On a Problem of Ammunition Rationing

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Abstract

Suppose you have u units of ammunition and want to destroy as many as possible of a sequence of attacking enemy aircraft. If you fire $v = v(u)$, $0 \le v \le u$, units of your ammunition at the first enemy, it survives with probability q^v, where $0 < q < 1$ is given, and then kills you. With the complementary probability, $1 - q^v$, you destroy the aircraft and you live to face the next enemy with only u–v units of ammunition remaining. It seems almost obvious that any strategy which maximizes the expected number of enemies destroyed before you die will fire more units at the first enemy as u increases, i.e., it seems obvious that $v'(u) \ge 0$ under optimal play. We show this to be false thereby disproving an appealing conjecture proposed by Weber.

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1. Introduction and summary. Consider the following optimization problem. Enemy aircraft pass overhead according to a Poisson process of constant rate λ. A defender has a stockpile of u units of ammunition with which to defend himself against the aircraft. If v of these ($0 \leq v \leq u$) are fired at an aircraft, then it will be destroyed with probability $1 - q^v$, $0 < q < 1$. Otherwise, the enemy aircraft kills the defender. Here, u and v may be treated as integer-valued (discrete) or real-valued (continuous) variables.

Two different objectives have been considered in the literature. One is to maximize, in a finite time-horizon setting, the probability that all enemy aircraft arriving within the first T time units are destroyed. The problem becomes that of determining a rationing function $K_0(u,t)$ which specifies the optimal amount of ammunition to be used against an aircraft arriving t time units before the horizon T when u is the remaining amount of ammunition. It seems intuitively clear that this function satisfies several monotonicity conditions:

(A) For fixed u, $K_0(u,t)$ is nonincreasing in t.

(B) For fixed t, $K_0(u,t)$ is nondecreasing in u.

(C) For fixed t, the amount held back, $u - K_0(u,t)$, is nondecreasing in u.

Of these three "conjectures", only the third has a trivial proof. Klinger and Brown (1968) showed Conjecture A holds providing Conjecture B holds. This unsatisfactory state of affairs was partly relieved when Samuel (1970) showed Conjecture A holds whether or not B holds. Recently, Simons and Yao (1990) gave a thorough review of this problem (which was described in a slightly different guise) and obtained some results that may shed light on the veracity of Conjecture B. However, this conjecture remains unproven.

The second objective is to maximize, over an infinite time horizon, the expected time until an enemy aircraft is missed (the defender's expected "survival time") or, equivalently, to maximize the expected number of successfully destroyed enemy aircraft. The problem becomes that of determining a rationing function K(u) which specifies the optimal amount of ammunition to use against an enemy aircraft encountered when u is the remaining amount of ammunition. While it might seem intuitively reasonable that this function should be nondecreasing in u, Weber (1985) discovered that K(u) is sometimes decreasing when u and K(u) are both restricted to integer values. Since this surprising behavior can easily be "explained" by the discreteness, Weber went on to conjecture that K(u) is nondecreasing when u and K(u) are both treated as continuous. This conjecture is the subject of this paper.

We shall show that Weber's conjecture is false; the function K(u) is sometimes decreasing. We remark that this fact casts some doubt on the validity of Conjecture B, in the finite horizon setting, even though the objectives in the two settings are quite different. It would seem to be more difficult to find a counterexample to Conjecture B in the finite horizon setting because, by analogy with the infinite horizon setting, if a counterexample exists, such an example might occur only when u and t are large, and q is small. These cases are difficult to investigate numerically, because they demand extensive computations.

We have tacitly assumed in stating the problem, that $K_0(u,t)$ and K(u) are unique. In fact, we shall see that K(u) is not unique.

Section 2 describes some relevant results concerning the optimal policy; it turns out that the task reduces to that of maximizing the function

$$R(x) := \sum_{n=1}^{\infty} \prod_{j=1}^{n} (1 - e^{-x_j}) \tag{1}$$

where the infinite-dimensional vectors $x = (x_1, x_2, \cdots)$ are in the domain

$$D(u) = \{(x_1, x_2, \cdots) \in \mathbb{R}^{\infty} : x_n \geq 0,\ n \geq 1, \text{ and } x_1 + x_2 + \cdots = u\}. \tag{2}$$

Here, x_n represents the amount of ammunition used on the n–th enemy aircraft, assuming all previous aircraft are successfully destroyed, and the value of q, described above, is equal to e^{-1}, which it may be without loss of generality. Thus the optimization problem is an infinite-dimensional nonlinear programming problem of the Kuhn-Tucker–Lagrange type.

At the maximum, the initial components x_i of x must be strictly positive, strictly decreasing in i, and satisfy a specifiable, non-linear, recursive relation; the remaining components must be zero. The task of showing K(u) is sometimes decreasing reduces to showing the *first* component of x = x(u) is sometimes decreasing in u. The precise solution for x(u) can only be worked out numerically.

An additional surprise is the fact that K(u) actually *jumps* downward, occasionally, as u increases; it is *not* a continuous function. The basis for this is rigorously established in Section 2.

For the reader's benefit, we outline the picture that emerges from the mathematics and numerical computations:

(a) The objective function R attains its supremum within each domain D(u), u > 0. I.e., for each u > 0, there is a point $x = (x_1, x_2, \cdots)$ that maximizes R subject to the constraint $x_1 + x_2 + \cdots = u$. Necessarily, it is of the form: $x_1 > x_2 > \cdots > x_n > 0 = x_{n+1} = x_{n+2} = \cdots$ for some n depending on u.

(b) The method of Lagrange multipliers leads to the consideration of a sequence of continuous functions $\{f_m(t), 0 < t \leq \log 2\}_{m=1}^{\infty}$, where $f_1(t) = t$, $f_2(t) = t + \log 2$, and for $m \geq 3$,

$$f_m(t) = f_{m-1}(t) + \log\{2 - (e^{f_{m-2}(t)} - 1)/(e^{f_{m-1}(t)} - 1)\}. \tag{3}$$

The nonzero x_i's appearing in (a) are of the form $x_n = f_1(t)$, $x_{n-1} = f_2(t)$, $\cdots$, $x_1 = f_n(t)$ for some t, $0 < t \leq \log 2$. Thus, for each u > 0, any point of maximum of R within the domain D(u) can be associated with a parameter pair (n,t).

(c) It is convenient to totally order the parameter pairs (n,t) lexicographically and to view x, R and u as functions of (n,t). More precisely, let

$$X(n,t) := (f_n(t), f_{n-1}(t), \cdots, f_1(t), 0, \cdots), \tag{4a}$$

$$\mathscr{R}(n,t) := R(X(n,t)) = \sum_{m=1}^{n} \prod_{j=m}^{n} (1 - e^{-f_j(t)}), \tag{4b}$$

$$U(n,t) := f_1(t) + \cdots + f_n(t). \tag{4c}$$

One could, equivalently, view X, $\mathscr{R}$ and U as functions of a real variable $z > 0$, where $z = t + (n-1)\cdot(\log 2)$.) The functions X, $\mathscr{R}$ and U are continuous in (n,t) (since, $f_1(0+) = 0$ and $f_n(0+) = f_{n-1}(\log 2)$ for $n \geq 2$). The latter two have the same range, $(0,\infty)$, but neither one is one-to-one, unfortunately. They begin as strictly increasing functions but decrease periodically when $n \geq 27$. Thus when u is sufficiently large, there can be several candidate pairs (n,t) which are solutions of the equation $U(n,t) = u$. It follows that the recursion described above, together with the constraint $x_1 + x_2 + \cdots = u$, does not *always* tell how to maximize R; something more is needed. It is easily shown by induction, using equations (3) and (4), that

$$\mathscr{R}(n,t) = (1 - e^{-f_n(t)})/(e^{(f_{n+1}(t) - f_n(t))} - 1)$$

and

$$\mathscr{R}(n,t) = (\mathscr{R}(n-1,t) + 1)^2/(\mathscr{R}(n-2,t) + 2).$$

(d) Counter-examples to Weber's conjecture can be demonstrated with small enough values of u that the lack of one-to-one-ness, described in (c), does not cause a problem: When $u = U(7,t)$, one obtains $K(u) = f_7(t)$, the only possibility for x_1 here. But the function $f_7(t)$ is *strictly decreasing* in t when t is near zero. (One can easily evaluate the derivative of $f_7(t)$ exactly with the aid of a symbolic mathematics program, and one finds that $f_7'(0+) = -\frac{2189508611878333}{54407292942326310} \doteq -.04$.) It follows that K(u) is, in places, a decreasing function of u.

(e) The counterpart of Conjecture C holds in the current setting, i.e., $u - K(u)$ is nondecreasing in u, which, in turn, implies that the number of nonzero components of the maximizing point x is nondecreasing in u. (See (a) above.)

(f) Numerical calculations for $27 \leq n \leq 100$ reveal the following behavior: For certain values of u, the equation $U(n,t) = u$ has three different solutions of the forms $(n-1,t_1)$, (n,t_2) and (n,t_3) with $0 < t_2 < t_3 < t_1 < \log 2$. This occurs because the functions $U(n,t)$ and $\mathcal{R}(n,t)$ are decreasing near $t = 0$, then turn around and increase until $t = \log 2$, ending up higher than they are at $(n-1,\log 2)$. The maximum of R in the domain D(u) arises through one of the two pairs $(n-1,t_1)$ and (n,t_3), never through the middle pair (n,t_2). The net effect is that, for each such n, there is an interval of parameter pairs $((n-1,t_4),(n,t_5))$ (under the lexicographic ordering), with $0 < t_5 < t_4 < \log 2$, which correspond to *no* constrained maximization of the objective function R. Moreover, one finds at the endpoints of this "hole" that $U(n-1,t_4) = U(n,t_5)$, $\mathcal{R}(n-1,t_4) = \mathcal{R}(n,t_5)$, and the latter is the value of the maximum of R in the domain $D(U(n,t_5))$. So, the *maximizing point of* R *within* D(u) *is not always unique.* Finally, all other pairs (n,t), outside the holes, *do* correspond to a constrained maximization of R.

(g) The existence of holes, as described in (f) (and established by numerical calculations), explains why K(u) can not be continuous everywhere; worse than the fact that the allocation function K(u) is sometimes strictly decreasing, it occasionally *jumps* downward as u increases. This behavior occurs at every point u where K(u) is *not* uniquely defined. In fact, it will be rigorously shown that there exists a value of u at which K(u) assumes multiple values. These jumps occur whenever one "jumps over a hole", i.e., whenever $u = U(n-1,t_4) = U(n,t_5)$, as described in (f).

(h) It is unknown whether the picture becomes even more complicated beyond $n = 100$.

2. Results for the infinite horizon setting. Enemy aircraft arrive according to a Poisson process with intensity λ, so that the interarrival times are independent and exponentially distributed with mean λ^{-1}.

Now let $u > 0$ be the initial amount of ammunition. Any policy for rationing this ammunition can be expressed as an infinite–dimensional vector $x = (x_1, x_2, \cdots)$ in the set D(u) defined in (2); the n–th component is the amount rationed by the policy to the n–th enemy aircraft. Letting M denote the number of enemy aircraft destroyed by a given policy, one finds that

$$E(M) = \sum_{n=1}^{\infty} P(M \geq n) = \sum_{n=1}^{\infty} \prod_{j=1}^{n} (1 - q^{x_j}) = R(-(\log q) \cdot x),$$

where R is as shown in (1). So the expected time until a bomber is first missed equals $\lambda^{-1}\{R(-(\log q) \cdot x) + 1\}$. This is to be maximized over all vectors x in D(u). Without loss of generality, one may take $\lambda = 1$ and $-\log q = 1$. Thus the problem under consideration is that of maximizing R(x) within the domain D(u) shown in (2).

We observe in passing that within D(u),

$$R(x) \leq \sum_{n=1}^{\infty} (1 - e^{-x_n}) < \sum_{n=1}^{\infty} x_n = u, \tag{5}$$

since $(1 - e^{-x_n}) \leq x_n$, for $n \geq 1$, with a strict inequality holding whenever $x_n > 0$. Thus, $\sup\{R(x) \colon x \in D(u)\} \leq u$. Also, for any $n \geq 1$,

$$R(x) = \sum_{k=1}^{n} \prod_{j=1}^{k} (1 - e^{-x_j}) + \prod_{j=1}^{n} (1 - e^{-x_j}) \cdot R(x_{n+1}, x_{n+2}, \cdots). \tag{6}$$

We shall now show:

Lemma 1. R *attains its supremum, denoted* $R^*(u)$, *within the domain* D(u), $u > 0$.

Proof. It is convenient to extend the domain of R from D(u) (defined in (2)) to:

$$D'(u) = \{(x_1,x_2,\cdots) \in \mathbb{R}^\infty : x_n \geq 0,\ n \geq 1, \text{ and } x_1 + x_2 + \cdots \leq u\}. \qquad (2')$$

It is easily shown, when $x_1 + x_2 + \cdots = v < u$, that R is strictly smaller at $(x_1,x_2,\cdots)$ than at $(x_1+u-v,x_2,x_3\cdots)$, a point in D(u). Thus a point x is in D(u) and it maximizes R within D(u) if it is in $D'(u)$ and it maximizes R within $D'(u)$.

Now R is continuous in $D'(u)$. This follows from the dominated convergence theorem and the simple inequality

$$\prod_{j=1}^{n} (1 - e^{-x_j}) \leq (1 - e^{-u})^n.$$

Finally, consider a sequence of points in $D'(u)$ along which R converges to its supremum, and let x be a limit point of this sequence. Clearly, x is in $D'(u)$, and it maximizes R. It follows that x is in D(u), and R attains its supremum as asserted. □

We shall need:

Lemma 2. *The function* $R^*(u)$, $u > 0$, *is continuous.*

Proof. This is a direct consequence of the continuity of R. □

Any point x in D(u) that maximizes R, i.e., for which $R(x) = R^*(u)$, will be called an *optimal point* for u.

The value of the "rationing function" K at u is simply the value of the first component of the optimal point for u. Since optimal points are not always unique, K is not unique. We will show that K is sometimes decreasing regardless of the version.

Some important facts about optimal points and the function K follow:

Lemma 3.

(i) *If* $x = (x_1,x_2,\cdots)$ *is an optimal point for* u, *and if* $x_1 + \cdots + x_n < u$, *then*

$(x_{n+1}, x_{n+2}, \cdots)$ *is an optimal point for* $u - (x_1 + \cdots + x_n)$.

(ii) *For* $u \geq w \geq 0$,

$$R^*(u) \geq (1 - e^{-(u-w)})\{1 + R^*(w)\} \qquad (R^*(0) \equiv 0),$$

with equality holding if and only if $w = u - K(u)$ *for some version of* K.

Proof. These easily follow from (6). □

The following proposition gives a rough description of optimal points:

Proposition 1.

(i) *Any optimal point* $x = (x_1, x_2, \cdots)$ *for* u *has only finitely many nonzero components.*

(ii) *Moreover, it must satisfy*

$$x_1 > x_2 > \cdots > x_n > 0 = x_{n+1} = x_{n+2} = \cdots \text{ for some } n \geq 1.$$

Proof. Fix $n \geq 1$, and let $u_n = x_{n+1} + x_{n+2} + \cdots$. We shall show that u_n is zero for some n when x is an optimal point, thus establishing (i). Then (ii) can easily be established by showing, for *any* x satisfying (2) and $0 < x_i \leq x_j$, with $j > i \geq 1$, that the value of R can be *increased* by decreasing x_j by a small amount and adding this amount to x_i.

According to (6) and to (5) (applied to the point $(x_{n+1}, x_{n+2}, \cdots)$),

$$R(x_1, x_2, \cdots) - \sum_{k=1}^{n} \prod_{j=1}^{k} (1 - e^{-x_j}) \leq \prod_{j=1}^{n} (1 - e^{-x_j}) \cdot u_n \leq (1 - e^{-x_n}) \cdot u_n,$$

where $u_n = x_{n+1} + x_{n+2} + \cdots$. Moreover,

$$R(x_1 + u_n, x_2, x_3, \cdots, x_n, 0, 0, \cdots) - \sum_{k=1}^{n} \prod_{j=1}^{k} (1 - e^{-x_j}) \geq e^{-x_1} - e^{-x_1 - u_n} = (1 - e^{-u_n}) \cdot e^{-x_1}.$$

If, now, x is an optimal point for u, so that $R(x) \geq R(x_1+u_n,x_2,x_3,\cdots,x_n,0,0,\cdots)$, these inequalities combine to give the inequalities:

$$(1-e^{-x_n})\cdot u_n \geq (1-e^{-u_n})\cdot e^{-x_1},\ n \geq 1. \tag{7}$$

Finally, either $u_n = 0$ for some n or, necessarily, $0 < u_n \to 0$ and $u_n^{-1}(1-e^{-u_n}) \to 1$ as $n \to \infty$. In the latter case, $x_n \to 0$, and one sees that (7) can not hold for every $n \geq 1$. This is a contradiction. Thus $u_n = 0$ for some n, which was to be shown. □

The next result asserts that the amount of ammunition held back, $u - K(u)$, for defending against future enemy aircraft, is a nondecreasing function of u, a direct analog of "Conjecture C" for the finite-horizon problem described in Section 1. It has an important corollary. Here, K(u) is *any* version of the rationing function that may exist.

Proposition 2.

(i) *The difference* $u - K(u)$, $u > 0$, *is nondecreasing.*

(ii) *The difference is identically zero for* $0 < u \leq \log 2$ *and strictly increasing elsewhere.*

Remark. Lemma 4 below shows $u - K(u) \equiv 0$ for $u \leq \log 2$, and strictly positive elsewhere.

Proof. The argument for a strict inequality, when $u > \log 2$, requires special knowledge appearing in Proposition 3 below. We will begin by showing that $u - K(u) \geq v - K(v)$ for $u > v > 0$. Suppose it is false, so that $u - v + K(v) < x \leq u$ with $x = K(u)$. Lemma 3 yields

$$\begin{aligned} R^*(u) &\geq (1-e^{-(u-v+K(v))})\{1 + R^*(v-K(v))\} \\ &= \frac{1-e^{-(u-v+K(v))}}{1-e^{-K(v)}} R^*(v) \\ &> \frac{1-e^{-x}}{1-e^{-(x+v-u)}} R^*(v) \geq (1-e^{-x})\cdot\{1 + R^*(u-x)\}. \end{aligned}$$

The strict inequality $R^*(u) > (1-e^{-x})\cdot\{1 + R^*(u-x)\}$ contradicts the assumption $x = K(u)$. So, $v - K(v) \leq u - K(u)$.

This inequality is strict when $u > \log 2$. Otherwise, let $c > 0$ be the common value and let $(x_1,\cdots,x_n,0,\cdots)$ be an optimal point for c. (Lemma 4 below shows that c must be positive.) Necessarily, $(K(v),x_1,\cdots,x_n,0,\cdots)$ and $(K(u),x_1,\cdots,x_n,0,\cdots)$ are optimal points for v and u, respectively. (See Lemma 3.) By Proposition 3 below, $K(v) = K(u)$, a contradiction of the assumption that $v - K(v) = u - K(u)$. This completes the proof. □

Corollary 2.1. *Any version of the rationing function* K *has a downward jump at* $u > 0$ *if there is more than one choice for the value of* $K(u)$.

Remark. The converse is also true. See Lemma 6 below.

Proof. Suppose there are two possible values for $K(u)$: $k_1(u)$ and $k_2(u) > k_1(u)$. Then for $v < u$, $K(v) \geq k_2(u) - u + v$. And for $v > u$, $K(v) \leq k_1(u) - u + v$. So,

$$\liminf_{v \uparrow u} K(v) \geq k_2(u) > k_1(u) \geq \limsup_{v \downarrow u} K(v),$$

which says K jumps downward at u by at least the amount $k_2(u) - k_1(u)$. □

The next several results yield much more precise information about optimal points.

According to Proposition 1, any optimal point for u must be in one of the sets

$$C_n(u) := \{(x_1,x_2,\cdots,x_n,0,0,\cdots): x_i \geq 0,\ x_1 + x_2 + \cdots + x_n = u\},\ n \geq 1.$$

Lemma 4.

(i) *The point* $(u,0,0,\cdots)$ *is the unique optimal point for* u *when* $0 < u \leq \log 2$.

(ii) *It is not an optimal point when* $u > \log 2$.

Proof. We first find the point that maximizes $R(x)$ within $C_2(u)$. The task reduces to maximizing

$$f(v) = (1 - e^{-v})(2 - e^{-(u-v)})$$

over $0 \leq v \leq u$. This function is strictly concave; the unique maximum point can be found explicitly. It follows that there is a unique maximizing point of $R(x)$ within $C_2(u)$, namely $(u,0,0,\cdots)$ for $0 < u \leq \log 2$, and $(\frac{1}{2}(u + \log 2),\frac{1}{2}(u - \log 2),0,0,\cdots)$ for $u > \log 2$. The latter establishes the negative assertion in (ii).

We are now ready to prove part (i). If the optimal point for u is in $C_n(u)$ for some $n \geq 2$, but not in $C_{n-1}(u)$, then it follows from Lemma 3 that $(x_{n-1},x_n,0,\cdots)$ is an optimal point for $x_{n-1} + x_n$, and, hence, a maximal point of R within $C_2(x_{n-1}+x_n)$. But when $0 < u \leq \log 2$, this contradicts the fact that $(x_{n-1}+x_n,0,\cdots)$ is the unique maximal point in $C_2(x_{n-1}+x_n)$ (since $0 < x_{n-1} + x_n \leq \log 2$). This establishes part (i). □

There is no general closed–form expression for optimal points. However, by using the method of Lagrange multipliers, one can derive a simple recursive relationship satisfied by all optimal points. Suppose $x = (x_1,\cdots,x_n,0,\cdots)$ is an optimal point for u with $n \geq 2$ and $x_1 > x_2 > \cdots > x_n > 0$, so that x is an interior point of $C_n(u)$. Applying the method of Lagrange multipliers to R, one obtains:

$$\frac{\partial}{\partial y_r}\left\{ \sum_{j=1}^{n} \prod_{i=1}^{j} (1 - e^{-y_i}) - \lambda \sum_{j=1}^{n} y_j \right\} \Bigg|_{(x_1,\cdots,x_n)} = 0,$$

i.e.,

$$\sum_{j=r}^{n} \prod_{i=1}^{j} (1 - e^{-x_i}) = \lambda(e^{x_r} - 1), \quad r = 1,\cdots,n. \tag{8}$$

Comparing the r–th and (r+1)–st equations gives

$$\prod_{i=1}^{r} (1 - e^{-x_i}) = \lambda(e^{x_r} - e^{x_{r+1}}), \quad r = 1,\cdots,n-1. \tag{9}$$

Again, comparing the r–th and (r+1)–st equations of the latter gives

$$1 - e^{-x_{r+1}} = (e^{x_{r+1}} - e^{x_{r+2}})/(e^{x_r} - e^{x_{r+1}}),$$

i.e.,

$$e^{x_r} = e^{x_{r+1}}(2\,e^{x_{r+1}} - e^{x_{r+2}} - 1)/(e^{x_{r+1}} - 1), \quad r = 1,\cdots,n-2. \tag{10}$$

Finally, comparing the last equations in (8) and (9) gives

$$1 - e^{-x_n} = (e^{x_n} - 1)/(e^{x_{n-1}} - e^{x_n}),$$

i.e.

$$x_{n-1} = x_n + \log 2. \tag{11}$$

Thus, once the value of x_n is specified, the values of x_{n-1}, x_{n-2}, $\cdots$, x_1 are determined sequentially by equations (10) and (11). The value of x_n does not depend on u uniquely, but it follows from Lemmas 3 and 4 that $0 < x_n \leq \log 2$. To summarize, we have shown:

Proposition 3.

(i) *For each* $u > 0$, *there is a pair* (n,t), *with* $n \geq 1$ *and* $0 < t \leq \log 2$, *such that*

$$(f_n(t),f_{n-1}(t),\cdots,f_1(t),0,\cdots) \tag{12}$$

is an optimal point for u, *where* $f_1(t) = t$, $f_2(t) = t + \log 2$, *and the remaining functions are defined recursively as shown in* (3).

(ii) Every optimal point $x = (x_1,x_2,\cdots)$ (*for some* $u > 0$) *has the form described in* (i).

We have now justified most of the mathematical description appearing in (a)–(c) of Section 1: The expression in (12) is what we defined in (4a) as X(n,t), and it is apparent that if U(n,t), defined in (4c), is equal to $u > 0$, then X(n,t) is a prime candidate for an "optimal point for u". It must be a correct candidate unless, perhaps, (n,t) is not the only pair which makes U equal to u. Likewise, $\mathcal{R}(n,t)$, defined in (4b), equals $R^*(u)$ (the maximum attainable value of R(x), $x \in D(u)$) when X(n,t) is an optimal point for u.

The justification for viewing the pairs (n,t) as ordered lexicographically was made in Section 1; it makes all three functions $X(n,t)$, $\mathscr{R}(n,t)$, $U(n,t)$ continuous on their common domain $\mathscr{J} := \{(n,t): n \geq 1, 0 < t \leq \log 2\}$. As noted earlier, the lexicographic ordering is clearly equivalent to the introduction of a *real* variable $z > 0$, where $z = t + (n-1)\cdot(\log 2)$.

The appropriate topology for $\mathscr{J}$ is the "order topology" induced by "open intervals"; it is topologically equivalent to that for the corresponding real interval $\{z: z > 0\}$.

Now let $\mathscr{J}_0 := \{(n,t) \in \mathscr{J}: X(n,t)$ is an optimal point for $U(n,t)\}$. The following lemma can be used to show that the sets $\mathscr{J}$ and $\mathscr{J}_0$ coincide for small values of (n,t).

Lemma 5. *If* U *is increasing for every point* $(n,t) \leq (n_0,\log 2)$ *of* $\mathscr{J}$, *then there exists a* $t_0 \in (0,\log 2]$ *such that the equation* $U(n,t) = u$ *has a unique solution for all* $u < U(n_0,t_0)$. *Therefore, each point* $(n,t) < (n_0,t_0)$ *must be in* $\mathscr{J}_0$, *and the only possibility for* $K(u)$ *is* $f_n(t)$.

Proof. An elementary induction argument shows that $f_n(t) > f_{n-1}(t)$, $n \geq 2$, on the common domain $0 < t \leq \log 2$. Thus $U(n,t)$ is strictly increasing in n.

Now let (n_1,t_0) be the minimal point of U over the interval $[(n_0,\log 2),(n_0+1,\log 2)]$. It must be shown for $(n,t) < (n_0,t_0)$ and $(m,s) > (n_0,\log 2)$, that $U(m,s) > U(n,t)$. Since $m \geq n_0+1$ and $n_1 \geq n_0$, $U(m,s) \geq U(n_0+1,s) \geq U(n_1,t_0) \geq U(n_0,t_0) > U(n,t)$. □

It is easily seen that the functions $f_1(\cdot)$, $f_2(\cdot)$, and $f_3(\cdot)$ are strictly increasing, and hence, the function U is increasing for every point $(n,t) \leq (3,\log 2)$. Moreover, the ranges of $f_1(\cdot)$, $f_2(\cdot)$ and $f_3(\cdot)$ are $(0,\log 2]$, $(\log 2, 2\log 2]$ and $(2\log 2, \log(20/3)]$, respectively. So, according to Lemma 5, there is a unique optimal point for each $u \leq U(2,\log 2) = 3\log 2$. Thus we have:

Proposition 4. *For* $\log 2 < u \leq 3\log 2$, $(\frac{1}{2}(u + \log 2),\frac{1}{2}(u - \log 2),0,0,\cdots)$ *is the unique optimal point for* u.

The form of the optimal point when $u \leq \log 2$ was addressed earlier in Lemma 4. While the full strength of Lemma 5 has not been used, it will be in Section 3.

Proposition 3 and the information on the ranges of f_1, f_2 and f_3 combine to give:

Proposition 5. *If* $x = (x_1,\cdots,x_n,0,\cdots)$ *is an optimal point for some u with* $x_n > 0$, *then*

$$x_1 > x_2 > \cdots > x_{n-2} > 2\log 2 \geq x_{n-1} > \log 2 \geq x_n.$$

Thus,

$$n < \frac{u}{\log 2} + 1.$$

The next three lemmas are technical lemmas that are needed before it can be shown that any version of the rationing function K has to have a downward jump.

Lemma 6. *The set* $\mathcal{J}_0$ *is a closed subset of* $\mathcal{J}$.

Proof. Let $\{(n_m,t_m), m \geq 1\}$ be a sequence of points in $\mathcal{J}_0$ with limit point (n_0,t_0) in $\mathcal{J}$ The task is to show that (n_0,t_0) is in $\mathcal{J}_0$. If not, there must be a point (n_*,t_*) in $\mathcal{J}_0$ with $U(n_*,t_*) = U(n_0,t_0)$ and, necessarily, $\mathcal{R}(n_*,t_*) > \mathcal{R}(n_0,t_0)$. Since $\mathcal{R}$ is continuous, $\mathcal{R}(n_m,t_m)$ must converge to $\mathcal{R}(n_0,t_0)$. But since U is continuous, $U(n_m,t_m)$ must converge to $U(n_0,t_0) = U(n_*,t_*)$, and, according to Lemma 2, $\mathcal{R}(n_m,t_m)$ must converge to $\mathcal{R}(n_*,t_*)$. This is a contradiction. So (n_0,t_0) has to be in $\mathcal{J}_0$. □

Lemma 7. *The function* U *is nonconstant in every open interval of* $\mathcal{J}$.

Proof. It is enough to consider intervals of the form $((n,t_1),(n,t_2))$ with $0 < t_1 < t_2 \leq \log 2$. Since $f_1(t) = t$, $f_2(t) = t + \log 2$, and $U(n,t) = f_1(t) + \cdots + f_n(t)$, the cases $n = 1, 2$ are trivial. For $n \geq 3$, one can use the recursive formula for $f_n(t)$, and apply induction, to show $U(n,t)$ is analytic in t. So if it were constant in some interval of t, it would need to be constant for all t. This is impossible, since $U(n,\log 2) - U(n,0+) = U(n,\log 2) - U(n-1,\log 2) = f_n(\log 2) > 0$, for $n \geq 2$. □

Lemma 8. *The function* U *is strictly increasing on* $\mathcal{J}$ *if and only if it is one-to-one on* $\mathcal{J}_0$.

Proof. If U is strictly increasing on $\mathcal{J}$, the desired conclusion is obvious. If U is not strictly increasing on $\mathcal{J}$, let (n_*,t_*) be its first local maximum, i.e., $U(n_*,t_*) > U(n,t)$ for $(n,t) < (n_*,t_*)$, and also for $(n_*,t_*) < (n,t) < (n_0,t_0)$, where (n_0,t_0), in $\mathcal{J}$, is suitably chosen. Note, (i) there must be a local maximum since U is continuous on $\mathcal{J}$ with range $(0,\infty)$, and (ii) all local maxima must be isolated points (see Lemma 7 and its proof). So there must be a *first* local maximum point (n_*,t_*), *at which* U *is strictly larger than it is at neighboring points.*

Lemma 6 guarantees that there is a closest point $(n_1,t_1) \leq (n_*,t_*)$ belonging to $\mathcal{J}_0$. It will be argued, now, that there must be another point (n_2,t_2) in $\mathcal{J}_0$ with $(n_2,t_2) > (n_1,t_1)$ and $U(n_2,t_2) = U(n_1,t_1)$, so that U can not be one–to–one on $\mathcal{J}_0$. Since U maps $\mathcal{J}_0$ onto $(0,\infty)$, the sets $\{(n,t)\in \mathcal{J}_0: U(n,t) = U(n_1,t_1)+ \epsilon\}$ can not be empty for any $\epsilon \in (0,1]$. Necessarily, such sets must occur to the right of the point (n_0,t_0) if $(n_1,t_1) = (n_*,t_*)$ (i.e. if $(n_*,t_*)\in \mathcal{J}_0$), *or* to the right of (n_*,t_*) if $(n_*,t_*)\notin \mathcal{J}_0$, always within some compact interval on account of the last statement in Proposition 5. Letting $\epsilon = 1/m$, $m \geq 1$, one can find a sequence of points in $\mathcal{J}_0$ with a limit point (n_2,t_2) (greater than (n_1,t_1)) in (the closed set) $\mathcal{J}_0$ such that $U(n_2,t_2) = U(n_1,t_1)$. □

We are finally ready to argue: *Any version of the rationing function* K *has to have a downward jump.* According to Corollary 2.1, it is enough to show that the value K(u) is not unique at some value of $u > 0$. Our numerical evidence shows conclusively that U is *not* strictly increasing on $\mathcal{J}$ So, according to Lemma 8, there are points (n_1,t_1) and $(n_2,t_2) > (n_1,t_1)$ in $\mathcal{J}_0$ with the same value of U. Let us suppose n_1 is chosen as small as possible, necessarily greater than 1 because of Lemma 4. We claim the value of K(u) for $u = U(n_1,t_1) = U(n_2,t_2)$ is not unique. If it were then one would have $f_{n_1}(t_1) = f_{n_2}(t_2)$ and $U(n_1-1,t_1) = U(n_2-1,t_2)$. This is a contradiction since, according to Lemma 3, the points (n_1-1,t_1) and (n_2-1,t_2) are in $\mathcal{J}_0$. Thus, the occurrence of at least one downward jump in the function K has been established rigorously, providing the numerical evidence supports our claim that U is *not* strictly increasing on $\mathcal{J}$. We address this issue in Subsection 3b below.

3. Numerical results and discussion.

Here, we discuss numerical results needed to establish two claims made in Section 1:

(a) The function K(u) decreases for $n = 7$ and small values of t; and

(b) the function K(u) jumps downward.

3a. Showing that K(u) decreases for some values of u. As noted in item (d) of Section 1, the limiting value of the derivative of $f_7(t)$ as t goes to zero can be computed *exactly*; it is a rational number, and, most importantly, *it is negative.* If it can be shown that U is increasing for $(n,t) \leq (7,\log 2)$, then, according to Lemma 5, there is a range of u–values on which K is decreasing.

It is possible to *prove* that U is increasing for $(n,t) \leq (7,\log 2)$ by using a symbolic math program. To this end, let $s = e^t - 1$ and define a new set of functions $g_m(s) = e^{f_m(t)} - 1$, $m \geq 1$, $0 < s \leq 1$. Thus $g_1(s) = s$, $g_2(s) = 2s+1$, and the recursion shown in (3) becomes for $m \geq 3$,

$$g_m(s) = 2\, g_{m-1}(s) + 1 - g_{m-2}(s) - g_{m-2}(s)/g_{m-1}(s).$$

It easily can be checked that

$$\frac{d}{dt}\, U(n,t) = \sum_{m=1}^{n} (s+1) \cdot \frac{d}{ds} \log(g_m(s) + 1).$$

For each n, the latter sum is expressible as a ratio of polynomials in s with integer coefficients.

The precise ratios, with explicit integer coefficients, can be found using a symbolic math program such as Macsyma. This has been done. For each $n \leq 7$, and for both polynomials, the coefficients are all strictly positive. So U is strictly increasing for $n \leq 7$.

For n = 1 and 2, the ratio of polynomials reduces to the integers 1 and 2, respectively. For n = 3 and 4, one obtains, respectively:

$$\frac{18s^2 + 20s + 5}{6s^2 + 7s + 2} \text{ and } \frac{1152s^6 + 4464s^5 + 7248s^4 + 6288s^3 + 3060s^2 + 786s + 82}{288s^6 + 1152s^5 + 1948s^4 + 1178s^3 + 922s^2 + 257s + 30}.$$

This pattern, with strictly positive coefficients, continues for n = 5, 6 and 7, but with higher degree polynomials and larger coefficients. For n = 7, the polynomials are both of degree 62, and some coefficients have over 40 digits. This completes the proof that K(u) is decreasing in the immediate neighborhood to the right of u = U(6,log 2).

It is possible, by the same approach, to examine the derivatives of $f_n(t)$ for $n \leq 6$, and, thereby, to *prove* K(u) is increasing for $u \leq U(6,\log 2) \doteq 11.68$. This has been done.

3b. Showing the function K(u) jumps downward. In view of the discussion at the end of Section 2, the function must jump downward at least once if the function U ever decreases in the variable (n,t).

It turns out that $U'(27,0+) \doteq -.13037$ (the limit of the derivative of U(27,t) as $t \downarrow 0$), so U(27,t) *is decreasing for small values of* t. This assertion is based on *floating-point* calculations. While we have no doubts concerning its validity, it would be correct to say that the "standard of proof" is not as high as that attained in Subsection 3a through the employment of *symbolic mathematical calculations.* One could, *in principle*, use the same methodology here. But symbolic mathematical calculations for n larger than about 10 are not practicable. This completes the "proof" that any version of the rationing function K has to jump downward at least once.

This jump is accompanied by a "hole" in $\mathcal{J}$ which contributes no points to $\mathcal{J}_0$. (This is the open interval $((n_1,t_1),(n_2,t_2))$ indirectly alluded to in the proof of Lemma 8. Here, $n_1 =$ 26 and $n_2 = 27$.) The location of the hole can be found numerically (as the solution of the two equations $U(26,t_1) = U(27,t_2)$ and $\mathcal{R}(26,t_1) = \mathcal{R}(27,t_2)$). It is the very small open interval ((26,.69313),(27,.00705)). (For comparison, $\log 2 \doteq .69315$.) The jump (from $f_{26}(.69313) \doteq 4.78548$ to $f_{27}(.00705) \doteq 4.78434$) is $-.00114$ approximately, which also is quite small.

Our numerical investigations, working with $n \leq 100$, suggest many more properties concerning the set $\mathcal{J}_0$, and the functions U and $\mathcal{R}$, than our limited theory can justify: We feel confident that the functions U and $\mathcal{R}$ increase in (n,t) through n = 26, so that the first evidence that U is not strictly increasing in (n,t) occurs when n = 27. From n = 27 onward (through n = 100 at least), $U(n,\cdot)$ and $\mathscr{R}(n,\cdot)$ start off the interval $0 < t \leq \log 2$ by decreasing but eventually increase; the negatives of $U(n,\cdot)$ and $\mathscr{R}(n,\cdot)$ seem to be unimodal. It appears that

$$\mathscr{I}_0 = \{(n,t) \leq (26,t_{26})\} \cup [(27,v_{27}),(27,t_{27})] \cup [(28,v_{28}),(28,t_{28})] \cup \cdots \tag{13}$$

with $0 < v_n < t_{n-1} < \log 2$ for $n \geq 27$. Moreover, it appears that U and $\mathscr{R}$ are nondecreasing, continuous functions on $\mathscr{I}_0$ with continuity extending across the boundaries of $\mathscr{I}_0$ in the sense that $U(n-1,t_{n-1}) = U(n,v_n)$ and $\mathscr{R}(n-1,t_{n-1}) = \mathscr{R}(n,v_n)$ for $n \geq 27$. Finally, it appears that U and $\mathscr{R}$ are strictly increasing at all other points of $\mathscr{I}_0$. These assertions imply that $K(\cdot)$ is non–unique and has a downward jump at a point u if it is in the set $\{U(n,v_n), n \geq 27\}$. We believe that K is unique and continuous at all other values of $u > 0$. See the comments in item (f) of Section 1 for some additional information.

A sample of specific calculations is shown in Table 1 below, and graphs of K(u) are shown in Figures 1–4 below. Figure 1 shows a macroscopic view. At first sight, it appears to be an increasing concave function; a closer look shows it to be "saw–toothed". Figures 2–4 show microscopic views over three different short ranges of u, corresponding to specified ranges of n. The function K(u) can be seen to be strictly increasing in Figure 2, continuous and periodically decreasing in Figure 3, and jumping downward in Figure 4.

n	t_{n-1}	v_n	Common U value	Common $\mathscr{R}$ value	Jump in K
30	.69277	.03170	22.64238	108.35070	−.004764
40	.68983	.09593	31.03308	159.69041	−.011628
50	.68583	.14123	39.49702	214.19119	−.014532
60	.68193	.17770	48.01552	271.18720	−.015981
70	.67758	.20276	56.57433	330.22463	−.016330

TABLE 1

Unfortunately, it is not easy to determine whether a given point (n,t) is in $\mathscr{I}_0$ (except when n is small), and it is fairly difficult to find the boundary points of $\mathscr{I}_0$, such as those shown in Table 1; we have no theory which simplifies these tasks. However, the following simple observation may be of some interest. Let $\mathcal{A}_n = \{t\in(0,\log 2]: (n,t)\in\mathcal{T}_0\}$. By Lemma 3, $(n-1,t)\in\mathcal{T}_0$ whenever $(n,t)\in\mathcal{T}_0$ $(n \geq 2)$, so that the sequence of sets $\mathcal{A}_n$ is nonincreasing in n. Numerical results indicate that for $n > 26$, $\mathcal{A}_n$ is bounded away from zero, so that it is a closed (compact) subset of $(0,\log 2]$. Since each $\mathcal{A}_n$ is nonempty (otherwise, $\mathcal{A}_n$ would be empty for all larger n), one can easily argue that the limit of $\mathcal{A}_n$ (i.e., the intersection of all $\mathcal{A}_n$) is a *nonempty* closed set. It would be interesting to identify this limit set. Our numerical results suggest that each $\mathcal{A}_n$ is an interval, as indicated in (13), so that the limit set is an interval of the form $[v_\infty,t_\infty]$. But we are unable to justify this claim.

Since $\{(u,R^*(u)): u > 0\}$ has to be a *proper* subgraph of $\{(U(n,t),\mathcal{R}(n,t)): (n,t) \in \mathcal{T}\}$ (because $\mathcal{T}_0$ is a proper subset of $\mathcal{T}$), we find it surprising that the plots of the two graphs are practically indistinguishable: If $U(n_1,t_1) = U(n_2,t_2) = u$, the values of $\mathcal{R}(n_1,t_1)$ and $\mathcal{R}(n_2,t_2)$ are very nearly equal, even when $(n_1,t_1) \in \mathcal{T}_0$ and $(n_2,t_2) \notin \mathcal{T}_0$. Apparently, the function $R(\cdot)$ is quite "flat" in the vicinity of the optimal point for u. For this reason, we suspect there exists a *monotone* variant of rationing function $K(\cdot)$, which performs very nearly as well as any (non–monotone) version of $K(\cdot)$, so that Weber's conjecture, while false, is "true" from a pragmatic standpoint. Then, apart from the compelling mathematical evidence presented above, it would have been extremely difficult to rule out the conjecture *with numerical computations alone.*

We conclude with the observation that the facts of the "infinite-horizon bomber problem", discussed here, have proven to be much more complicated than we anticipated when we began our study; it has some unexpected surprises. Perhaps this explains why important details of the structurally more complex "finite-horizon bomber problem", introduced by Klinger and Brown more than 20 years ago, are still not sorted out.

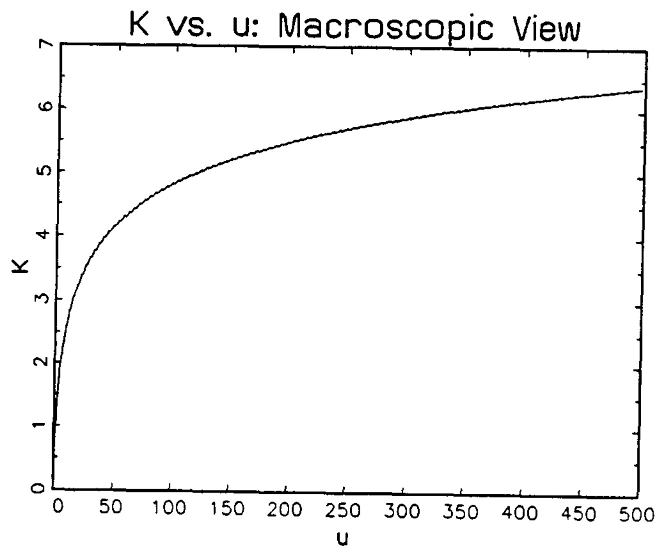

FIGURE 1

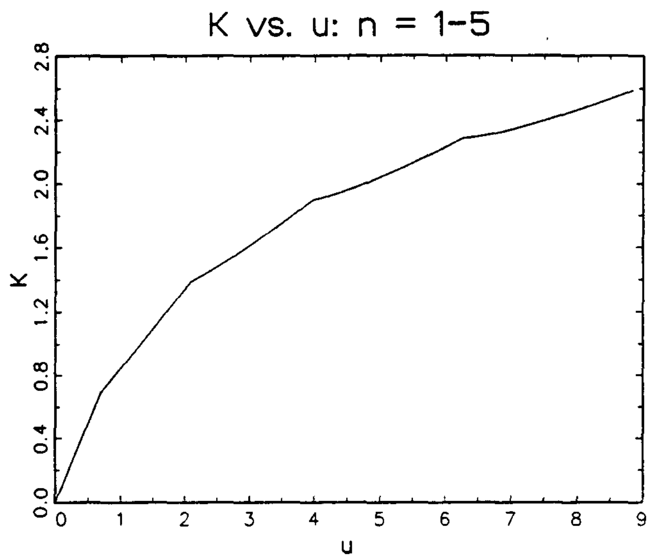

FIGURE 2

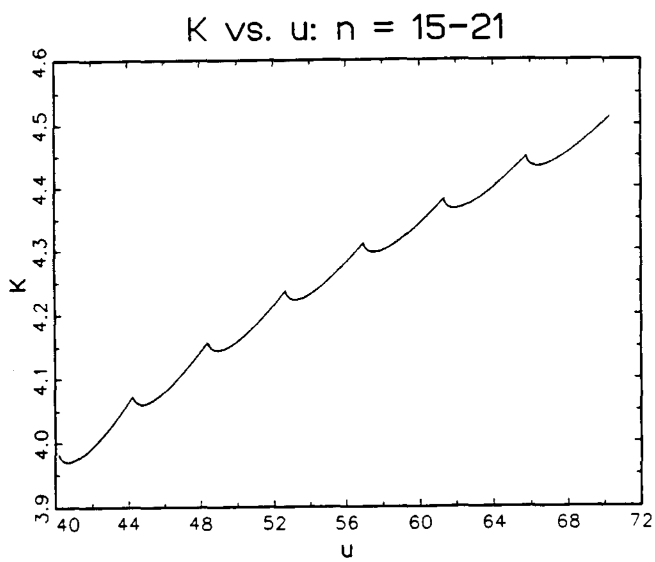

FIGURE 3

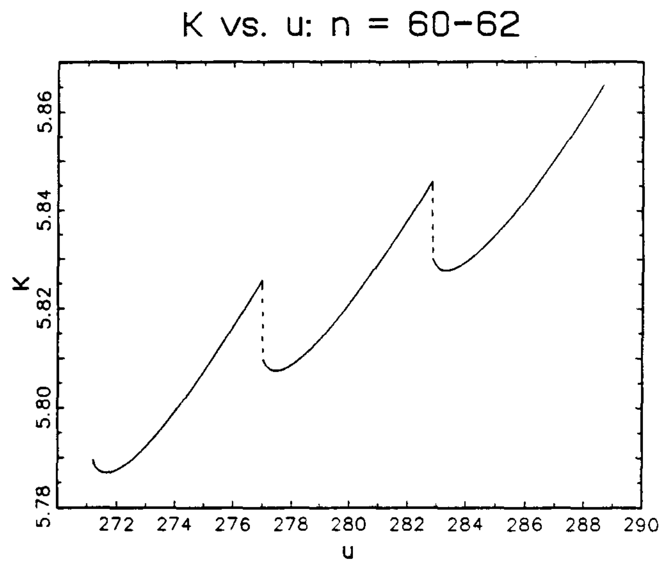

FIGURE 4

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