Geometric Construction of Some Two-Class and Three-Class Association Schemes and Codes from Non-Degenerate and Degenerate Hermitan Varieties. I.M. Chakravarti

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Geometric Construction of Some Families of Two-class and Three-class Association Schemes and Codes from Non-degenerate and Degenerate Hermitian Varieties

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0. Summary.

Taking a non-degenerate Hermitian variety V_1 defined by the equation $x_o^{s+1} + x_1^{s+1} + x_2^{s+1} = 0$, as the projective set in a hyperplane $\mathscr{H} = PG(2,s^2)$, Mesner (1967) obtained a two-class association scheme with parameters $v = s^6$, $n_1 = (s^2-1)(s^3+1)$, $p_{11}^1 = s^2(s^2+1) - s^3 - 2$, $p_{11}^2 = s^2(s^2 - 1)$.

We generalize his construction in two ways. First, we show that his construction works for a non-degenerate Hermitian variety in a projective space of any dimension. Secondly, we allow degenerate Hermitian varieties also as projective sets.

In the first case, we take as our projective set a non-degenerate Hermitian variety V_{N-2} defined by the equation $x_0^{s+1} + \dots + x_{N-1}^{s+1} = 0$ in a hyperplane $\mathscr{H} = PG(N-1, s^2)$ of $PG(N,s^2)$ (which is the set of points of $PG(N,s^2)$ not on $\mathscr{H}$). Two points a and b are defined to be *first associates* if the line $\overline{ab}$ is incident with a point on V_{N-2}; *second associates* if the line is incident with a point on $\mathscr{H}$, not on V_{N-2}. We show that this gives a family of two-class association schemes with parameters, $v = s^{2N}$, $n_1 = (s^N-(-1)^N)(s^{N-1}-(-1)^{N-1})$, $p_{11}^1 = s^{2N-2}-(-s)^{N-1}(s-1)-2$, $p_{11}^2 = s^{2N-2}-(-s)^{N-1}$. This family of association schemes has the same set of parameters as those derived as restrictions of the Hamming association schemes to two-weight codes defined as linear spans of

coordinate vectors of points on a non-degenerate Hermitian variety in $PG(N-1,s^2)$. The relations of these codes to orthogonal arrays and difference sets are described in Calderbank and Kantor (1986) and Chakravarti (1990).

In the second case, we take a degenerate Hermitian variety V_1^0 which is the intersection of a non-degenerate Hermitian variety V_2 in $PG(3,s^2)$ with one of its tangent planes, say $\mathcal{T} \equiv PG(2,s^2)$ at the point C on V_2. The points of $PG(3,s^2)$ which are not on $\mathcal{T}$ define a 3-dimensional affine space $E_3 \equiv EG(3,s^2)$.

Two points a and b of E_3 are defined to be *first associates* if the line $\overline{ab}$ meets V_1^0 at a *regular point*; *second associates* if the line $\overline{ab}$ meets $\mathcal{T}$ at an *external point* (not on V_1^0); third associates if the line $\overline{ab}$ passes through the singular point C. Then we show that it is a three-class association scheme on $v = s^6$ points of $EG(3,s^2)$, with

$$n_1 = (s^3+s^2)(s^2-1), \quad n_2 = (s^4-s^3)(s^2-1), \quad n_3 = (s^2-1),$$

$$P_1 = (P^1_{ij}) = \begin{bmatrix} s^2(2s^2-s-2) & s^4(s-1) & s^2-1 \\ s^4(s-1) & s^3(s-1)(s^2-s-1) & 0 \\ s^2-1 & 0 & 0 \end{bmatrix},$$

$$P_2 = (P^2_{ij}) = \begin{bmatrix} s^3(s+1) & s^2(s+1)(s^2-s-1) & 0 \\ s^2(s+1)(s^2-s-1) & s^2(s^2-2)+(s^2-s-1)(s^2-s-2)s^2 & s^2-1 \\ 0 & s^2-1 & 0 \end{bmatrix},$$

$$P_3 = (P^3_{ij}) = \begin{bmatrix} s^2(s+1)(s^2-1) & 0 & 0 \\ 0 & s^3(s^2-1)(s-1) & 0 \\ 0 & 0 & s^2-2 \end{bmatrix},$$

1. Introduction

The geometry of Hermitian varieties in finite dimensional projective spaces have been studied by Jordan (1870), Dickson (1901), Dieudonné (1971), and recently, among others by Bose (1963, 1971), Segre (1965, 1967), Bose and Chakravarti (1966) and Chakravarti (1971). In this paper, however, we have used results given in the last two articles.

If h is any element of a Galois field $GF(s^2)$, where s is a prime or a power of a prime, then $\bar{h}=h^s$ is defined to be conjugate to h. Since $\bar{h}^2=h$, h is conjugate to $\bar{h}$. A square matrix $H = (h_{ij})$, $i,j = 0,1,\ldots,N$, with elements from $GF(s^2)$ is called Hermitian if $h_{ij} = \bar{h}_{ji}$ for all i,j. The set of all points in $PG(N,s^2)$ whose row-vectors $\underline{x}^T = (x_0,x_1,\ldots,x_N)$ satisfy the equation $\underline{x}^T H \underline{x}^{(s)} = 0$ are said to form a Hermitian variety V_{N-1}, if H is Hermitian and $\underline{x}^{(s)}$ is the column vector whose transpose is $(x_0^s, x_1^s,\ldots,x_N^s)$. The variety V_{N-1} is said to be non-degenerate if H has rank N+1. The Hermitian form $\underline{x}^T H \underline{x}^{(s)}$ where H is of order N+1 and rank r can be reduced to the canonical form $y_0\bar{y}_0 + \ldots + y_r\bar{y}_r$ by a suitable non-singular linear transformation $\underline{x} = A\underline{y}$. The equation of a non-degenerate Hermitian variety V_{N-1} in $PG(N,s^2)$ can then be taken in the canonical form $x_0^{s+1} + x_1^{s+1} + \ldots + x_N^{s+1} = 0$.

Consider a Hermitian variety V_{N-1} in $PG(N,s^2)$ with equation $\underline{x}^T H \underline{x}^{(s)} = 0$. A point C in $PG(N,s^2)$ with row-vector $\underline{c}^T = (c_0,c_1,\ldots,c_N)$ is called a *singular* point of V_{N-1} if $\underline{c}^T H = \underline{0}^T$ or equivalently, $H \underline{c}^{(s)} = \underline{0}$. A point of V_{N-1} which is not singular is called a *regular* point of V_{N-1}. Thus a non-singular point is either a regular point of V_{N-1} or a point not on V_{N-1}. It is clear that a non-degenerate V_{N-1} cannot possess a singular point. On the other hand, if V_{N-1} is degenerate and rank $H = r < N+1$, the singular points of V_{N-1} constitute a (N-r)-flat called the *singular space* of V_{N-1}.

Let C be a point with row vector $\underline{c}^T$. Then the *polar space* of C with respect to the Hermitian variety V_{N-1} with equation $\underline{x}^T H \underline{x}^{(s)} = 0$, is defined to be the set of points of $PG(N,s^2)$ which satisfy $\underline{x}^T H \underline{c}^{(s)} = 0$.

When C is a singular point of V_{N-1}, the polar space of C is the whole space $PG(N,s^2)$. When, however, C is neither a regular point of V_{N-1} or an external point, $\underline{x}^T H \underline{c}^{(s)} = 0$ is the equation of a hyperplane which is called the *polar hyperplane* of C with respect to V_{N-1}. Let C and D be two points of $PG(N,s^2)$. If the polar hyperplane of C passes through D, then the polar hyperplane of D passes through C. Two such points C and D are said to be *conjugates* to each other with respect to V_{N-1}. Thus the points lying in the polar hyperplane of C are all the points which are conjugates to C. If C is a *regular* point of V_{N-1}, the polar hyperplane of C passes through C; C is thus self-conjugate. In this case, the polar hyperplane is called the *tangent hyperplane* to V_{N-1} at C.

When V_{N-1} is non-degenerate, there is no singular point. To every point, there corresponds a unique polar hyperplane, and at every point of V_{N-1}, there is a unique tangent hyperplane. If C is an external point, its polar hyperplane will be called a *secant hyperplane*.

The number of points in a non-degenerate Hermitian variety V_{N-1} in $PG(N,s^2)$ is $\Phi(N,s^2) = (s^{N+1} - (-1)^{N+1})(s^N - (-1)^N)/(s^2-1)$.

A polar hyperplane $\mathcal{L}_{N-1}$ of an external point $\mathcal{D}$ (also called a secant hyperplane) in $PG(N,s^2)$ intersects a non-degenerate Hermitian variety V_{N-1}, in a non-degenerate Hermitian variety V_{N-2} of rank N. It has $(s^N - (-1)^N)(s^{N-1})/(s^2-1)$ points.

A tangent hyperplane $\mathcal{T}_{N-1}$ to a non-degenerate V_{N-1} at a point C, intersects V_{N-1} in a degenerate V_{N-2} of rank N-1. The singular space of V_{N-2} consists of the single point C. Every point of V_{N-2} lies on a line joining C to the points of a non-degenerate V_{N-3} lying on an (N-2) - dimensional flat disjoint with C.

The number of points in a degenerate Hermitian variety V_{N-1} of rank $r < N+1$ in $PG(N,s^2)$ is $(s^2-1)\, f(N-r,s^2)\, \Phi(r-1,s^2) + f(N-r,s^2) + \Phi(r-1,s^2)$, where $f(k,s^2) = (s^{2(k+1)}-1)/(s^2-1)$. Thus the number of points in a degenerate V_{N-2} of rank N-1, is

$$(s^2-1)f(0,s^2)\Phi(N-2,s^2) + f(0,s^2) + \Phi(N-2,s^2)$$

$$= 1 + (s^{N-1} - (-1)^{N-1})(s^{N-2} - (-1)^{N-2})s^2/(s^2-1).$$

For the definition of an association scheme and related results, please see Bose and Mesner (1959).

2. Two-class association scheme from a non-degenerate Hermitian variety in $PG(N-1,s^2)$.

Let V_{N-2} be a non-degenerate Hermitian variety defined by the equation

$$x_o^{s+1} + x_1^{s+1} + \ldots + x_{N-1}^{s+1} = 0,$$

in a $\mathcal{H} = PG(N-1,s^2)$. Consider $\mathcal{H}$ as the hyperplane at infinity in a $PG(N,s^2)$. Then the affine space complementary to $\mathcal{H}$ in $PG(N,s^2)$ is $EG(N,s^2)$.

Suppose d_O is a point on V_{N-2}. The tangent hyperplane $\mathcal{T}(d_o)$ at d_o intersects V_{N-2} in a degenerate V_{N-3} with d_o as the point of singularity. V^o_{N-3} consists of d_o and all the points on the lines joining d_o to the points of a non-degenerate Hermitian variety V_{N-4}. Thus the number of generator lines through d_o is the same as the number of points on V_{N-4}, which is

$$(s^{N-2} - (-1)^{N-2})(s^{N-3} - (-1)^{N-3})/(s^2-1).$$

The number of tangent lines through d_o, = the number of lines in $\mathcal{T}(d_o)$ through d_o - number of generator lines through d_o =

$$(s^{2N-4}-1)/(s^2-1) - (s^{N-2}-(-1)^{N-2})(s^{N-3}-(-1)^{N-3})/(s^2-1)$$

$$= (s^{2N-5} + (-s)^{N-3})/(s+1).$$

The number of secants (lines which are neither tangents nor generators) through d_0 =

$$\text{Number of lines through } d_0 \text{ in } PG(N-1,s^2)$$

$$- \text{Number of lines through } d_0 \text{ on } \mathcal{T}(d_0)$$

$$= (s^{2N-2}-1)/(s^2-1) - (s^{2N-4}-1)/(s^2-1) = s^{2N-4}$$

Each secant line meets V_{N-2} at s+1 points.

Suppose d is an external point of $\mathcal{H}$, that is, a point of $\mathcal{H}$ which is not on V_{N-2}. The polar of d intersects V_{N-2} in a non-degenerate Hermitian variety V_{N-3}. Each one of the points on V_{N-3} is conjugate to d. Hence the tangent hyperplanes at each one of these points will pass through d. Thus the number of tangent lines through d is the same as the number of points conjugate to d, which is

$$(s^{N-1} - (-1)^{N-1})\ (s^{N-2} - (-1)^{N-1})/(s^2-1).$$

Hence the number of secant lines through d

$$= \frac{1}{(s^2-1)(s+1)} \{(s^N - (-1)^N)(s^{N-1} - (-1)^{N-1})-(s^{N-1} - (-1)^{N-1})(s^{N-2} - (-1)^{N-2})\}$$

$$= s^{N-2}(s^{N-1} - (-1)^{N-1})/(s+1).$$

Define two points a and b of $EG(N,s^2)$ to be *first associates* if the line $\bar{ab}$ meets $\mathcal{H}$ at a point of V_{N-2}; *second associates* if the line $\bar{ab}$ meets $\mathcal{H}$ at an external point of $\mathcal{H}$.

Then

$$p_{11}^{1}(a,b) = (s^2-2) + s^2(s^2-1)\,\frac{(s^{N-2}-(-1)^{N-2})(s^{N-3}-(-1)^{N-3})}{s^2-1} + s(s-1)s^{2N-4}$$

$$= s^{2N-2} - (-s)^{N-1}(s-1) - 2,$$

which is independent of the pair of points a and b.

Also,

$$p_{11}^{2}(a,b) = s(s+1)\,\frac{s^{N-2}(s^{N-1}-(-1)^{N-1})}{s+1}$$

$$= s^{2N-2} - (-s)^{N-1},$$

which is again independent of the pair of points a and b. Thus this is a two-class association scheme with

$$v = s^{2N},\ n_1 = (s^N-(-1)^N)(s^{N-1}-(-1)^{N-1}),$$

$$p_{11}^{1} = s^{2N-2}-(-s)^{N-1}(s-1)-2 \quad \text{and} \quad p_{11}^{2} = s^{2N-2}-(-s)^{N-1}.$$

It is known (Calderbank and Kantor (1986), Chakravarti (1990)) that the code generated by the linear span of the coordinate vectors of the points on a non-degenerate Hermitian variety V_{N-2} in $PG(N-1,s^2)$ is a two-weight projective code C in s^2 symbols, with weights $w_1 = s^{2N-3}$ and $w_2 = s^{2N-3}+(-s)^{N-2}$ with respective frequencies

$$f_{w_1} = (s^N-(-1)^N)(s^{N-1}-(-1)^{N-1}) \quad \text{and} \quad f_{w_2} = (s-1)(s^{2N-1}+(-s)^{N-1}).$$

This code in its turn determines another code C' in s symbols, with parameters

$$n' = (s^N-(-1)^N)(s^{N-1}-(-1)^{N-1})/(s-1),\quad k = 2N$$

$$w_1' = s^{2N-2},\quad w_2' = s^{2N-2}-(-s)^{N-1},$$

$$f_{w_1'} = (s^N-(-1)^N)(s^{N-1}-(-1)^{N-1}) \quad \text{and}$$

$$f_{w_2'} = (s-1)(s^{2N-1}+(-s)^{N-1}) .$$

The graph on s^{2N} vertices corresponding to the codewords of C' over GF(s), is strongly regular, that is, it is the graph of a two-class association scheme with parameters,

$$v = s^{2N}, \quad n_1 = (s^N-(-1)^N)(s^{N-1}-(-1)^{N-1})$$

$$p_{11}^1 = s^{2N-2}-(-s)^{N-1}(s-1)-2 \quad \text{and} \quad p_{11}^2 = s^{2N-2}-(-s)^{N-1}$$

(Chakravarti 1990). This is the restriction of the Hamming association scheme $\mathcal{H}_{n'}(s)$ to the code C'; which has the same parameters as the one derived earlier by a Mesner-type construction.

3. Three-class association scheme from a degenerate Hermitian variety in $PG(3,s^2)$.

Let V_2 be a non-degenerate Hermitian variety in $PG(3,s^2)$. Let V_1^0 denote the degenerate Hermitian variety which is derived as an intersection of V_2 with one of its tangent planes, say $\mathcal{T} = PG(2,s^2)$ at the point C on V_2. Then C is the point of singularity of V_1^0. V_1^0 consists of C and the points on lines joining C to the points of a non-degnerate V_0, which has (s+1) points. Thus the number of points on V_1^0 is $1 + s^2(s+1) = 1 + s^2 + s^3$.

The points on $PG(3,s^2)$ which are not on $\mathcal{T}$ form a 3-dimensional affine space $EG(3,s^2)$ which has s^6 points. Every line of $EG(3,s^2)$ meets $\mathcal{T}$ (the plane at infinity) exactly at one point.

Two points a and b of $EG(3,s^2)$ are defined to be *first associates* if the line $\overline{ab}$ joining a and b, meets $\mathcal{T}$ at a *regular point* of V_1^0; *second associates* if the line $\overline{ab}$ meets $\mathcal{T}$ at a point external to V_1^0; *third associates* if the line $\overline{ab}$

passes through the point of singularity C. To show that this defines a three-class association scheme, we do enumerations and use geometric arguments similar to those of Mesner (1967).

Since there are s^3+s^2+1 points on V_1^0 of which only one point C is singular, the remaining s^3+s^2 are regular points. Thus the number of first associates of a given point is

$$n_1 = (s^3 + s^2)(s^2-1).$$

Now the number of points on $\mathcal{J}$ which are external to V_1^0 is $(s^4+s^2+1)-(s^3+s^2+1) = s^4-s^3$. Thus the number of second associates of a given point is

$$n_2 = (s^4-s^3)(s^2-1).$$

The number of third associates of a given point a is equal to the number of affine points (other than a) on the line joining a to C. Hence $n_3 = s^2-1$.

The following results which we need for proving the constancy of the parameter $p^i_{jk}(a,b)$, can be found in Bose and Chakravarti (1966) and Chakravarti (1971).

(i) There are s+1 lines through C, the point of singularity, which are generators, that is, each line intersects V_1^0 at s^2 points other than C. The remaining s^2-s lines on $\mathcal{J}$, passing through C are tangent lines at C, that is, each line meets V_1^0 only at C.

(ii) Suppose D is a regular point on V_1^0, that is, D $\neq$ C. Then there is exactly one generator through D, DC, which meets V_1^0 at s^2+1 points and there are s^2 lines through D and $\mathcal{J}$, which are secants, that is, each line meets V_1^0 at (s+1) points.

(iii) Suppose D is a point on the plane $\mathcal{J}$, but external to V_1^0. Then DC is a tangent to V_1^0 at C, that is, it meets V_1^0 only at C. The remaining s^2 lines through D on $\mathcal{T}$, are all secants, that is, each line meets V_1^0 at s+1 points.

Let $f_D(u)$ denote the number of lines on $\mathcal{T}$ passing through D, each one of which meets V_1^0 at exactly u points, $u = 0,1,\ldots,s^2+1$.

Suppose a and b are first associates, that is the line $\overline{ab}$ meets V_1^0 at a regular point D ($D \neq C$). Then

$$p_{11}^1(a,b) = s^2-2 + (s^2-1)(s^2-2)\ f_D(s^2+1) + s(s-1)\ f_D(s+1),$$

where s^2-1 is the number of affine points on the line $\overline{ab}$; $(s^2-1)(s^2-2)$ is the number of ordered pairs of points (e,f) that one can form from the s^2-1 points (other than C and D) on the generator C,D. The intersection of the lines $\overline{ae}$ and $\overline{bf}$ is an affine point which is a first associate of both a and b. Similarly, each secant contributes s(s-1) affine points which are first associates of both a and b. But there is only one generator through D and s^2 secants through D. Thus

$$f_D(s^2+1) = 1 \quad \text{and} \quad f_D(s+1) = s^2.$$

Hence

$$p_{11}^1(a,b) = s^2-2 + (s^2-1)(s^2-2) + s(s-1)s^2$$

$$= s^2(2s^2-s-2),$$

which is independent of a and b.

Suppose now that the line $\overline{ab}$ meets $\mathcal{T}$ at a point D not on V_1^0. Then a and b are second associates. Let us calculate $p_{11}^2(a,b)$. Through D, there are (s^2+1) lines of which one, namely $\overline{CD}$, is a tangent to V_1^0 and the other s^2 lines are secants to V_1^0, that is, each line meets V_1^0 at s+1 points.

Hence

$$p_{11}^2(a,b) = (s+1)s\ f_D(s+1) = (s+1)s\ s^2 = s^3(s+1),$$

which is independent of a and b.

Now suppose that the line $\overline{ab}$ meets V_1^0 at C. Thus a and b are third associates. Through C there are s+1 generator lines each one of which meets V_1^0 at s^2+1 points including C. The remaining s^2-s lines through C on $\mathcal{F}$, are tangents. That is, each line meets V_1^0 only at C. Thus

$$p_{11}^3(a,b) = s^2(s^2-1)\ f_C(s^2+1) = s^2(s^2-1)(s+1).$$

In this manner, using the geometric results quoted before, we have calculated all the $p_{jk}^i(a,b)$ i,j,k = 1,2,3 parameters and these are independent of the pair of points a and b. Hence this is a three-class association scheme. The parameters are

$$v = s^6,\ n_1 = (s^3+s^2)(s^2-1),\ n_2 = (s^4-s^3),\ n_3 = s^2-1,$$

$$P_1 = (P_{ij}^1) = \begin{bmatrix} 2s^4-s^3-2s^2 & s^5-s^4 & s^2-1 \\ s^5-s^4 & s^3(s^2-s-1)(s-1) & 0 \\ s^2-1 & 0 & 0 \end{bmatrix},$$

$$P_2 = (P_{ij}^2) = \begin{bmatrix} s^3(s+1) & s^2(s+1)(s^2-s-1) & 0 \\ s^2(s+1)(s^2-s-1) & s^2(s^2-2)+(s^2-s-1)(s^2-s-2)s^2 & s^2-1 \\ 0 & s^2-1 & 0 \end{bmatrix},$$

$$P_3 = (P_{ij}^3) = \begin{bmatrix} s^2(s^2-1)(s+1) & 0 & 0 \\ 0 & s^3(s^2-1)(s-1) & 0 \\ 0 & 0 & s^2-2 \end{bmatrix}$$

The linear span of the coordinate vectors of the points of the degenerate Hermitian variety V_1^0 provides a three-weight projective code with s^6 codewords and $n = 1 + s^2 + s^3$ (Chakravarti (1990)). Whether the restriction of the Hamming association scheme $H_n(s)$ to this code provides a three-class association scheme and if yes, whether the three-class association already derived is related to this code, are still unsettled.

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