ON THE INVARIANCE PRINCIPLE FOR EXCHANGEABLE RANDOM VARIABLES

by

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ABSTRACT: The following note provides a more general version of Corollary 1 in Weber (1980) and explains why one can get by with only conditions on the full n sum in the case of exchangeable random variables.

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Abbreviated title: Invariance Principle for Exchangeable Random Variables

AMS 1980 subject classification: 60F05.

1. INTRODUCTION

Let $\{X_{ni}: 1 \le i \le m, m > n \ge 1\}$ be an array of zero mean random variables such that for each n, $X_{n1}, X_{n2}, \ldots, X_{nm}$ are exchangeable. Define the σ-fields

$$\mathcal{F}_{nj} = \sigma\{X_{n1}, X_{n2}, \ldots, X_{nj}, \Sigma_{i=j+1}^{m} X_{ni}\}, \quad j = 1,2,\ldots,m.$$

Our first result explains why one can replace the condition $\Sigma_{i=1}^{[nt]} X_{ni}^2 \xrightarrow{P} t$, used in invariance principles for martingale difference arrays (see, e.g. Scott (1973) Theorem 2), by the condition $\Sigma_{i=1}^{n} X_{ni}^2 \xrightarrow{P} 1$ when the $\{X_{ni}\}$ are exchangeable. We then use this result to obtain sufficient conditions for an invariance principle for exchangeable random variables which are weaker than those given in Weber (1980).

2. MAIN RESULTS

First we need the following lemma.

<u>Lemma 1.</u> Let $\{a_{ni} : 1 \le i \le n, n \ge 1\}$ be an array of positive numbers satisfying

(i) $\max_{i \le n} a_{ni} \to 0$, and

(ii) $\Sigma_{i=1}^{n} a_{ni} \to 1$, as $n \to \infty$.

Then we have, for any $\ell > 1$,

$$\Sigma_{i=1}^{n} a_{ni}^{\ell} \to 0, \tag{1}$$

and for any integer $k \ge 1$,

$$\Sigma_{i_1=1}^{n} a_{ni_1}\left[\Sigma_{i_2 \ne i_1} a_{ni_2}\right] \cdots \left[\Sigma_{i_k \ne i_1, i_2, \ldots, i_{k-1}} a_{ni_k}\right] \to 1, \tag{2}$$

as $n \to \infty$.

Proof. Note

$$\Sigma_{i=1}^{n} a_{ni}^{\ell} \leq \max_{i \leq n} a_{ni}^{\ell-1} \Sigma_{i=1}^{n} a_{ni} \to 0 \quad \text{as } n \to \infty.$$

Since

$$(\Sigma_{i=1}^{n} a_{ni})^{k} = \Sigma_{i=1}^{n} a_{ni}^{k} + k \Sigma_{i_1=1}^{n} a_{ni_1}^{k-1} \left(\Sigma_{i_2 \neq i_1} a_{ni_2} \right)$$

$$+ \ldots + \Sigma_{i=1}^{n} a_{ni_1} \left[\Sigma_{i_2 \neq i_1} a_{ni_2} \right] \cdots \left[\Sigma_{i_k \neq i_1, i_2, \ldots, i_{k-1}} a_{ni_k} \right],$$

by (1) and (ii) we have that (2) holds.

Theorem 1. Given $\{X_{ni}\}$ as above, if

(i) $\max_{i \leq n} |X_{ni}| \xrightarrow{P} 0$, and

(ii) $\Sigma_{i=1}^{n} X_{ni}^{2} \xrightarrow{P} 1$,

then for $t \in [0,1]$,

$$\Sigma_{i=1}^{[nt]} X_{ni}^{2} \xrightarrow{P} t, \quad \text{as } n \to \infty.$$

Proof. Let $\{y_{ni} : 1 \leq i \leq n\}$ be an array of real numbers such that $\max_{i \leq n} |y_{ni}| \to 0$ and $\Sigma_{i=1}^{n} y_{ni}^{2} \to 1$. Let π_n be a random permutation of $\{1,2,\ldots,n\}$ and in the first case suppose that $X_{ni} = y_{n,\pi_n(i)}$. We claim that for $t \in [0,1]$ and $k \geq 1$

$$\lim_{n\to\infty} E\left[\Sigma_{i=1}^{[nt]} X_{ni}^{2}\right]^{k} = t^{k}, \tag{3}$$

and

$$\lim_{n\to\infty} P\left[\Sigma_{i=1}^{[nt]} X_{ni}^{2} \leq u\right] = I(u \geq t). \tag{4}$$

To see that (3) holds, note

$$\lim_{n\to\infty} E\left[\Sigma_{i=1}^{[nt]} X_{ni}^2\right]^k = \lim_{n\to\infty}\left[[nt]\ E\ X_{n1}^{2k} + k[nt]([nt]-1)\ E\ X_{n1}^{2(k-1)}X_{n2}^2\right.$$

$$\left. + \dots + [nt]([nt]-1)\dots([nt]-k+1)\ E\left[X_{n1}^2\ X_{n2}^2 \dots X_{nk}^2\right]\right]$$

$$= \lim_{n\to\infty}\left[[nt]\ \frac{1}{n}\ \Sigma_{i=1}^n\ y_{ni}^{2k} + k\ \frac{[nt]([nt]-1)}{n(n-1)} \sum_{i_1=1}^n y_{ni_1}^{2(k-1)}\left[\sum_{i_2\neq i_1} y_{ni_2}^2\right]\right.$$

$$+ \dots + \frac{[nt]\dots([nt]-k+1)}{n(n-1)\dots(n-k+1)} \sum_{i_1=1}^n y_{ni_1}^2\left[\sum_{i_2\neq i_1} y_{ni_2}^2\right]$$

$$\left.\dots\left[\Sigma_{i_k\neq i_1,i_2,\dots,i_{k-1}}\ y_{ni_k}^2\right]\right]$$

$= t^k$, using Lemma 1.

Now (4) follows by using the method of moments as described, for example, in Billingsley (1979, p. 342). Finally, using Lemma 1.1 of Kallenberg (1973) we immediately obtain the result for an arbitrary array of row-wise exchangeable random variables.

The above result can now be used to obtain the following invariance principle for exchangeable random variables under weaker conditions than those in Corollary 1 of Weber (1980).

<u>Theorem 2</u>. Given $\{X_{ni}, \mathcal{F}_{ni}\}$ as above, if

(i) $n\ E\ X_{n1}^2 \to 1$,

(ii) $n^2\ E\ X_{n1}\ X_{n2} \to 0$,

(iii) $\Sigma_{j=1}^n\ X_{nj}^n \overset{P}{\to} 1$,

(iv) $\max_{j\leq n} |X_{nj}| \overset{P}{\to} 0$, and

(v) $n/m \to 0$,

then

$$W_n(t) = \sum_{j=1}^{[nt]} X_{nj} \Rightarrow W(t) \quad \text{as } n \to \infty,$$

where W(t) is a standard Wiener process on D[0,1].

Proof. Let $Y_{nj} = X_{nj} - E(X_{nj}|\mathcal{F}_{n,j-1})$ so that $\{\Sigma_{j=1}^{k} Y_{nj}, \mathcal{F}_{nk}\}$ is a martingale. Now, by Theorem 2 of Scott (1973), $\Sigma_{j=1}^{[nt]} Y_{nj} \Rightarrow W(t)$ if

(a) $\sum_{j=1}^{n} E\, Y_{nj}^2 \to 1,$

(b) $\max_{1 \le n} |Y_{nj}| \xrightarrow{P} 0,$ and

(c) $\sum_{j=1}^{[nt]} Y_{nj}^2 \xrightarrow{P} t.$

Since $E(X_{nj}|\mathcal{F}_{n,j-1}) = (m-j+1)^{-1} \Sigma_{i=j}^{m} X_{ni}$,

$$E\left[\sum_{j=1}^{n} E^2(X_{nj}|\mathcal{F}_{n,j-1})\right] = \sum_{j=1}^{n} E\, X_{n1}^2/(m-j+1) + \sum_{j=1}^{n} (m-j)E\, X_{n1} X_{n2}/(m-j+1)$$

$$\le E\, X_{n1}^2 \log(n+1) + n|E\, X_{n1} X_{n2}|$$

$$\to 0 \quad \text{as} \quad n \to \infty \quad \text{by (i), and (ii).}$$

Thus

$$\sum_{j=1}^{n} E^2(X_{nj}|\mathcal{F}_{n,j-1}) \xrightarrow{L_1} 0, \tag{5}$$

and so (a) follows from (i).

Next,

$$\max_{j \le n}|Y_{nj}| \le \max_{j \le n}|X_{nj}| + \max_{j \le n}|E(X_{nj})|\mathcal{F}_{n,j-1})|$$

and

$$\max_{j \le n}|E(X_{nj}|\mathcal{F}_{n,j-1})| \le \left[\sum_{j=1}^{n} E^2(X_{nj}|\mathcal{F}_{n,j-1})\right]^{1/2}$$

so (b) follows from (iv) and (5).

Using Theorem 1 we have that (iii), (iv) and (5) together imply (c). Thus to complete the proof we need to show that

$$\sum_{j=1}^{[nt]} E(X_{nj}|\mathcal{F}_{n,j-1}) \xrightarrow{P} 0 \tag{6}$$

But

$$E\left[\sum_{j=1}^{[nt]} E(X_{nj}|\mathcal{F}_{n,j-1})\right]^2$$

$$= E\left[\sum_{j=1}^{[nt]} E^2(X_{nj}|\mathcal{F}_{n,j-1})\right] + 2\sum_{j=1}^{[nt]-1}\sum_{k=j+1}^{[nt]} E[X_{nk}\, E(X_{nj}|\mathcal{F}_{n,j-1})]$$

The first of these terms goes to 0 using (5) and the second sum is bounded by

$$\sum_{j=1}^{[nt]-1}\sum_{k=j+1}^{[nt]} (m - j+1)^{-1}\, [E\, X_{n1}^2 + (m-j)\, E\, X_{n1}X_{n2}]$$

$$\le [nt]^2\, E\, X_{n1}^2/(m-1) + [nt]^2 |E\, X_{n1}\, X_{n2}|$$

$$\to 0, \text{ using (i), (ii) and (v).}$$

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