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BAYES PARAMETER ESTIMATION FOR THE BIVARIATE WEIBULL MODEL OF MARSHALL-OLKIN FOR CENSORED DATA

by

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Summary & Conclusions

Industrial applications frequently require statistical procedures be applied to analyze life test data from various sources, particularly in situations where observations are limited. This article develops a Bayes parameter estimation method for bivariate censored data collected from both system and component levels. The bivariate Weibull distribution of Marshall and Olkin is used to model lifetimes of components in a two-component system. Closed form Bayes estimators of the model parameters are proposed along with their variances and high probability density intervals.

Key Words – Bayes estimator, Marshall and Olkin bivariate Weibull,
Bivariate censored data, System and component data.

Reader Aids –

Purpose: Widen the state of the art

Special Math needed for explanations: Probability and statistics

Special Math needed to use results: Statistics

Results useful to: Statisticians, reliability theoreticians and engineers.

1. Introduction

Because time and cost limitations in collecting life-test results, it is frequently necessary to develop statistical inference procedures using information obtained from various sources. Recently, Viveros [21] studied large-sample interval estimations for mean lifetimes of components based on combined series system data. Kalbfleish and Lawless [9] proposed a reliability estimation method to integrate field-performance data together. Nair [17] suggested the use of degradation data measured from deterioration of electronic devices over several time periods. When life test data are observed for a system and its components, it is often desirable to combine all available information to improve statistical inferences, particularly in situations where observations are limited. Miyamura [16] analyzed life test measurements for air conditioners viewing them as series systems of electromagnetic valves and other components. Considering a thermal battery as a parallel system consisting of two bridgewires, Easterling and Prairie [7] studied life test data collected at the component level (bridgewires) as well as the system level (battery). Mastran [15] and Martz, Waller and Fickas [14] presented Bayesian approaches to permit the reliability assessment using test and prior data at both component and system levels.

We consider the following bivariate censored data from system and components: life tests of n prototypes of a two-component system are conducted and multiple time-censored data (x_i, y_i), $i = 1, \ldots, n$, are observed, where $x_i = \min(x_i^0, t_{xi})$, $y_i = \min(y_i^0, t_{yi})$ and t_{xi}, t_{yi} are censoring time of the ith observation for the first and the second components on test, respectively. Additionally, data $x_{n+j} = \min(x_{n+j}^0, t_{1i})$, $y_{n+k} = \min(y_{n+k}^0, t_{2i})$, $j = 1, \cdots, \ell$, $k = 1, \cdots, m$ are collected from separate component testings. The censoring times t_{xi}, t_{yi}, t_{1i}, t_{2i} are fixed and may be equal.

In many multi-component systems, a common cause failure or a similar environmental factor might cause the dependence between lifetimes of components [cf. Esary and Proschan, 8]. In studies of breakdown times of dual generators in a power plant or twin engines in a 2-engine airplane, both generators might fail simultaneously due to a common cause such as a mis-operation in the central control system. In all the aforementioned studies of system-component data, the component lifetimes were assumed to be s-independent for the sake of simplicity of mathematical treatment. In this paper, we use the bivariate Weibull distribution (BVW) due to Marshall-Olkin [12] to model component lifetimes and study inference procedures of the model parameters. The survival function (SF) of the BVW is given by

$$\bar{F}(x, y) = \Pr(X^0 > x, Y^0 > y) = \exp\left\{- \lambda_1 x^\beta - \lambda_2 y^\beta - \lambda_3 \left[max(x, y)\right]^\beta\right\}, \quad x, y > 0,$$

$$0 < \lambda_i < \infty, i = 1, 2, \quad 0 \leq \lambda_3 < \infty, \quad 0 < \beta < \infty. \tag{1.1}$$

Taking $y = 0$ $(x = 0)$ in the joint SF $\bar{F}(x, y)$, we obtain the marginal SF's of $X^0(Y^0)$ of the BVW distribution of the form

$$\bar{F}_{X^0}(x) = \exp\left[-(\lambda_1 + \lambda_3) x^\beta\right], \quad \bar{F}_{Y^0}(y) = \exp\left[-(\lambda_2 + \lambda_3) y^\beta\right]. \tag{1.2}$$

The dependence of the component lifetimes arises from simultaneous failures of both components (see Lu [10] for model derivation). A special case, $\beta = 1$, of the bivariate Weibull model BVW leads to Marshall-Olkin's bivariate exponential distribution (BVE). In the study of bivariate lifetime models, the BVE model has received the most attention, both in its theoretical development and its application.

There are no closed form results for maximum likelihood estimates (MLE) of the parameters for either the BVE model or the BVW model. Although moment [cf. Bemis, *et. al.*, 2] and likelihood based [cf. Proschan and Sullo, 19] estimators have been proposed for the BVE, the finite sample statistical properties of the estimators are analytically intractable. Moreover, in the study of bivariate lifetimes, classical methods use only system testing data for evaluating the system performance. In this article, component life-test results, which might be available from test data collected in different environments or from engineering judgments are used as prior information for the unknown parameters of the model of system lifetimes. The objective of this article is to utilize a Bayesian approach to handle bivariate censored data, and to provide closed form point and interval estimations of the parameters of the bivariate Weibull distribution BVW.

In Section 2, a data set is presented for the illustration of our procedures. The probability density function (pdf) and conditional survival probability $\Pr(X^0 > x \mid Y^0 = y)$ of the BVW are also given in this section. These formulate the likelihood function of the model parameters for the censored paired data. In section 3, based on life test results at component level, prior distributions for the parameters, which places continuous distribution on the reciprocal of scale parameters and a discrete distribution on the shape parameter (cf. Soland [20]; Martz and Waller [13]) are established. The construction of the likelihood function by using the information collected from system life tests is given in Section 4. Section 5 contains a derivation of posterior distributions of the parameters. Closed form Bayes estimators, their variances and high probability density intervals are proposed with numerical examples.

Notation

X^0, Y^0 lifetimes of components

X, Y censored observations for component lifetimes

$\bar{F}(x, y)$ joint survival function (Sf) of X^0 and Y^0

$f(x, y)$ joint probability density function (pdf) of X^0 and Y^0

$\bar{F}_{X^0 \mid Y^0 = y}(x)$ conditional survival probability $\Pr(X^0 > x \mid Y^0 = y)$

$\underset{\sim}{z}$ life-test information collected at system level

$g(\underset{\sim}{\Lambda}, \beta)$ joint prior density of $\underset{\sim}{\Lambda}$ and β

$g(\underset{\sim}{\Lambda}, \beta \mid \underset{\sim}{z})$ joint posterior density of $\underset{\sim}{\Lambda}$ and β

$g(\underset{\sim}{\Lambda} \mid \beta_j)$ conditional prior density of $\underset{\sim}{\Lambda}$ given β_j

$g(\underset{\sim}{\Lambda} \mid \beta_j, \underset{\sim}{z})$ conditional posterior density of $\underset{\sim}{\Lambda}$ given β_j

$\Pr(\beta = \beta_j \mid \underset{\sim}{z})$ posterior mass function of $\beta = \beta_j$

$C(\underset{\sim}{\alpha}_j, \underset{\sim}{\xi}_j)$ constant term in prior densities

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

2. A DATA SET AND PROBABILITY FUNCTIONS OF THE BVW

To illustrate the procedure of computing the proposed Bayes estimators, we consider a data set taken from Nelson [18], where times to failure on 10 motors with a new Class H insulation at 220°C are recorded for three causes: turn, phase and ground. Specifically, for the failure times (in weeks) of motors 1 through 10 of the turn cause, the observations are 14.7, 20.3, 20.3, 20.3$^+$, 20.3, 20.3, 25.9, 25.9, 25.9, 25.9, where a week is defined as 120 = 5 × 24 hours and "+" denotes time without failure. The chosen time scale, a week, is purely used to reduce the size of the normalizing constants in the prior and posterior distributions. The corresponding observations from phase and ground causes are (20.3, 20.3, 20.3, 23.1$^+$, 20.3$^+$, 34.3$^+$, 34.3$^+$, 34.3$^+$, 25.9, 34.3$^+$) and (20.3, 20.75, 20.3, 23.1, 20.3, 34.3$^+$, 34.3$^+$, 34.3$^+$, 25.9$^+$, 34.3$^+$), respectively. Note that the motor 3 has the same failure times, 20.3, for all three causes. Because the failure times from these causes might be equal, *i.e.* $\Pr(X = Y) \neq 0$, the Marshall-Olkin model BVW is appropriate for analyzing this data set. Other bivariate Weibull models (cf. Lu [10], [11]) require that $\Pr(X = Y)$ be zero. Furthermore, separately fitting the univariate Weibull distribution to the observations from these causes, we obtain the following MLE's of the model parameters: $\hat{\theta}_t = 23.853$, $\hat{\beta}_t = 7.665$ (turn), $\hat{\theta}_p = 38.461$,

$\hat{\beta}_p = 2.944$ (phase) and $\hat{\theta}_g = 36.088$, $\hat{\beta}_g = 2.758$ (ground). Because the likelihood ratio test of the hypothesis H: $\beta = \beta_p = \beta_g$ is not rejected (with $-2\log\lambda_n = .0126$ and $\widetilde{\beta} = 2.838$), the motor data from the phase and ground causes are used as our system life-test results for further study. To illustrate the integration of component and system data, we simulate 10 and 9 observations (censored at 34.3) for phase and ground causes, respectively, based on the Weibull model with the parameter values equal to the MLE's calculated above. The simulation yielded the following results: 25.768, 34.3^+, 34.3^+, 17.220, 34.3^+, 29.466, 12.933, 24.974, 34.3^+, 33.852 for phase cause and 16.749, 34.3^+, 21.910, 32.524, 29.364, 26.581, 34.3^+, 34.3^+, 13.239 for ground cause. Note that there are 6 failures in both types of component testings. Based on the simulated data, the MLE's are $\hat{\theta}_{pc} = 35.3566$, $\hat{\beta}_{pc} = 3.019$ (phase) and $\hat{\theta}_{gc} = 33.088$, $\hat{\beta}_{gc} = 2.993$ (ground).

The analytical treatment of the bivariate Weibull distribution BVW is difficult due to the existence of a singular component to a two-dimensional Lebesgue measure. A mixture of one- and two- dimensional Lebesgue measures [cf. Bemis, *et. al.*, 2; Bhattacharyya and Johnson, 3] leads to the following pdf of (1.1):

$$f(x,\, y) = \sum_{\alpha=1}^{3} f_\alpha(x,\, y)\; R_\alpha(x,\, y), \tag{2.1}$$

where

$$f_1(x,\, y) = \lambda_1 \gamma_2 \beta^2 x^{\beta-1} y^{\beta-1} \exp(-\,\lambda_1 x^\beta - \gamma_2 y^\beta)$$

$$f_2(x,\, y) = \lambda_2 \gamma_1 \beta^2 x^{\beta-1} y^{\beta-1} \exp(-\,\gamma_1 x^\beta - \lambda_2 y^\beta)$$

$$f_3(x,\, y) = \lambda_3 \beta x^{\beta-1} \exp(-\,\lambda x^\beta)$$

and R_α is an indicator for different domain of $(x,\, y)$: $R_1 = 1$ if $0 < x < y < \infty$, $R_2 = 1$ if $0 < y < x < \infty$, $R_3 = 1$ if $0 < x = y < \infty$, R_1, R_2, $R_3 = 0$ otherwise, and $\gamma_1 = \lambda_1 + \lambda_3$, $\gamma_2 = \lambda_2 + \lambda_3$, $\lambda = \lambda_1 + \lambda_2 + \lambda_3$.

The following conditional survival probabilities, given in Eq. (1.9) of Barlow and Proschan [1], are essential to construct the likelihood function of the parameters for the (two-dimensional type I) censored data.

$$\bar{F}_{X^0 | Y^0 = y}(x) = \Pr(X^0 > x \mid Y^0 = y)$$

$$= \begin{cases} \exp(-\lambda_1 x^\beta) & \text{for } y > x, \\ \lambda_2 \gamma_2^{-1} \exp(-\gamma_1 x^\beta + \lambda_3 y^\beta) & \text{for } y \leq x. \end{cases} \quad (2.2)$$

Results for $\Pr(Y^0 > y \mid X^0 = x)$ are defined similarly.

3. PRIOR DISTRIBUTIONS BASED ON COMPONENT TESTING

In a Bayesian framework all parameters λ_i, β are treated as unknown random variables denoted as Λ_i, β, $i = 1, 2, 3$, respectively. Let $\underset{\sim}{\Lambda}$ be a vector of $(\Lambda_1, \Lambda_2, \Lambda_3)'$. In this section, information collected from life tests at the component level, $\underset{\sim}{x}_c = (x_{n+1}, x_{n+2}, \cdots, x_{n+\ell})'$ and $\underset{\sim}{y}_c = (y_{n+1}, y_{n+2}, \cdots, y_{n+m})'$, is utilized to formulate the joint prior distribution on $(\underset{\sim}{\Lambda}, \beta)$. The prior distribution to be considered is the one originally proposed by Soland [20] for the univariate Weibull distribution, which is a family of joint prior distributions that places a continuous distribution on the scale parameter and a discrete distribution on the shape parameter.

Suppose that β has k values β_j in $(0, \infty)$ with the probability p_j', $j = 1, \ldots, k$, where $\sum_{j=1}^{k} p_j' = 1$ and k is the number of components in the system, $k = 2$ in this case. One can use ML estimates of the shape parameters of marginal Weibull distributions (1.2), for the values of β_j. The weights p_j' are decided according to either the sample sizes, $p_1' = \ell/(\ell + m)$, $p_2' = m/(\ell + m)$ or variances, $p_1' = v_2/(v_1 + v_2)$, $p_2' = v_1/(v_1 + v_2)$, where v_j, $j = 1, 2$ are variances of the estimators. Note that if the shape parameter β is known (set to 1), the distributions of the observations $\underset{\sim}{x}_c$ and $\underset{\sim}{y}_c$ are both gamma. Hence, the conditional prior distribution of $\underset{\sim}{\Lambda}$ given β_j is related to gamma distributions with density of the form

$$g(\underset{\sim}{\Lambda} \mid \beta_j) = C(\underset{\sim}{\alpha}_j, \underset{\sim}{\xi}_j) \times (\lambda_1 + \lambda_3)^{\alpha_{1j}} (\lambda_2 + \lambda_3)^{\alpha_{2j}} \times \exp(-\lambda_1 \xi_{1j} - \lambda_2 \xi_{2j} - \lambda_3 \xi_{3j}), \quad (3.1)$$

where $\xi_{3j} = \xi_{1j} + \xi_{2j}$ and $(\lambda_i + \lambda_3)$, $i = 1, 2$ are the reciprocal scale parameters of the marginals (1.2) of the BVW model. Note that the conditional prior density $g(\underset{\sim}{\Lambda} \mid \beta_j)$ depends on β_j only through the dependence of the prior parameters $\underset{\sim}{\alpha}_j$ and $\underset{\sim}{\xi}_j$, where $\underset{\sim}{\alpha}_j = (\alpha_{1j}, \alpha_{2j})'$, $\underset{\sim}{\xi}_j = (\xi_{1j}, \xi_{2j})'$. The prior parameters $\underset{\sim}{\alpha}_j$ and $\underset{\sim}{\xi}_j$ can be estimated by using the information $\underset{\sim}{x}_c$ and $\underset{\sim}{y}_c$. For example, $\xi_{1j} = \sum_{k=1}^{\ell} x_{n+k}^{\beta_j} = \sum_{k \in D} (x_{n+k}^0)^{\beta_j} + \sum_{k \in C} t_{1k}^{\beta_j}$, $\xi_{2j} = \sum_{k=1}^{m} y_{n+k}^{\beta_j} = \sum_{k \in D} (y_{n+k}^0)^{\beta_j} + \sum_{k \in C} t_{2k}^{\beta_j}$, where D and C denote the sets of units for which lifetimes are observed and censored, respectively. And, the observed number of lifetimes are $\alpha_{1j} = \sum_{k=1}^{\ell} \delta_{n+k}$, $\alpha_{2j} = \sum_{k=1}^{m} \delta_{n+k}$, where $\delta_{n+.}$ is the failure indicator.

The normalizing constant $C(\underset{\sim}{\alpha}_j, \underset{\sim}{\xi}_j)$ of (3.1) can be obtained by requiring that the integration of the density $g(\underset{\sim}{\Lambda} \mid \beta_j)$ must be equal to one. Applying binomial series to $(\lambda_1 + \lambda_3)^Q$ and $(\lambda_2 + \lambda_3)^R$, we have the following results:

$$(\lambda_1 + \lambda_3)^Q = \sum_{i=0}^{Q} \begin{bmatrix} Q \\ i \end{bmatrix} \lambda_1^{Q-i} \lambda_3^i, \quad (\lambda_2 + \lambda_3)^R = \sum_{k=0}^{R} \begin{bmatrix} R \\ k \end{bmatrix} \lambda_2^{R-k} \lambda_3^k, \tag{3.2}$$

for any constants Q and R. This leads to the constant

$$C^{-1}(\underset{\sim}{\alpha}_j, \underset{\sim}{\xi}_j) = \sum_{i=0}^{\alpha_{1j}} \sum_{k=0}^{\alpha_{2j}} \begin{bmatrix} \alpha_{1j} \\ i \end{bmatrix} \begin{bmatrix} \alpha_{2j} \\ k \end{bmatrix} \xi_{1j}^{-\eta_{1j}} \xi_{2j}^{-\eta_{2j}} \xi_{3j}^{-\eta_{3j}} \Gamma(\eta_{1j}) \Gamma(\eta_{2j}) \Gamma(\eta_{3j}), \tag{3.3}$$

where $\eta_{1j} = \alpha_{1j} - i + 1$, $\eta_{2j} = \alpha_{2j} - k + 1$, $\eta_{3j} = i + k$. Note that $C(\underset{\sim}{\alpha}_j, \underset{\sim}{\xi}_j)$ is a linear combination of three products of gamma functions.

EXAMPLE 1: Based on the component testing results given in Section 2, we assign the values of the parameters for the prior distribution as follows: since the MLE's of the shape parameter are calculated as $\hat{\beta}_{pc} = 3.019$ and $\hat{\beta}_{gc} = 2.993$, we assume the prior distribution of β has two values $\beta_1 = 3.019$ and $\beta_2 = 2.993$ with probabilities decided from the sample sizes as 10/19 and 9/19, respectively. Given the values of the β_j's, j =1, 2, the conditional prior distributions of $\underset{\sim}{\Lambda}$ are assumed to have form (3.1) with the parameter values $\alpha_{11} = 6$, $\alpha_{21} = 6$, $\xi_{11} = 283805.4$, $\xi_{21} = 231763.6$

(for β_1) and $\alpha_{12} = 6$, $\alpha_{22} = 6$, $\xi_{12} = 259385.3$, $\xi_{22} = 211938.3$ (for β_2), respectively. The reciprocal of normalizing constants (3.3) of the prior distribution are thus calculated as $C_1^{-1} = .116406\text{e}{-69}$ and $C_2^{-1} = .408654\text{e}{-69}$ for β_1 and β_2, respectively, where .116406e–69 denotes $.116406 \times 10^{-69}$. □

REMARKS:

1. Sometimes, one might have other prior knowledge for the model parameters. Life tests of other systems, which have parts of similar components to the ones considered here, might be useful. For instance, in other systems the component lifetimes might have Weibull distributions with parameters (λ_i^{-1}, β), or $[(\lambda_i+\lambda_4)^{-1}, \beta]$, $i = 1, 2$, respectively. Although this is different from the marginals (1.2) of the system life, the method to construct the prior distribution is quite similar. One can simply replace $(\lambda_i+\lambda_3)$ by λ_i or $(\lambda_i+\lambda_4)$ in (3.1) and use binomial series (3.2) to get the information for model parameters for system life.

2. Other options exist for setting up the prior of the parameters of the Weibull distribution. Instead of using discrete prior distribution for the shape parameter β, one might consider a uniform prior for β (cf. Canavos and Tsokos [6]), or a location-scale conjugate joint prior for $(\underset{\sim}{\Lambda}, \beta)$ (cf. Bury [5]). However, there are no closed form results in these approaches. Numerical methods are needed to carry out the integration for prior and posterior distributions.

4. LIKELIHOOD FUNCTION BASED ON SYSTEM TESTING

In this section, we formulate the likelihood function of the parameters based on censored data obtained at system level. We define the following indicators:

$$C_{1i} = I(X_i > t_{xi}),\ C_{2i} = I(Y_i > t_{yi}),\quad D_{ki} = 1 - C_{ki},\ \ k = 1, 2,\ \ i = 1, \cdots, n.$$

In system testing, one might observe (1) the failures of both components, (2) the failure of one component and the survival of the other component, or (3) the survival of both components. The likelihood for the observations collected from Cases 1 and 3 are simply products of the joint pdf (2.1) and the survival function (1.1). In the case that only one component is failed, the likelihood is products of the conditional survival probabilities, *e.g.* $\Pr(X^0 > x_i \mid Y^0 = y_i)$, and the corresponding marginal pdf $f_{Y^0}(y_i)$. We write the likelihood in a general notation:

$$L(\underset{\sim}{\lambda}) = \prod_{i=1}^{n} \Big\{ \big[f(x_i, y_i)\big]^{D_{1i}D_{2i}} \times \big[\bar{F}(x_i, y_i)\big]^{C_{1i}C_{2i}} \times \Big[\bar{F}_{X^0|Y^0=y}(x_i)\, f_{Y^0}(y_i)\Big]^{C_{1i}D_{2i}} \times \Big[\bar{F}_{Y^0|X^0=x}(y_i)\, f_{X^0}(x_i)\Big]^{D_{1i}C_{2i}} \Big\}. \tag{4.1}$$

For the bivariate Weibull distribution BVW, we apply the joint pdf (2.1), survival function (1.1) and the conditional survival probability (2.2) to derive the likelihood function for paired data. As an example, considering that only the second component is failed, we have

$$\Big[\bar{F}_{X^0|Y^0=y}(x_i)\, f_{Y^0}(y_i)\Big]^{C_{1i}D_{2i}}$$
$$= (\lambda_2\beta)^{(1 - R_{1i})\, C_{1i}D_{2i}}\, (\gamma_2\beta)^{R_{1i}C_{1i}D_{2i}}\, y_i^{(\beta-1)C_{1i}D_{2i}} \times \tag{4.2}$$
$$\exp\Big[-(1 - R_{1i})\, C_{1i}D_{2i}(\gamma_1 x_i^{\beta} + \lambda_2 y_i^{\beta}) - R_{1i}C_{1i}D_{2i}(\lambda_1 x_i^{\beta} + \gamma_2 y_i^{\beta})\Big],$$

where R_{1i}, R_{2i}, R_{3i} are indicators for domains of $(x_i,\ y_i)$ as defined in Section 2. Other terms of (4.1) are similarly defined. With some simplifications, a collection of all these distributional functions leads to the following likelihood function based on the data collected at system level:

$$L(\underset{\sim}{\lambda}) = \lambda_1^{n_1}\, \lambda_2^{n_2}\, \lambda_3^{n_3}\, (\lambda_1 + \lambda_3)^{n_4} (\lambda_2 + \lambda_3)^{n_5} \beta^{n_6} \prod_{i=1}^{n} \Big[x_i^{(\beta-1)D_{1i}}\, y_i^{(\beta-1)D_{2i}(1-R_{3i}D_{1i})}\Big]$$
$$\times \exp\Big[-\lambda_1 x_s - \lambda_2 y_s - \lambda_3(x_s + y_s - t_s)\Big], \tag{4.3}$$

where $\quad n_1 = \sum_{i=1}^{n} \Big\{R_{1i}D_{1i}D_{2i} + (1 - R_{2i})D_{1i}C_{2i}\Big\}, \qquad n_3 = \sum_{i=1}^{n} R_{3i}D_{1i}D_{2i},$

$n_4 = \sum_{i=1}^{n} R_{2i}D_{1i}$, and $n_2 = \sum_{i=1}^{n}\left\{R_{2i}D_{1i}D_{2i} + (1 - R_{1i})C_{1i}D_{2i}\right\}$ is the summation of two counts: (1) when both components are failed [cf. Eq. (2.1)], the case of $X_i < Y_i$, and (2) when only the second component is failed [cf. Eq. (4.2)], the case of $X_i \leq Y_i$; the power of the parameter $\gamma_2 = \lambda_2 + \lambda_3$ is the count $n_5 = \sum_{i=1}^{n} R_{1i}D_{2i}$, which is the sum of two summations from the cases $R_{1i}D_{1i}D_{2i}$ [cf. Eq. (2.1)] and $R_{1i}C_{1i}D_{2i}$ [cf. Eq. (4.2)]; and $n_6 = \sum_{i=1}^{n}\left\{(1 - R_{3i})D_{1i}D_{2i} + (1 - C_{1i}C_{2i})\right\}$; the summary statistics from the failure times are defined as $x_s = \sum_{i=1}^{n} x_i^{\beta}$, $y_s = \sum_{i=1}^{n} y_i^{\beta}$, $t_s = \sum_{i=1}^{n} [\min(x_i, y_i)]^{\beta}$.

EXAMPLE 2: Based on system testing results given in Section 2, one can verify that the indicators R_{1i}, R_{2i}, R_{3i}, $i = 1, 2, \ldots, 10$ have values: (0, 1, 0, 0, 0, 0, 0, 0, 1, 0), (0, 0, 0, 1, 1, 0, 0, 0, 0, 0) and (1, 0, 1, 0, 0, 1, 1, 1, 0, 1), respectively. Several summary statistics in the likelihood function (4.3) are calculated as follows: $n_1 = 2$, $n_2 = 2$, $n_3 = 2$, $n_4 = 0$, $n_5 = 1$, $n_6 = 7$ and for $\beta = \beta_1 = 3.019$, $x_{s1} = \sum_{i=1}^{n} x_i^{\beta_1} = 239626.5$, $y_{s1} = \sum_{i=1}^{n} y_i^{\beta_1} = 240232.7$, and for $\beta = \beta_2 = 2.993$, $x_{s2} = 219274.1$, $y_{s2} = 219829.6$. Because the x_i's are always less than the corresponding y_i's in this data set, we have $t_{sj} = \sum_{i=1}^{n} [\min(x_i, y_i)]^{\beta_j} = x_{sj}$, $j = 1, 2$. □

5. POSTERIOR DISTRIBUTIONS AND BAYES ESTIMATORS

In this section, we derive the posterior distributions of the model parameters of the BVW. The posterior distribution follows the results given in Martz and Waller [13] for the univariate Weibull distribution. It is computationally appealing and leads to closed form Bayes estimators. Applying Bayes theorem, we obtain the marginal posterior distribution of β given the system data ($\underset{\sim}{z}$) as follows:

$$\Pr(\beta = \beta_j \mid \underset{\sim}{z}) = p_j'' = p_j' A_j / \sum_{j=1}^{k} p_j' A_j, \tag{5.1}$$

where

$$A_j = V(\underset{\sim}{z};\, \beta_j) \times C(\underset{\sim}{\alpha}_j, \underset{\sim}{\xi}_j) \times \beta_j^{\,n_6} \sum_{i=0}^{Q} \sum_{k=0}^{R} \begin{bmatrix} Q \\ i \end{bmatrix} \begin{bmatrix} R \\ k \end{bmatrix} \times$$

$$U_1^{-S_1} U_2^{-S_2} U_3^{-S_3} \Gamma(S_1)\Gamma(S_2)\Gamma(S_3),$$

$C(\underset{\sim}{\alpha}_j, \underset{\sim}{\xi}_j)$ is given in (3.3), and

$$V(\underset{\sim}{z};\, \beta_j) = \prod_{i=1}^{n}\left[x_i^{(\beta_j-1)D_{1i}}\; y_i^{(\beta_j-1)D_{2i}(1-R_{3i}D_{1i})}\right],$$

$$Q = n_4 + \alpha_{1j},\;\; R = n_5 + \alpha_{2j}, \tag{5.2}$$

$$S_1 = n_1 + Q - i + 1,\quad S_2 = n_2 + R - k + 1,\quad S_3 = n_3 + i + k,$$

$$U_1 = x_s + \xi_{1j},\;\; U_2 = y_s + \xi_{2j},\;\; U_3 = x_s + y_s - t_s + \xi_1 + \xi_2 + \xi_3.$$

Note that Eq. (5.1) is quite similar to Eq. (9.11) of Martz and Waller [13] except that Equation (5.1) has more terms in the prior and likelihood functions and the parameters $\lambda_j + \lambda_3$, $j = 1, 2$, have to be related to λ_k, $k = 1, 2, 3$, through the binomial series [cf. Eq. (3.2)]. Under the squared error loss, the Bayes estimator $\widetilde{\beta}$ is the expected value of the posterior distribution of β, and is equal to $\sum_{j=1}^{k} p_j''\beta_j$. The variance of the Bayes estimator $\widetilde{\beta}$ is calculated as $\sum_{j=1}^{k}\beta_j^2 p_j''(1 - p_j'') - 2\sum_{j<j'} \beta_j p_j''\beta_{j'}p_{j'}''$ from the variance of the quantity $\sum_{j=1}^{k}\beta_j I_j$, where I_j is the indicator function of the event $\beta = \beta_j$. For example, when $k = 2$ (β has two prior values), the Bayes estimator $\widetilde{\beta}$ is $\beta_1 p_1'' + \beta_2 p_2''$ and its variance is $(\beta_1 - \beta_2)^2 p_1'' p_2''$, where $p_2'' = 1 - p_1''$.

The posterior density of $\underset{\sim}{\Lambda}$ given β_j is given as a linear combination of several gamma distributions [cf. Eq. (9.13) in Martz and Waller, 13]:

$$g(\underset{\sim}{\Lambda} \mid \beta_j, \underset{\sim}{z}) = C(\underset{\sim}{z}) \times \sum_{i=0}^{Q}\sum_{k=0}^{R}\begin{bmatrix} Q \\ i \end{bmatrix}\begin{bmatrix} R \\ k \end{bmatrix} \lambda_1^{S_1-1}\lambda_2^{S_2-1}\lambda_3^{S_3-1} \times$$

$$\exp\left[-\lambda_1 U_1 - \lambda_2 U_2 - \lambda_3 U_3\right], \tag{5.3}$$

and the normalizing constant is defined as

$$C^{-1}(\underset{\sim}{z}) = H(S_1, S_2, S_3, \underset{\sim}{U};\; Q, R)$$

$$= \sum_{i=0}^{Q}\sum_{k=0}^{R}\begin{bmatrix} Q \\ i \end{bmatrix}\begin{bmatrix} R \\ k \end{bmatrix} U_1^{-S_1} U_2^{-S_2} U_3^{-S_3}\, \Gamma(S_1)\Gamma(S_2)\Gamma(S_3), \tag{5.4}$$

where $\underset{\sim}{U} = (U_1, U_2, U_3)'$ and Q, R, S_m, U_m, $m = 1, 2, 3$ are given in (5.2). The H function is defined to simplify expressions and is nothing but a linear combination of gamma functions. The joint posterior distribution of $(\underset{\sim}{\Lambda}, \beta)$ can be obtained explicitly by using the product of (5.1) and (5.3). To derive the marginal conditional posterior density of Λ_1 given β_j, we integrate the joint conditional posterior density of $\underset{\sim}{\Lambda}$ given in (5.3),

$$g(\lambda_1 | \beta_j, \underset{\sim}{z}) = C(\underset{\sim}{z}) \times \sum_{i=0}^{Q} \sum_{k=0}^{R} \begin{bmatrix} Q \\ i \end{bmatrix} \begin{bmatrix} R \\ k \end{bmatrix} \lambda_1^{S_1 - 1} \exp\left[- \lambda_1 U_1\right] U_2^{-S_2} U_3^{-S_3} \Gamma(S_2)\Gamma(S_3). \tag{5.5}$$

When the squared error loss function is used, the Bayes estimator for λ_1 is obtained by taking the expectation of Λ_1 with respect to its posterior. The results is

$$\tilde{\lambda}_1 = E(\Lambda_1 \mid \underset{\sim}{z}) = \int_0^\infty \lambda_1 \left[\sum_{j=1}^{m} \Pr(\beta = \beta_j \mid \underset{\sim}{z}) \, g(\lambda_1 \mid \beta_j, \underset{\sim}{z}) \right] d\lambda_1 = \sum_{j=1}^{k} p_j'' B_{1j}, \tag{5.6}$$

where $\Pr(\beta = \beta_j \mid \underset{\sim}{z}) = p_j''$ is defined in (5.1), and

$$B_{1j} = H(S_1 + 1, S_2, S_3, \underset{\sim}{U}; \ Q, R) / H(S_1, S_2, S_3, \underset{\sim}{U}; \ Q, R), \tag{5.7}$$

with Q, R, S_m, U_m, $m = 1, 2, 3$ given in (5.2). The numerator of B_{1j}, an integration of density (5.5) multiplied by λ_1, is obtained from the use of gamma function with the coefficient $S_1 + 1$. And, the denominator is just the normalizing constant. The variance $\tilde{\lambda}_1$ is similarly calculated as follows:

$$\begin{aligned} Var(\Lambda_1 \mid \underset{\sim}{z}) &= \int_0^\infty \lambda_1^2 \left[\sum_{j=1}^{m} \Pr(\beta = \beta_j \mid \underset{\sim}{z}) \, g(\lambda_1 \mid \beta_j, \underset{\sim}{z}) \right] d\lambda_1 - \left[E(\Lambda_1 \mid \underset{\sim}{z}) \right]^2 \\ &= \sum_{j=1}^{m} p_j'' \, B_{2j}, \end{aligned} \tag{5.8}$$

where

$$B_{2j} = H(S_1 + 2, S_2, S_3, \underset{\sim}{U}; \ Q, R) / H(S_1, S_2, S_3, \underset{\sim}{U}; \ Q, R). \tag{5.9}$$

Marginal conditional posterior densities, Bayes estimators and their variances for λ_2 and λ_3 are derived similarly.

EXAMPLE 3: From the prior and likelihood given in Examples 1 and 2, we use (5.2) and (5.1) to compute $V_1 = .668317e{+}19$, $V_2 = .382438e{+}19$ and $A_1 = .894223e{-}18$, $A_2 = .894406e{-}18$. Then, the weights p_j'' of the posterior distribution of β are obtained as $p_1'' = .526265$ and $p_2'' = .473735$. The Bayes estimator and its variance of β are calculated as $\beta_1 p_1'' + \beta_2 p_2'' = 3.007$ and $(\beta_1 - \beta_2)^2 p_1'' p_2'' = .000169$. The posterior conditional distribution of the scale parameters $\underset{\sim}{\Lambda}$ given β_j is a linear combination of several gamma distributions. From the marginal conditional distribution (5.5) of Λ_1 given β_j, we calculate the Bayes estimator (5.6) of Λ_1 as .177200e-04 with its variance (cf. 5.8) as .350581e-09. The B functions given in (5.7) and (5.9) have the following values: $B_{11} = .169681e{-}04$, $B_{21} = .320819e{-}09$ (for β_1) and $B_{12} = .185552e{-}04$, $B_{22} = .383642e{-}09$ (for β_2). Similarly, we report the Bayes estimators of the parameters Λ_2 and Λ_3 as .219147e-04 and .181702e-05, respectively, and their variances as .530282e-09 and .497062e-11, respectively.

The scale parameters, $\theta_j = (\lambda_j + \lambda_3)^{-1/\beta}$, $j = 1, 2$, of the marginal distribution have no closed form Bayes estimators. To compare the estimates obtained from the proposed Bayesian and ML approaches, we calculate simple estimates $\tilde{\theta}_j = (\tilde{\lambda}_j + \tilde{\lambda}_3)^{-1/\tilde{\beta}}$, where $\tilde{\lambda}_j$'s and $\tilde{\beta}$ are the aforementioned Bayes estimators. The MLE's $\hat{\theta}_1 = 36.031$, $\hat{\theta}_2 = 35.361$, $\hat{\beta} = 2.943$ are obtained by fitting independent Weibull models (with the equal shape parameters) to the combined system and component data. Compared to the MLE's, our simple estimators $\tilde{\theta}_1 = 36.832$, $\tilde{\theta}_2 = 34.525$ and $\tilde{\beta} = 3.007$ are quite reasonable. □

Since the aforementioned marginal posterior density for Λ_1 is quite complicated for interval estimation for Λ_1, one might use the following approximation procedure. First, note that the marginal posterior density of Λ_1

$$g(\lambda_1 \mid \underset{\sim}{z}) = \sum_{j=1}^{m} \Pr(\beta = \beta_j \mid \underset{\sim}{z})\, g(\lambda_1 \mid \beta_j, \underset{\sim}{z}), \tag{5.10}$$

is a weighted finite sum of gamma densities, where $\Pr(\beta = \beta_j \mid \underset{\sim}{z})$ and $g(\lambda_1 \mid \beta_j, \underset{\sim}{z})$ are given in (5.1) and (5.5) respectively. One can approximate $g(\lambda_1 \mid \underset{\sim}{z})$ by a gamma distribution $G(\alpha_{10}, \beta_{10})$ with the following parameter values:

$$\alpha_{10} = \frac{E(\Lambda_1 \mid \underset{\sim}{z})}{Var(\Lambda_1 \mid \underset{\sim}{z})}, \quad \beta_{10} = \frac{E(\Lambda_1 \mid \underset{\sim}{z})^2}{Var(\Lambda_1 \mid \underset{\sim}{z})}. \tag{5.11}$$

The distribution $G(\alpha_{10}, \beta_{10})$ has the same mean and variance as the exact posterior distribution. Since $2\alpha_0\Lambda_1$ has an approximately χ^2 distribution with the degree of freedom (df) $2\beta_0$, we can construct a $100(1 - \alpha)\%$ high probability density (HPD) interval $[(2\alpha_0)^{-1}C_L, (2\alpha_0)^{-1}C_U]$ of Λ_1, where C_L and C_U satisfies the following conditions (cf. Box and Tiao [4], pp 90 and 255):

$$P(C_L \leq \chi^2_{2\beta_0} \leq C_U) = 1 - \alpha \quad \text{and} \quad C_L^{n/2}\exp(-\,C_L/2) = C_U^{n/2}\exp(-\,C_U/2). \tag{5.12}$$

EXAMPLE 4: To construct a HPD interval of the parameter λ_1, we use a gamma distribution with the scale parameter $\alpha_{10} = 50544.67$ and shape parameter $\beta_{10} = .895652$, calculated from Equation (5.11), to approximate the marginal posterior density (5.10) of Λ_1. We then transform this gamma distribution to a χ^2 distribution with the df $r = 2\beta_0 = 1.791303$ and find the lower and upper limits $C_{L1} = .559179\text{e-}06$ and $C_{U1} = .907436\text{e-}04$, respectively, from the χ^2_r distribution with the condition (5.12) satisfied. The parameters of the approximate gamma distribution for the marginal posterior density of Λ_2 and Λ_3 are (41326.5, .905658) and (365552, .664215), respectively. The HPD intervals of the parameters λ_2 and λ_3 are constructed similarly as (.7044756e-06, .110218e-03) and (.241101e-07, .141659e-04), respectively. □

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