_i)/k$, LDL_i = central line + Z_{iL}, UDL_i = central line + Z_{iU}, where Z_{iL} and Z_{iU} are the lower and upper $\alpha/(2k)$ quantiles of the Z_i statistic. Plotting the individual $\log \hat{b}_i$ on the chart, one is able to see whether the difference $\log \hat{b}_i - (\Sigma \log \hat{b}_i)/k$ is significantly large. Of course, instead of plotting on the log-scale, one can transform the whole chart back to the original scale of $\hat{b}_i$ for ease of interpretation. For example, on the log-scale for the first tester, the central line is calculated as - 1.7145 and the lower, upper decision limits based on the Bonferroni intervals are - 2.5001 and - 1.0572, respectively. On the original scale of $\hat{b}_i$'s, the LDL, central line, and UDL are reported as 0.08208, 0.18005 and 0.34743, respectively. Since $\hat{b}_1 = 0.18603$ lies inside its decision limits, we conclude that $b_1 = b = (\prod_{i=1}^{k} b_i)^{1/k}$, the geometric average of b_i's. The ANOM chart for the analysis of scale parameters is sketched in Figure 3.2 in the original scale of $\hat{b}_i$. Based on the Bonferroni intervals, we note that the hypothesis H_0: $\log b_i = (\sum_{i=1}^{k} \log b_i)/k$, $i = 1, 2, \ldots, 10$ is rejected at $\alpha = 0.10$, but not rejected at $\alpha = 0.05$.

[Please put Figure 3.1 and Table 3.2 here]

If the maximum modulus intervals are used, the limits of the chart are given as central line $\pm Z_M$, where Z_M is the $100(1-\alpha)$th quantile of the distribution of $\max(|Z_i|, i = 1, 2, \ldots, 10)$, which is reported at the bottom of Table 3.2. For example, with $\alpha = 0.05$ we have LDL and UDL calculated as - 2.4905 and - 0.9385 in the $\log \hat{b}_i$ scale, and 0.08286, 0.39121 in the $\hat{b}_i$ scale. The null hypothesis H_0 is not rejected at $\alpha = 0.05$ or 0.10. Comparing the upper control limits derived from these finite sample intervals, we note that all maximum modulus limits are outside the corresponding Bonferroni limits. This pattern is reversed in the lower decision limits.

[Please put Figure 3.2 here]

Next, we derive the large sample distribution of the pivotal statistics Z_i's for a comparison to their finite sample results. Relying on the normal approximation (Lawless, p. 172) of MLE's of the parameters of the EV distribution, we obtain the asymptotic distribution of MLE's $\hat{b}_i$, $i = 1, 2, \ldots, k$, where $k = 10$, as normal with mean vector $(b_1, b_2, \ldots, b_{10})$, zero covariances and variances $\eta_i = \hat{b}_i^2 [n + \sum_{i=1}^{k} \hat{w}_{ij}^2 \, exp(\hat{w}_{ij})]^{-1}$, $i = 1, 2, \ldots, k$, where $n = 10$ = number of failures, $\hat{w}_{ij} = (y_{ij} - u_i)/b_i$, y_{ij} = log-lifetimes. Applying the *delta* method on the transformations $f_i(x_1, x_2, \ldots, x_k) = [logx_i - \sum_{i=1}^{k} (logx_i)/k] - [logb_i - \sum_{i=1}^{k} (logb_i)/k]$, we obtain the asymptotic distribution of Z_i, $i = 1, 2, \ldots, k$ as normal with zero means, zero covariances and standard deviations as (.2067, .1820, .2266, .2035, .2045, .2155, .2374, .2074, .2123, .1893). The decision limits for the ANOM chart based upon the Bonferroni intervals are computed as central line $\pm Z^*_{Li}$, where Z^*_{Li}'s are the upper $\alpha/(2k)$ quantiles of the marginal normal distributions. The limits for 90% overall coverage probability are plotted in Figure 3.2 for a comparison to their finite sample derivations. Note that the lengths of the large sample limits are considerably shorter than the ones provided from finite sample simulations. In particular, the estimate of the scale parameter $\hat{b}_6$ is outside its large sample limits but not its finite sample limits.

The empirical coverage probabilities for the finite sample Bonferroni, maximum modulus and large sample Bonferroni intervals are computed for comparison. We calculate the proportions that all MLE's $(\hat{\xi}_i - \hat{\bar{\xi}})$ lie inside the corresponding $100(1 - \alpha/k)\%$ confidence interval $(\hat{\xi}_{im} - \hat{\bar{\xi}}_m - Z_{iU}, \hat{\xi}_{im} - \hat{\bar{\xi}}_m - Z_{iL})$, $i = 1, 2, 3, 4$, $m = 1, 2, \ldots, 500$, where $\xi_i = logb_i$. They are (88.5714%, 88.5%, 66.2857%) and (97.8571%, 97.25%, 75.4286%) for 90% and 95% overall coverage probabilities for the aforementioned three types of intervals. We find that the coverage probabilities for the large sample intervals are considerably low compared the C.P. of the finite sample limits, and the C.P.'s of the finite sample Bonferroni, maximum modulus intervals are quite close.

4. ANALYSIS OF PERCENTILE LIFETIMES OF STEEL STRENGTH

In this section, we use the ANOM chart to compare the strengths of four different formulations of steel. The fatigue testing results for 42 steel specimens from four types of steels are listed in Table 4.1, which is taken from page 313 of Nelson's (1990) book on accelerated life testing. We note that the specimens are tested at different stress levels and totally 26.2% of the observations are censored. Since the lower percentiles, such as 5th or 10th percentiles, are of much interest in reliability studies (Lawless, 1982, p. 251), we develop an ANOM chart to compare the 10th percentile lifetimes of steel strength at normal use stress level, say 40.0 ksi.

The distributional result of percentile lifetimes depends on the choice of life distribution. For instance, a study given in Lawless (1982, p. 250-1) illustrates the differences of the point and interval estimations of the 1st, 10th and 50th percentiles of normal, extreme-value and other life distributions. They observe that for the 1st percentile lifetime $y_{.01}$, the MLE of $y_{.01}$ under the normal model is not even inside the 90% confidence interval for $y_{.01}$ under the extreme-value model. Fitting the lognormal, Weibull and generalized gamma distributions to the steel strength data as in Section 2 for model selection, we found that the lognormal is the best choice for the analyses. In Nelson's (1990) study, he also assumed the lognormal model for the distribution of this steel data. Based on log-transformed normal data, the likelihood ratio statistic for testing the equality of scale parameters is computed as 0.82858. With p-value .93457 for the LR statistic from χ^2_4 distribution, we do not reject H_0: $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$ at significance level $\alpha = 0.01$. Under the assumption of equal scale parameters, the study of percentiles $y_{pi} = \mu_{0i} + z_p \sigma$ in the normal distribution can be handled by studying the location parameters μ_{0i}, which has been addressed in Section 2. Henceforth, we will develop the procedure of constructing an ANOM chart for percentile lifetimes for the general case of scale parameters without restricting them to be equal. The estimation results of the model parameters such as intercept, slope, scale, mean at $x_0 = 40$ ksi and $y_{.10}$ at x_0, of the normal distributions for the log-life of steel data are reported in Table 4.2.

[Please put Table 4.1 and Table 4.2 here]

With possible unequal scale parameters, there are no pivotal quantities for the estimates $\hat{y}_{pi} - \hat{\bar{y}}_p$, where $\hat{\bar{y}}_p = (\sum_{i=1}^{k} \hat{y}_{pi})/k$ and $k = 4$. The distributions of the statistics $T_i = [(\hat{y}_{pi} - \hat{\bar{y}}_p) - (y_{pi} - \bar{y}_p)]/\hat{\sigma}_i$, $i = 1, 2, 3, 4$ are studied with the parametric bootstrap technique to determine their finite sampling behavior, and are also investigated with normal approximation for its large sample results. The histograms of the estimates of intercepts, slopes and scale parameters do not show any unusual pattern and appear normally distributed. Figure 4.1 presents the histograms of the T_i statistics based on 1000 simulations. The shape of the histograms of T_i are generally not symmetric, for example, the histogram of T_2 is skewed to the right with the coefficient of skewness 0.7333 and the coefficient of kurtosis 5.2061 (more peaked around its center than the density of the normal curve). Various summary statistics such as mean, s.d., and quantiles of the simulated T_i's are listed in Table 4.3. In general, the means are not too far from zero. Since the second sample (steel A-I.H.) contains 40% censored data, we note that its s.d. from the simulated data is considerably larger than the s.d.'s of the other samples.

[Please put Figure 4.1 and Table 4.3 here]

Taking the quantiles from Table 4.3, the central line and decision limits for the ANOM chart based on Bonferroni intervals are computed as follows: Central line = $\hat{\bar{y}}_p$, LDL = $\hat{\bar{y}}_p + T_{iL}\hat{\sigma}_i$ and UDL = $\hat{\bar{y}}_p + T_{iU}\hat{\sigma}_i$, where $\hat{\bar{y}}_p = (\sum_{i=1}^{k} \hat{y}_{pi})/k$, T_{iL} and T_{iU} are the lower and upper $\alpha/(2k)$ quantiles of the distribution of T_i's and $\hat{\sigma}_i$'s are the MLE's of σ_i's given in Table 4.2. For example, from the first sample, we obtain the MLE's $\hat{y}_{p1}$ = 13.3402, $\hat{\sigma}_1$ = 1.3079 and the overall average $\hat{\bar{y}}_p$ = 20.2278 of 10th percentiles from four samples. For the Bonferroni intervals, the quantiles T_{1L} and T_{1U} for the 95% overall coverage probability are computed to be - 4.4958 and 3.7261 respectively. The decision limits are then calculated as LDL = 14.3478 and UDL = 25.1012. Since the 10th percentile lifetime of the first steel type $\hat{y}_{p1}$ = 13.3402 lies outside the decision limits, we conclude that $y_{p1} \neq \bar{y}_p$ at the overall significance level $\alpha = 0.05$. In Figure 4.2 we plot the ANOM chart for testing H_0: $y_{pi} \neq \bar{y}_p$, $i = 1, 2, 3, 4$ with $\alpha = .05$ and .10. In conclusion, considering the ANOM limits from various significance levels, we reject H_0 at $\alpha = .02$, but not $\alpha = .01$, where the corresponding limits for y_{p1} are (14.0224, 26.9212) and (12.8623, 29.4321), respectively.

For the maximum modulus approach, the quantiles M_T's for the 80%, 90%, 95%, 98% and 99% overall coverage probabilities of $\max(|T_1|, |T_2|, |T_3|, |T_4|)$ are reported as 5.3848, 7.1593, 8.8251, 10.7814 and 12.3435, respectively. According to the decision limits $\hat{\bar{y}}_p \pm \hat{\sigma}_i M_T$ plotted in Figure 4.2, we conclude that the hypothesis H_0 is not rejected at $\alpha = 0.10$.

[Please put Figure 4.2 here]

The large sample distributions of the T_i statistics are derived by the following steps. First, we replace the regression parameters in the derivatives of log-likelihood given in Eq. (6.5.6) of Lawless (1982, p. 315) by their MLE's $(\hat{\alpha}_i, \hat{\beta}_i, \hat{\sigma}_i)$, $i = 1, 2, 3, 4$ to obtain the observed Fisher information matrix I_0. When sample sizes are large enough, the joint distribution of MLE's is approximated by a normal distribution with zero means and variance-covariance matrix I_0^{-1}. The *delta* method is then employed to transform the distribution of MLE's $(\hat{\alpha}_i, \hat{\beta}_i, \hat{\sigma}_i)$, $i = 1, 2, 3, 4$ to the distribution of the MLE's of y_{pi}, the 10th percentile lifetime of ith steel type at $x_0 = 40.0$ ksi. Now, the distribution of the T_i statistics can be readily derived. Since several data points are censored, the derivatives of log-likelihood involve the survival function of the normal distribution, which needs to be evaluated numerically. The variances of the T_i's and the decision limits of the ANOM chart are computed in one sequence utilizing S programs. We conclude that the distribution of T_i's are closed to normal with zero means and standard deviations $\sigma_i = 1.0012$, 2.3037, 1.1987, 1.1734 for $i = 1, 2, 3, 4$ respectively. The correlation coefficients are –.7527, .4722, .2421, –.7127, –.6838, .1104 for r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}, respectively. We note that the standard deviation of the 2nd sample is quite large and the correlations of the 2nd sample to other samples are considerably higher than the others due to the heavy censoring effect. This observation matches the finite sample simulation results given in Table 4.3. Computing decision limits $\hat{\bar{y}}_p \pm \hat{\sigma}_i T^*_{Li}$, where T^*_{Li}'s are the quantiles of the asymptotic normal distributions, we have the 95% Bonferroni intervals (16.9567, 23.4989), (12.1456, 28.3100), (16.4891, 23.9665), (15.9056, 24.5500) for the ith sample, $i = 1, 2, 3, 4$, respectively. The 90% Bonferroni limits are plotted in Figure 4.2 for comparison with the limits obtained from the finite sample simulations. Note that the lengths of the large sample intervals are shorter than the intervals given by the Bonferroni procedure but are slightly larger than the intervals given by maximum modulus in finite samples.

The empirical coverage probabilities for the finite sample Bonferroni, maximum modulus and large sample Bonferroni intervals are computed for comparison of their performances. We calculate the proportions that all MLE's $(\hat{y}_{pi} - \hat{\bar{y}}_p)$ lie inside the corresponding $100(1 - \alpha/k)\%$ confidence interval $(\hat{y}_{pim} - \hat{\bar{y}}_{pm} - \hat{\sigma}_m T_{iU}$, $\hat{y}_{pim} - \hat{\bar{y}}_{pm} - \hat{\sigma}_m T_{iL})$, $i = 1, 2, 3, 4$, $m = 1, 2, \ldots, 750$. And, they are (91.6%, 90.0%, 51.3333%) and (95.0667%, 94.9333%, 57.2%) for 90% and 95% overall coverage probabilities for the aforementioned three intervals. We find that the coverage probabilities for the large sample intervals are considerably low compared the C.P. of the finite sample limits, and the C.P.'s of the finite sample Bonferroni, maximum modulus intervals are quite close. In general, the normal approximations of the distribution of the scale parameters and percentile lifetimes are not as good as the normal approximations for the location parameters (c.f. Lawless, 1982, p. 178). However, the procedure of large sample approximation is easier to employ than the procedures obtained through finite sample simulations.

5. CONCLUSION AND FUTURE WORK

With the acceptance of Shewhart control charts in engineering applications, we believe that the ANOM charts will be widely used in the future to report findings from all types of industrial experiments. In this article, we extend the ANOM idea to the analysis of the reliability data for comparing the mean, variance and percentile lifetimes of products in a one-way classification. It is interesting to see some applications of ANOM charts to compare product lifetimes from multi-factor experiments, especially from the screening experiments which are advocated by Taguchi (see the chapter of life-testing in Taguchi and Wu, 1979). The decision limits are usually not the same across all products due to the differences of sample size, degree of censoring and scale parameters. The complications of censored data and non-normal life distribution are possibly handled by simple procedures such as imputation from conditional means (c.f. Schmee and Hahn, 1979) and Box-Cox transformation (c.f. Johnson, 1982). Henceforth, the usual ANOM charts developed for normally distributed complete data can be used. The performance of these simple methods needs to be investigated. The large sample approximation is another alternative and is more convenient to use. However, the performance of asymptotic approach depends on the sample size, the degree of censoring and the parameter of interest.

ACKNOWLEDGMENT

We are grateful to J. T. Arnold and D. A. Dickey for permitting the use of Sun Sparc stations to conduct simulations.

APPENDIX: DERIVATIVES OF LOG-LIKELIHOOD OF INTERVAL CENSORED DATA

We introduce the following notations for the convenience of presenting the derivatives of log-likelihood for interval censored data given in Section 2. The indicator δ_{ij} is defined as $\delta_{ij} = 1$ if $y_{ij} < t = log(1700)$ and $\delta_{ij} = 0$ otherwise. The standardized observations are denoted as $z_{ijm} = (y_{ijm} - u_i)/b$, $c_i = (t - u_i)/b$, where $y_{ijm} = y_{ijL}$ and

y_{ijU} denote the lower and upper bounds of the interval data. We define $G_i = exp(c_i)$, $E_{ijL} = exp(z_{ijL})$, $E_{ijU} = exp(z_{ijU})$, $D_{ijL} = exp(-E_{ijL})$, $D_{ijU} = exp(-E_{ijU})$ and $W_{ij} = D_{ijL} - D_{ijU}$. Then, the writing of derivatives can be simplified as follows:

$$\frac{\delta logL}{\delta u_i} = \sum_{j=1}^{n} \delta_{ij}(b\,W_{ij})^{-1} \times \left[D_{ijL}\,E_{ijL} - D_{ijU}\,E_{ijU}\right] + \sum_{j=1}^{n}(1-\delta_{ij})\,G_i/b,$$

$$\frac{\delta logL}{\delta b} = \sum_{i=1}^{k}\sum_{j=1}^{n} \delta_{ij}(b\,W_{ij})^{-1} \times \left[D_{ijL}\,E_{ijL}\,z_{ijL} - D_{ijU}\,E_{ijU}\,z_{ijU}\right] + \sum_{i=1}^{k}\sum_{j=1}^{n}(1-\delta_{ij})\;G_i\;c_i/b,$$

$$\frac{-\partial logL}{\partial u_i^2} = -\sum_{j=1}^{n} \delta_{ij}(b^2\;W_{ij})^{-1} \times \left[D_{ijL}(E_{ijL}^2 - E_{ijL}) - D_{ijU}(E_{ijU}^2 - E_{ijU})\right] + \sum_{j=1}^{n} \delta_{ij}(b\;W_{ij})^{-2} \times (D_{ijL}\,E_{ijL} - D_{ijU}\,E_{ijU})^2 + \sum_{j=1}^{n}(1-\delta_{ij})\;G_i/b^2,$$

$$\frac{-\partial logL}{\partial u_i \partial b} = -\sum_{j=1}^{n}\delta_{ij}(b^2\;W_{ij})^{-1} \times \left[D_{ijL}E_{ijL}(E_{ijL}z_{ijL} - z_{ijL} - 1) - D_{ijU}E_{ijU}(E_{ijU}z_{ijU} - z_{ijU} - 1)\right] + \sum_{j=1}^{n}\delta_{ij}(b\;W_{ij})^{-2}\,(D_{ijL}\,E_{ijL} - D_{ijU}\,E_{ijU})\,(D_{ijL}\,E_{ijL}\,z_{ijL} - D_{ijU}\,E_{ijU}\,z_{ijU}) + \sum_{j=1}^{n}(1-\delta_{ij})\;G_i(c_i + 1)/b^2,$$

$$\frac{-\partial logL}{\partial b^2} = -\sum_{i=1}^{k}\sum_{j=1}^{n} \delta_{ij}(b^2\,W_{ij})^{-1}\left\{\left\{D_{ijL}\,E_{ijL}\,z_{ijL}\left[E_{ijL}\,z_{ijL} - (z_{ijL} + 2)\right]\right\} - \left\{D_{ijU}\,E_{ijU}\,z_{ijU}\left[E_{ijU}\,z_{ijU} - (z_{ijU} + 2)\right]\right\}\right\} + \sum_{i=1}^{k}\sum_{j=1}^{n}\delta_{ij}(b\;W_{ij})^{-2}\,(D_{ijL}\,E_{ijL}\,z_{ijL} - D_{ijU}\,E_{ijU}\,z_{ijU})^2 + b^{-2}\sum_{i=1}^{k}\sum_{j=1}^{n}(1-\delta_{ij})\;G_i\;c_i(2 + b^{-1}).$$

REFERENCES

Hoadley, A. B. and Kettenring, J. R. (1990), "Communications Between Statisticians and Engineers/Physical Scientists," (with discussion) *Technometrics*, 32, 243-273.

Hogg, R. V., *et. al.* (1985), "Statistical Education for Engineers: An Initial Task Force Report," *The American Statistician*, 39, 168-175.

Johnson, R. A. (1982), "Transformation of Survival Data," in *Survival Analysis*, edited by John Crowley and Richard A. Johnson. The Institute of Mathematical Statistics, Hayward, California.

Lawless, J. F. (1982), *Statistical Models and Methods for Lifetime Data.* Wiley, New York.

McCool, J. I. (1981), "Analysis of Variance for Weibull Populations," in *The Theory and Applications of Reliability*, vol. 1, edited by Tsokos, C. P. and Shimi, I. N., Academic Press, Inc., 335-379.

Nelson, L. S. (1983). "Exact Critical Values for Use With the Analysis of Means," *Journal of Quality Technology*, 15, 40-44.

Nelson, P. R. (1982), "Exact Critical Points for the Analysis of Means," *Communications in Statistics – part A, Theory and Methods*, 11(6), 669-709.

Nelson, P. R. (1988), "Testing for Interactions Using the Analysis of Means," *Technometrics*, 30, 53-61.

Nelson, W. (1990), *Accelerated Life Testing.* Wiley, New York.

Ott, E. R. (1967), "Analysis of Means – A Graphical Procedure," *Industrial Quality Control*, 24, 101-109.

Ott, E. R. (1975), *Process Quality Control.* McGraw-Hill, New York.

Phadke, M. S. (1986), "Design Optimization Case Studies," *AT&T Technical Journal*, 65, 51-68.

Prentice, R. L. (1974), "A Log-gamma Model and Its Maximum Likelihood Estimation," *Biometrika*, 61, 539-544.

Raming, P. F. (1983), "Applications of the Analysis of Means," *Journal of Quality Technology*, 15, 19-25.

Ryan, T. P. (1989), *Statistical Methods for Quality Improvement.* Wiley, New York.

S, An Interactive Environment for Data Analysis and Graphics, (1989), developed by Becker, R. A. and Chambers, J. M. in Bell Telephone Laboratories, Inc., Murray Hill, NJ.

SAS/STAT User's Guide, 6.03 Edition (1988), SAS institute Inc., Cary, NC.

Schmee, J. and Hahn, G. (1979), "A Simple Method for Regression Analysis with Censored Data," *Technometrics*, 21, 417-432.

Seber, G. A. F. (1977), *Linear Regression Analysis.* Wiley, New York.

Schilling, E. G. (1973), "A Systematic Approach to the Analysis of Means," *Journal of Quality Technology*, 5, 93-108.

Taguchi, G. and Wu, Y. (1979), *Introduction to Off-line Quality Control.* Central Japan Quality Control Association, (available from American Supplier Institute, Dearborn, MI).

Ullman, N. R. (1989), "The Analysis of Means (ANOM) for Signal and Noise," *Journal of Quality Technology*, 21, 111-127.

Table 2.1. Lifetime of Router Bits

Replications	*Bit 1*	*Bit 2*	*Bit 3*	*Bit 4*
1	[3.0, 4.0]	[0.0, 1.0]	[0.0, 1.0]	[17.0, ∞]
2	[0.0, 1.0]	[0.0, 1.0]	[0.0, 1.0]	[2.0, 3.0]
3	[3.0, 4.0]	[0.0, 1.0]	[2.0, 3.0]	[17.0, ∞]
4	[2.0, 3.0]	[0.0, 1.0]	[3.0, 4.0]	[0.0, 1.0]
5	[17.0, ∞]	[0.0, 1.0]	[0.0, 1.0]	[17.0, ∞]
6	[14.0, 15.0]	[0.0, 1.0]	[0.0, 1.0]	[17.0, ∞]
7	[3.0, 4.0]	[17.0, ∞]	[3.0, 4.0]	[17.0, ∞]
8	[3.0, 4.0]	[0.0, 1.0]	[17.0, ∞]	[0.0, 1.0]

NOTE: The data is reported in units of 100 inches.

Table 2.2. Parameter Estimation for Life Distribution of Router Bits

Weibull Distribution				*Lognormal Distribution*			
Log-Likelihood = – 52.9534				*Log-Likelihood* = – 52.1168			
Parameter	*Estimate*	*Std. Err.*	*Pr>Chi*	*Parameter*	*Estimate*	*Std. Err.*	*Pr>Chi*
u_1	6.4260	0.9884	.0001	u_1	6.0273	0.9944	.0001
u_2	4.1128	1.1592	.0004	u_2	2.1899	1.6426	.1825
u_3	5.3626	1.0151	.0001	u_3	4.4381	1.1207	.0001
u_4	9.1626	1.5874	.0001	u_4	7.5128	1.1777	.0001
b	2.6106	0.6313		b	2.7167	0.7407	

Table 2.3. The Summary Statistics of V_i, $i = 1.2, \ldots, 5$ for Router Bit Data

Statistic	V_1	V_2	V_3	V_4	V_5
min	-1.7098	-1.6129	-1.6516	-1.6231	.0187
max	1.9220	1.3118	2.0621	1.7801	2.0621
mean	0.0080	-0.0063	-0.0021	0.0004	.5762
s.d.	0.4007	0.3816	0.3856	0.4996	.2728
μ_3/σ^3	-0.0015	-0.0577	-0.0067	0.2410	
μ_4/σ^4	3.3709	3.0719	3.3326	3.0461	
Quantiles#					
99%L	-1.3612	-1.1836	-1.3069	-1.4197	.7938(1)
98%L	-1.2101	-1.1041	-1.1668	-1.3675	.9582(2)
95%L	-1.0472	-0.9814	-0.9954	-1.2105	1.0956(3)
90%L	-0.9194	-0.8841	-0.8814	-1.0574	1.2455(4)
80%L	-0.7971	-0.7753	-0.7615	-0.9156	1.3517(5)
80%H	0.8069	0.7215	0.7564	1.0858	
90%H	0.9056	0.8272	0.8582	1.2016	
95%H	1.0057	0.9386	0.9576	1.2883	
98%H	1.1637	1.1163	1.1268	1.4197	
99%H	1.2423	1.1708	1.2552	1.4279	

NOTE: #: $100(1-\alpha)\%$L is the $100[\alpha/(4 \times 2)]$ quantiles of 5000 simulated samples
The statistics V_5 is the maximum module $V_5 = \max(|V_1|, |V_2|, \ldots, |V_4|)$.
The quantities (1) to (5) are 80^{th}, 90^{th}, 95^{th}, 98^{th} and 99^{th} quantiles of V_5.

Table 3.1. Fatigue Failure Times (in Hours) for Ten Different Testers

	Tester									
Replications	1	2	3	4	5	6	7	8	9	10
1	140.3	193.0	73.5	196.5	145.7	171.9	183.2	244.0	187.4	186.0
2	158.0	172.5	263.7	218.9	116.5	188.1	222.4	179.2	202.0	202.0
3	183.9	173.3	192.3	196.9	150.5	191.6	197.5	176.2	175.0	200.9
4	132.7	204.7	37.1	253.3	141.6	154.3	211.0	207.7	171.7	137.1
5	117.8	172.0	160.3	212.5	129.0	171.3	178.0	148.2	230.5	195.8
6	98.7	152.7	159.2	239.6	178.4	157.4	130.5	121.6	174.2	162.4
7	164.8	234.9	133.5	181.3	133.4	132.2	160.9	195.0	220.2	134.6
8	136.6	216.5	200.7	193.0	120.2	156.7	197.6	133.6	166.3	174.5
9	93.4	422.6	189.6	178.3	192.6	194.8	90.0	167.2	239.8	272.9
10	116.0	262.6	157.1	262.8	179.0	173.3	213.0	98.5	223.7	173.8
Parameter Estimation										
u_i	4.980	5.507	5.167	5.421	5.072	5.178	5.263	5.210	5.351	5.296
b_i	.1860	.3406	.3540	.1260	.1578	.0918	.1630	.2215	.1178	.2057

Table 3.2. Quantiles of Z_i Statistics from 5000 Simulations

C.I.[+] %-tile[#]	99% *.05%*	98% *.1%*	95% *.25%*	90% *.5%*	80% *1%*	80% *99%*	90% *99.5%*	95% *99.75%*	98% *99.98%*	99% *99.95%*
Testers										
1	-.8751	.8403	-.7855	-.7485	-.7088	.5783	.6299	.6574	.7086	.7578
2	-.9881	-.9366	-.8379	-.7957	-.6720	.5286	.5628	.6948	.7370	.9623
3	-1.0406	-.9993	-.8686	-.8059	-.7427	.5800	.6245	.6890	.7274	.8419
4	-1.0878	-.9007	-.8911	-.7569	-.6515	.5531	.5781	.5939	.6861	.7170
5	-.9784	-.9518	-.8621	-.7427	-.6612	.5665	.6242	.7067	.7319	.8659
6	-1.2246	-1.0030	-.9292	-.7503	-.6755	.5346	.5849	.6763	.7113	.8162
7	-.9923	-.9658	-.8674	-.7679	-.7032	.5409	.5937	.6617	.7354	.7717
8	-1.0225	-.9641	-.8502	-.7656	-.6401	.5654	.6910	.7716	.7981	.8016
9	-1.1334	-.9163	-.8823	-.7530	-.6977	.5642	.6463	.6899	.7218	.8255
10	-.9941	-.9034	-.8427	-.7462	-.6674	.5137	.5695	.6039	.6481	.6848

NOTE: +: The overall coverage probability for ten confidence intervals.

#: The quantiles of individual V_i, $i = 1, \ldots, 10$ statistics.

The 80th, 90th, 95th, 98th and 99th quantiles of the max $(|V_1|, |V_2|, \ldots, |V_{10}|)$ are computed as .6046, .7024, .7760, .8824, .9623, respectively.

Table 4.1. Steel Strength Data (in 100,000 Cycles)

Steel A (Std)		Steel A (I. H.)		Steel B (Std)		Steel B (I. H.)	
Stress	*Cycles*	*Stress*	*Cycles*	*Stress*	*Cycles*	*Stress*	*Cycles*
57.0	0.036	85.0	0.042	75.0	0.073	85.0	0.126
52.0	0.105	82.5	0.059	72.5	0.115	82.5	0.165
42.0	10.+	80.0	0.165	70.0	0.144	80.0	0.313
47.0	0.188	80.0	0.223	67.5	0.184	75.0	0.180
42.0	2.221	75.0	0.191	65.0	0.334	65.0	10.+
40.0	0.532	75.0	0.330	62.5	0.276	70.0	10.+
40.0	10.+	74.0	10.+	60.0	0.846	91.6	0.068
45.0	0.355	72.0	10.+	57.5	0.555	72.5	1.020
47.0	0.111	70.0	10.+	57.5	3.674	75.0	0.6009
45.0	0.241	65.0	10.+	57.5	10.+		
				55.0	0.398		
				55.0	10.+		
				55.0	10.+		

NOTE: The symbol + denotes Type I censored data.

Table 4.2. Estimation of the Model Parameters of Steel Strength Distribution

	Estimates					*Standard Error*		
Sample	*Intercept*	*Slope*	*Scale*	μ_0^+	$y_p^{\#}$	*Intercept*	*Slope*	*Scale*
1	27.5106	-0.3124	1.3079	15.0166	13.3402	3.8192	0.0826	0.3488
2	51.1589	-0.4870	1.4044	31.6788	29.8788	9.0050	0.1150	0.4396
3	28.0825	-0.2316	1.2485	18.8196	17.2193	3.3731	0.0534	0.2989
4	30.6623	-0.2235	0.9756	21.7235	20.4730	3.6376	0.0452	0.2772

NOTE: +: μ_o is the mean at stress level $X_0 = 40.0$ ksi.

#: y_p is the 10th percentile of life distribution at stress X_0.

Table 4.3. The Summary Statistics of Simulated T_i, i = 1, 2, 3, 4 Quantities

	T_1	T_2	T_3	T_4
mean	-0.5220	0.6724	0.1233	0.1098
s.d.	1.3420	5.0568	1.7985	2.9473
skewness+	0.2782	0.7333	0.3208	0.0936
kurtosis#	5.9084	5.2061	3.4982	4.0183
maximum	7.0711	22.2659	6.6499	13.5563
minimum	-6.2457	-10.9680	-5.4869	-9.0730
Quantiles				
*99%L**	-5.6316	-10.7194	-5.3175	-8.5298
98%L	-4.7446	-10.5318	-4.5794	-8.1881
95%L	-4.4958	-8.9734	-4.1470	-7.4853
90%L	-3.7524	-7.2227	-3.8111	-6.7090
80%L	-3.2705	-6.3497	-3.3377	-5.5910
80%H	2.0823	9.5048	4.1585	6.3247
90%H	2.5560	12.0582	4.5995	7.3486
95%H	3.7261	14.8509	5.3297	8.1950
98%H	5.1177	18.3894	6.2948	11.5459
99%H	7.0375	19.4132	6.5651	12.8217

NOTE: +: Coefficient of skewness μ_3/σ^3; #: Coefficient of kurtosis μ_4/σ^4; *: 100(1 − α)%L means the 100 [$\alpha/(4\times2)$] quantiles of 1000 simulated samples.

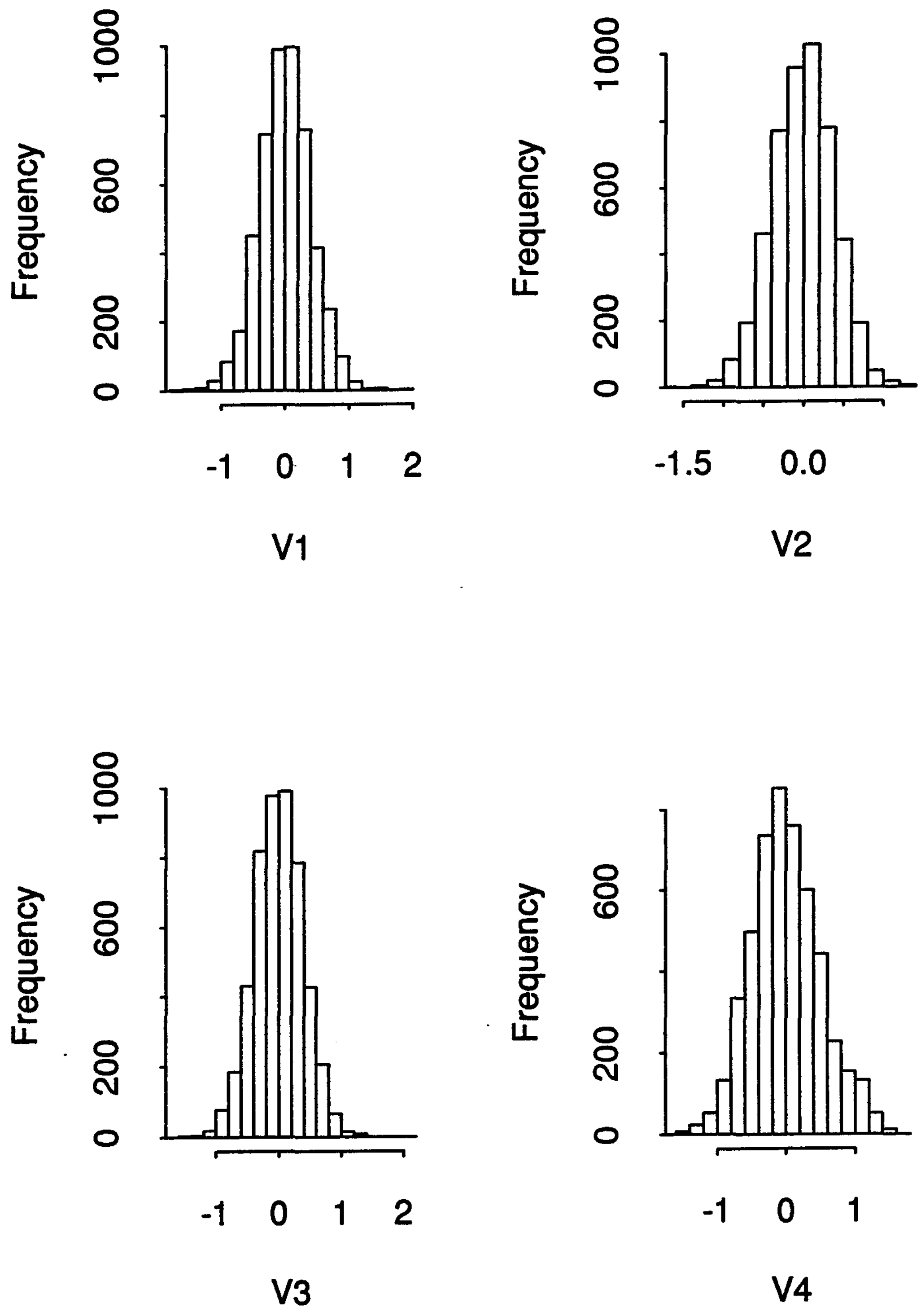

Figure 2.1. Histograms of Pivotal Statistics V_i , $i = 1, 2, 3, 4$ for Mean Lifetimes of Router Bits.

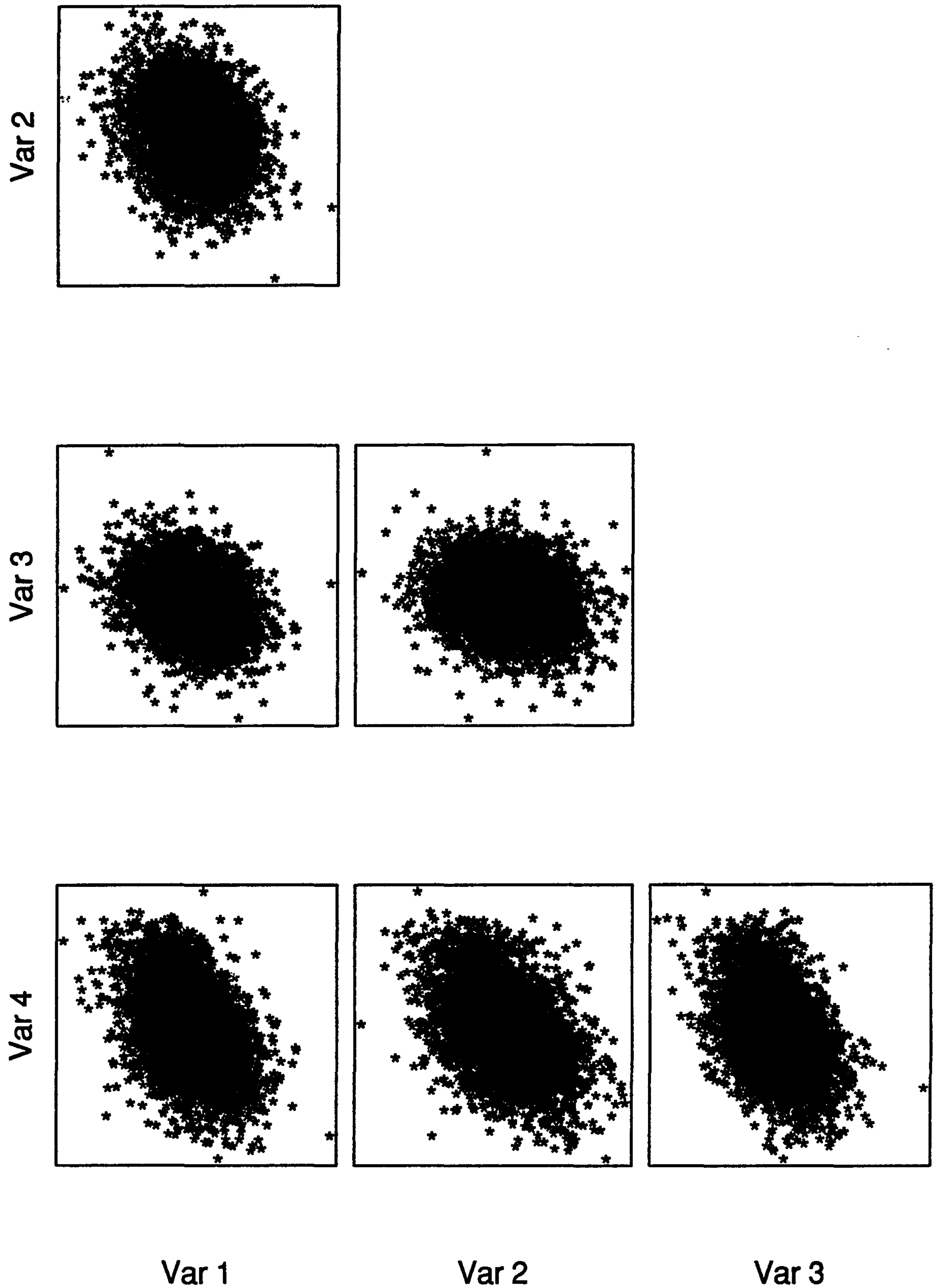

Figure 2.2. Pairwise Scatter Plots of Pivotal Statistics V_i, $i = 1, 2, 3, 4$.

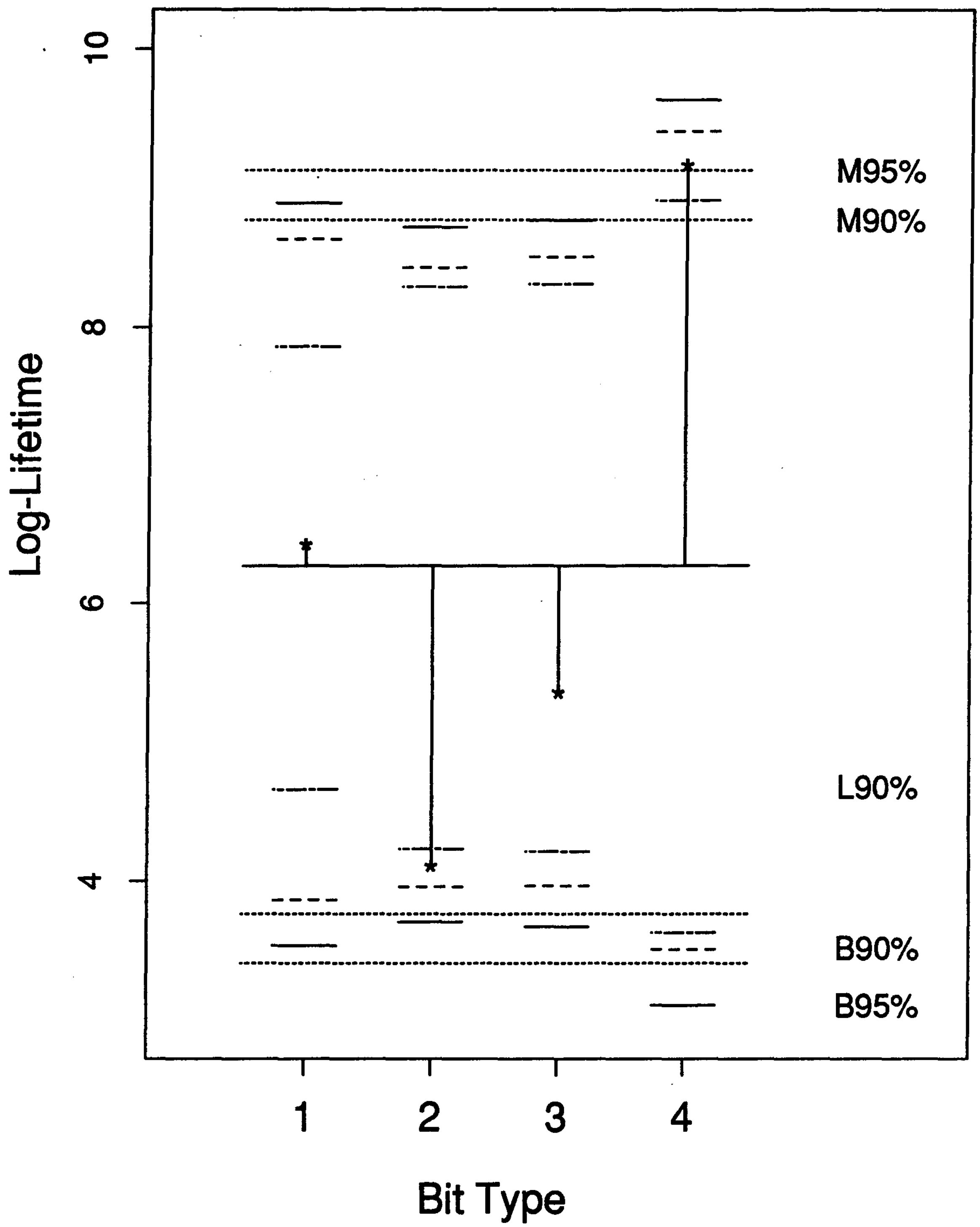

Figure 2.3. Analysis of Mean Time to Failure of Router Bit Lives.

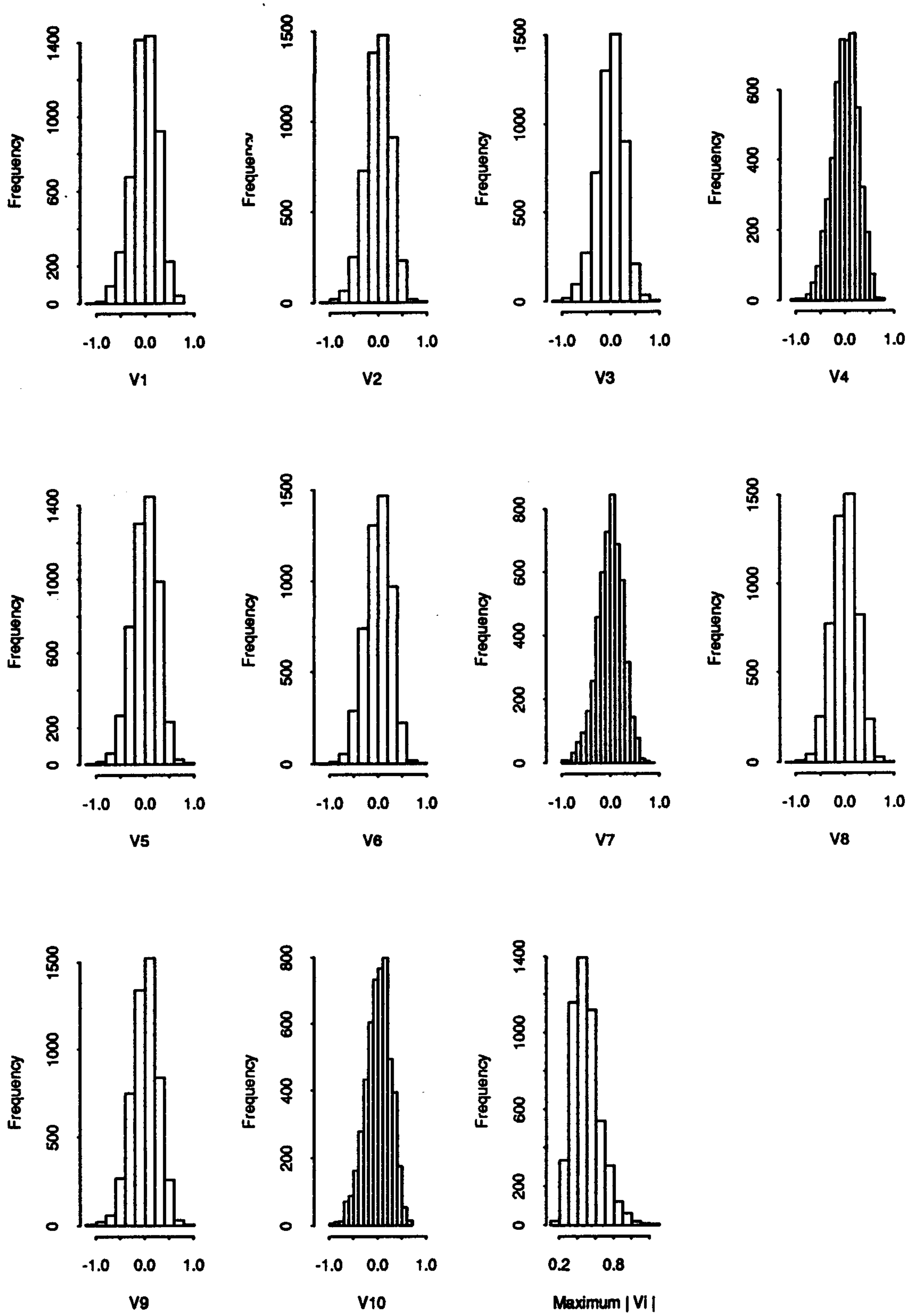

Figure 3.1. Histograms of Pivotal Statistics Z_i, $i = 1, 2, 3, 4$ for Dispersions of Fatigue Failure Times.

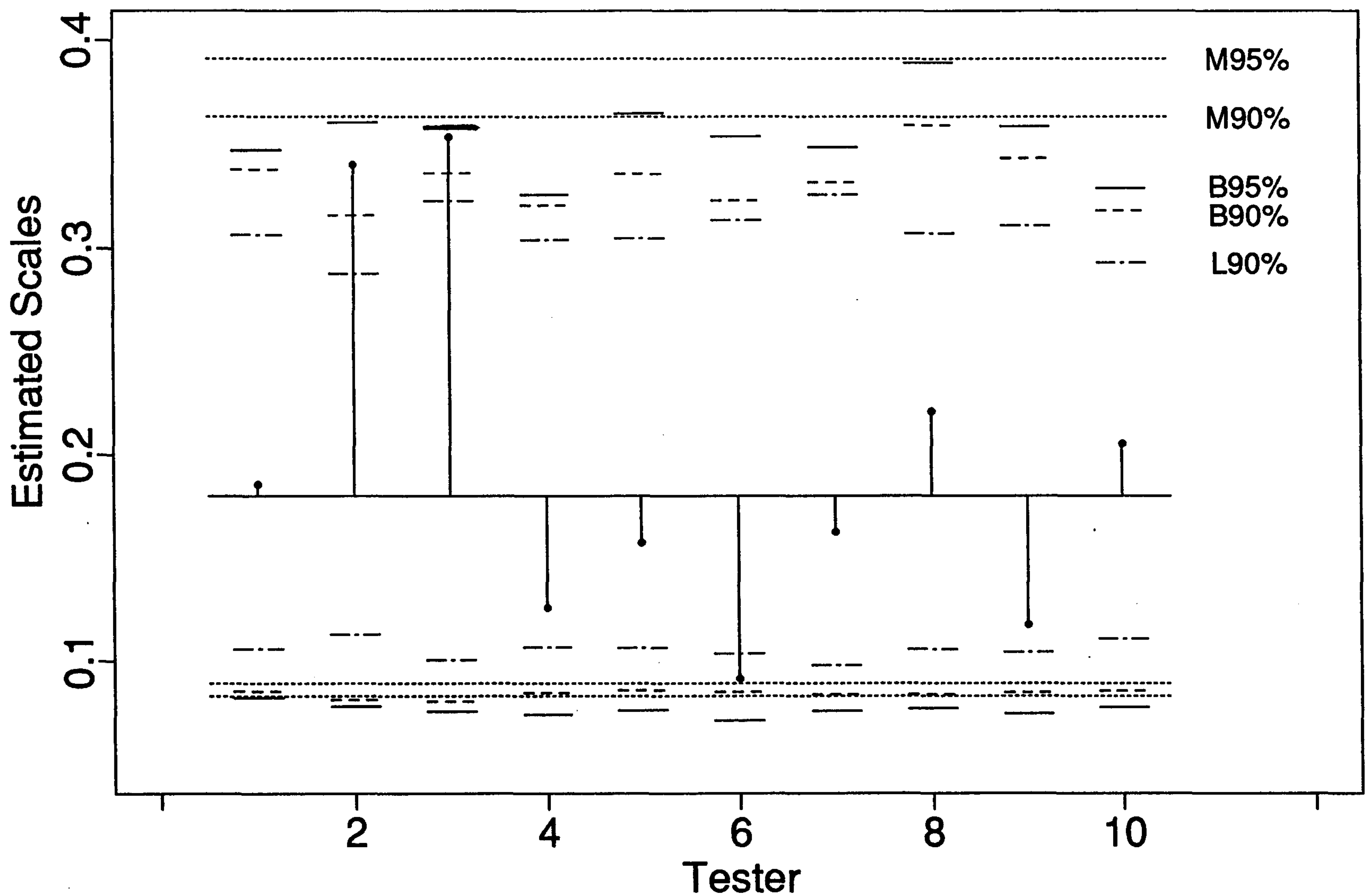

Figure 3.2. Analysis of Variations of Fatigue Failure Times in Ten Testers.

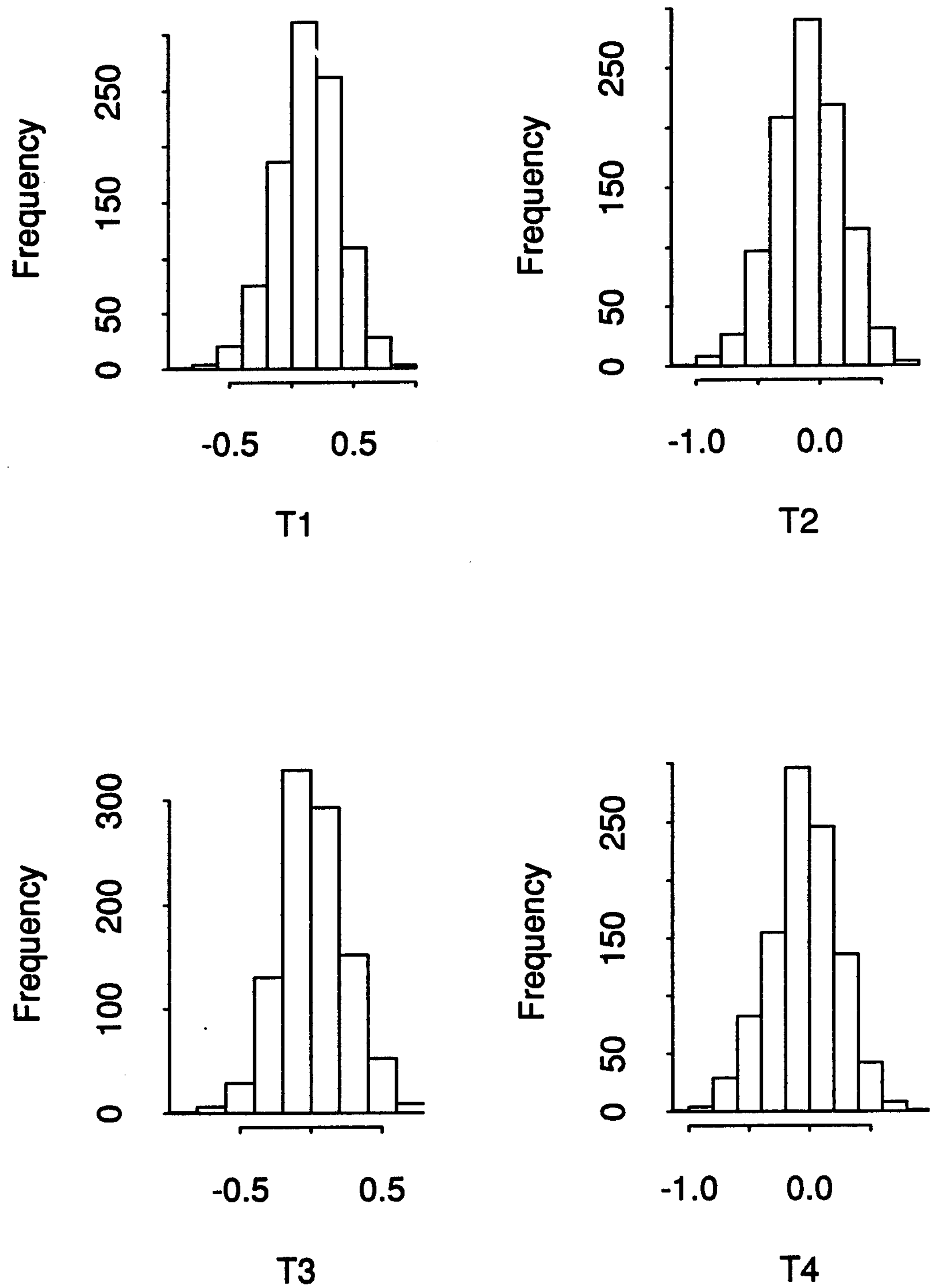

Figure 4.1. Histograms of Statistics $T_i = 1, 2, 3, 4$ for Percentile Lifetimes of Steel Strengths.

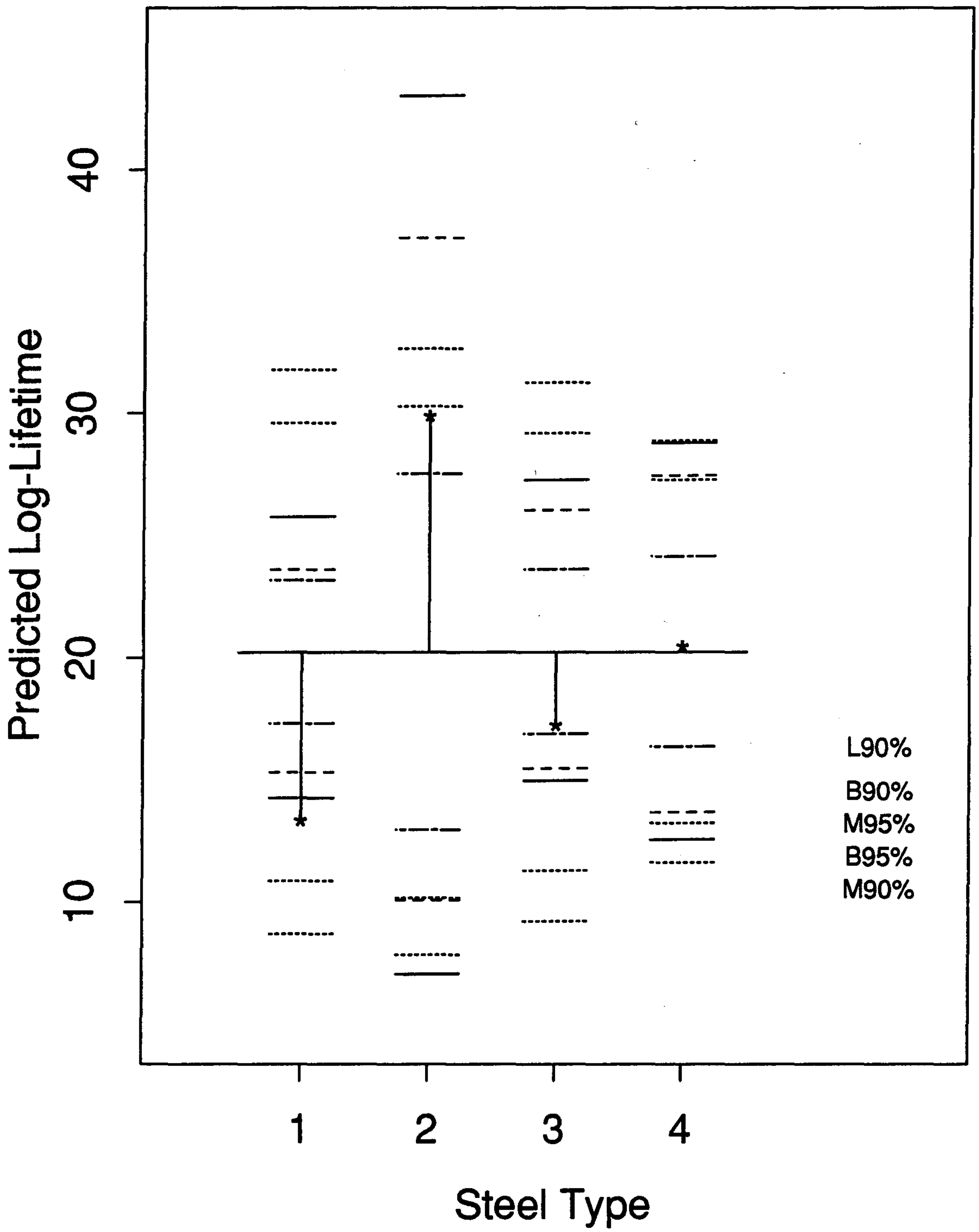

Figure 4.2. Analysis of 10th Percentile Lifetimes of Steel Strengths.