FLEXIBLE PROCESS CAPABILITY INDICES

Norman L. Johnson

Department of Statistics
University of North Carolina
Chapel Hill, NC 27599-3260

Samuel Kotz

Dept. of Management Science and Statistics
University of Maryland
College Park, MD 20742

W.L. Pearn

Dept. of Industrial Engineering and Management
National Chiao Tung University
Hsinchu, Taiwan 300 50
Republic of China

Key Words and Phrases: *asymmetry*; *Chi-squared distribution*; *mixture distributions*; *quality control*

ABSTRACT

A new process capability index (PCI) is proposed, which takes into account possible asymmetry in the distribution of the measured process characteristic (X). The distribution of a natural estimator of this index is investigated.

INTRODUCTION

Interest in the use of process capability indices (PCIs) has been increasing in recent years, at an accelerating rate (see, e.g., Kane (1986)). Many different views have been expressed (e.g. Beazley and Marcucci (1988), Bissell (1990), Boyles (1991), Kitska (1991), Spiring (1991)) on their usefulness and applicability. There have been difficulties in obtaining sampling distributions for estimators of some PCIs, though, happily, these have now been largely overcome, at least on the assumption of normality of the process distribution (e.g. Chan et al. (1988), Chen and Owen (1989), Zhang et al. (1990), Boyles (1991), Kotz and Johnson (1992), Pearn et al. (1992)). There have also been objections to the widespread use of this assumption, both for establishing relationship with expected proportion of nonconforming (NC) product, and also, even when this is regarded as irrelevant (a view not shared by the present authors), for consideration of sampling distributions.

In the present paper we present new PCIs, specifically designed to make some allowance for possible asymmetry in the process distribution - in particular for difference between variability of the measured characteristic (X) for values less than and greater than a target value (T). Although our proposals are aimed at producing PCIs which can be used for asymmetric distributions we will base our analysis of sampling distribution on assumptions of normality. Indeed we will restrict our detailed discussion to further, assuming that the process mean of X (μ) is equal to the target value, T. We will, however, present the structure of relevant distributions for more general cases.

PROCESS CAPABILITY INDICES

We first present, for reference purposes, a summary of some current PCIs. We use the following notation:

USL : upper specification limit

LSL : lower specification limit

$$d = \frac{1}{2}(USL-LSL)$$

T : a target value for X

μ, σ : mean and standard deviation respectively, of the population distribution of the measured characteristic (X)

$X_1, \ldots, X_n$: random variables representing results of measurements of X on a random sample of size n

$$\bar{X} = n^{-1} \sum_{i=1}^{n} X_i; \quad S = \sum_{i=1}^{n} (X_i - T)^2; \quad V^* = n^{-1}S;$$

$$V = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

An early PCI was

$$C_p = \frac{USL-LSL}{6\sigma} = \frac{d}{3\sigma}$$

A natural estimator of C_p is $\hat{C}_p = \frac{d}{3\hat{\sigma}}$ with $\hat{\sigma} = V^{½}$. This index does not take account of the values of μ or T.

The index

$$C_{pk} = \frac{\min(USL-\mu,\ \mu-LSL)}{3\sigma} = \frac{d-|\mu-½(USL+LSL)|}{3\sigma}$$

takes μ into account but not T. A natural estimator of C_{pk} is

$$\hat{C}_{pk} = \frac{d-|\bar{X}-½(USL+LSL)|}{3\hat{\sigma}}$$

The index

$$C_{pm} = \frac{USL-LSL}{6\{E[(X-T)^2]\}^{½}} = \frac{d}{3\{\sigma^2+(T-\mu)^2\}^{½}}$$

takes T into account.

A natural estimator of C_{pm} is

$$\hat{C}_{pm} = \frac{d}{3\sqrt{V^*}}$$

It is a defect of C_{pm} that it does not distinguish between $T-\mu = \delta$ and $T-\mu = -\delta$. Unless T = ½(USL+LSL) this can lead to products with expected proportion NC over 50% and less than 0.3% having identical values for C_{pm}! (Take $T = \frac{1}{4}[3 \times USL+LSL]$ and $\delta = \frac{1}{2}d$.)

The PCI

$$C_{pmk} = \frac{\min(USL-\mu,\ \mu-LSL)}{3\{\sigma^2+(T-\mu)^2\}^{½}},$$

with natural estimator

$$\hat{C}_{pmk} = \frac{d-|\bar{X}-½(USL+LSL)|}{3\sqrt{V^*}},$$

introduced by Pearn et al. (1992), combines features of C_{pk} and C_{pm}. These latter two indices can be regarded as 'second generation' indices and C_{pmk} as a 'third generation' index following C_p.

However none of these PCI's attempts to take into account possible asymmetry in the distribution of X. In the next section we suggest a way in which this might be done, and in the following section study the distribution of the new index under some special conditions.

A FLEXIBLE PCI

We will construct a PCI taking into account possible differences in variability of X for values above and below the target value T.

As one-sided PCIs we could use

$$CU_{jkp} = \frac{1}{3\sqrt{2}} (USL-T)/\{E_{X>T}[(X-T)^2]\}^{½} \qquad (1.1)$$

and

$$CL_{jkp} = \frac{1}{3\sqrt{2}} (T-LSL)/\{E_{X \le T}[(X-T)^2]\}^{½}, \qquad (1.2)$$

where

$$E_{X>T}[(X-T)^2] = E[(X-T)^2 | X > T] \Pr[X > T] \qquad (2.1)$$

and

$$E_{X<T}[(X-T)^2] = E[(X-T)^2 | X < T] \Pr[X < T] \qquad (2.2)$$

(We assume Pr[X=T] = 0).

The multiplier $1/(3\sqrt{2})$ while the earlier PCI's use $\frac{1}{3}$ arises from the fact that for a <u>symmetrical</u> distribution with variance σ^2 and expected value T we would have

$$E_{X>T}[(X-T)^2] = E_{X<T}[(X-T)^2] = \frac{1}{2}\sigma^2, \qquad (3)$$

Finally we define

$$C_{jkp} = \min(CU_{jkp}, CL_{jkp})$$

$$= \frac{1}{3\sqrt{2}} \min\left[\frac{USL-T}{\{E_{X>T}[(X-T)^2]\}^{½}}, \frac{T-LSL}{\{E_{X<T}[(X-T)^2]\}^{½}}\right] \qquad (4)$$

Note that if we have $T = \frac{1}{2}(USL + LSL)$ so that USL-T = T-LSL = d then

$$C_{jkp} = \frac{d}{3\sqrt{2}} \max[E_{X>T}[(X-T)^2], E_{X<T}[(X-T)^2]]^{-½}. \qquad (5)$$

ESTIMATION OF C_{jkp}

A natural estimator of C_{jkp} is

$$\hat{C}_{jkp} = \frac{1}{3\sqrt{2}} \min\left[\frac{USL-T}{\sqrt{(S_+/n)}}, \frac{T-LSL}{\sqrt{S_-/n}}\right], \qquad (6)$$

where $S_+ = \Sigma_{X_i>T} (X_i-T)^2$; $S_- = \Sigma_{X_i<T}(X_i-T)^2$.

Our analysis will be based on the very reasonable condition $LSL < T < USL$. We can express $\hat{C}_{jkp}$ as

$$\hat{C}_{jkp} = \frac{d\sqrt{n}}{(3/2)\sigma} \min\left[\frac{d^{-1}(USL-T)}{\sqrt{S_+}/\sigma}, \frac{d^{-1}(T-LSL)}{\sqrt{S_-}/\sigma}\right], \qquad (6)'$$

where σ is an arbitrary constant.

To study the distribution of $\hat{C}_{jkp}$ it will be convenient to consider the statistic

$$\hat{D} = \frac{n}{18} \left(\frac{d}{\sigma}\right)^2 \hat{C}_{jkp}^{-2} = \max(a_1 S_+ \sigma^{-2}, a_2 S_- \sigma^{-2}), \qquad (7)$$

where $a_1 = \left[\frac{USL-T}{d}\right]^{-2}$ and $a_2 = \left[\frac{T-LSL}{d}\right]^{-2}$.

The distribution of $\hat{D}$ will be, in general, quite complicated. We will discuss a special case in which the distribution of X is, indeed, normal with expected value T and variance σ^2. Although this is not, in fact, an asymmetrical distribution, consideration of this case can provide an initial point of reference. Later we will indicate ways in which our results can be extended to somewhat broader situations - though not as broad as we would wish.

With the stated assumptions, we know that

(i) the distribution of $(X_i-T)^2$ is that of $\chi_1^2 \sigma^2$. (We use the symbol 'χ_ν^2' to denote 'χ^2 with ν degrees of freedom').

(ii) This is also the conditional distribution of $(X_i-T)^2$, whether $X_i > T$ or $X_i < T$.

(iii) The number, K, of X's which exceed T has a binomial distribution with parameters n, ½. (Denoted Bin(n,½)). Hence,
(iv) Given K, the conditional distributions of $S_+ \sigma^{-2}$ and $S_- \sigma^{-2}$ are those of χ^2_K, χ^2_{n-K} respectively, and $S_+ \sigma^{-2}$ and $S_- \sigma^{-2}$ are mutually independent. And also,
(v) The distribution of $H = S_+/(S_+ + S_-)$ is that of $\chi^2_K/(\chi^2_K + \chi^2_{n-K})$ which is Beta(½K, ½(n-K)) so that the density function of H is

$$f_H(h) = \{B(\tfrac{1}{2}K, \tfrac{1}{2}(n-K))\}^{-1} h^{\frac{1}{2}K-1}(1-h)^{\frac{1}{2}(n-K)-1} \quad (0 < h < 1) \qquad (8)$$

for K = 1,2,...n-1, and H and $S_+ + S_-$ are mutually independent and further
(vi) The conditional distributions of $S_+ \sigma^{-2}$ and $S_- \sigma^{-2}$, given K and H, are those of $H \chi^2_n$ and $(1-H) \chi^2_n$, respectively.

From (7)

$$\hat{D} = \begin{cases} a_1 S_+ \sigma^{-2} & \text{for } S_+/S_1 > a_2/a_1 \\ a_2 S_- \sigma^{-2} & \text{for } S_+/S_- < a_2/a_1 \end{cases} . \qquad (9)$$

So $\hat{D}$ is distributed as

$$\begin{cases} a_1 H \chi^2_n & \text{for } H/(1-H) > a_2/a_1 \quad \text{i.e.} \quad H > a_2/(a_1+a_2) \\ a_2(1-H)\chi^2_n & \text{for } H < a_2/(a_1+a_2) \end{cases} .$$

The overall distribution of $\hat{D}$ can be represented as

$$\hat{D} \quad \begin{Bmatrix} a_1 H & \text{for } H > b \\ a_2(1-H) & \text{for } H < b \end{Bmatrix} \chi^2_n \bigwedge_H \text{Beta}(\tfrac{1}{2}K, \tfrac{1}{2}(n-K)) \bigwedge_K \text{Bin}(n,\tfrac{1}{2}), \qquad (10)$$

where the symbol $\bigwedge_Y$ means 'mixed with respect to Y' having the distribution that follows, and $b = a_2/(a_1+a_2)$.

Without loss of generality, we will assume that USL-T ≤ T-LSL. Then, for a symmetrical distribution with mean T and variance σ^2,

$$C_{jkp} = \frac{d}{3\sigma}\frac{USL-T}{d} = \frac{d}{3\sigma}\frac{1}{\sqrt{a_1}} \tag{11}$$

and $\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} = 2$.

Hence
$$\frac{\hat{C}_{jkp}}{C_{jkp}} = \sqrt{\frac{n\ a_1}{2}}\,\hat{D}^{-½} \tag{12}$$

and

$$E\left[\left(\frac{\hat{C}_{jkp}}{C_{jkp}}\right)^r\right] = \left(\frac{n\ a_1}{2}\right)^{½r} E[\hat{D}^{-½r}]. \tag{13}$$

Now, from (8), (9) and (10), (noting that $E[\chi_n^{-½r}] = \{2^{½r}\,\Gamma(½r)\}^{-1}\,\Gamma(½(n-r))$),

$$E[\hat{D}^{-½r}] = \frac{\Gamma(½(n-r))}{2^{½r}\,\Gamma(½n)}\frac{1}{2^n}\left[a_2^{-½r} + \sum_{k=1}^{n-1}\frac{\binom{n}{k}}{B(½k,\ ½(n-k))}\cdot\right.$$
$$\left.\{a_1^{-½r}\,B_{1-b}(½(n-k),\ ½(k-r)) + a_2^{-½r}\,B_b(½k,\ ½(n-k-r))\} + a_1^{-½r}\right],$$

whence

$$E\left[\left(\frac{\hat{C}_{jkp}}{C_{jkp}}\right)^r\right] = \frac{n^{½r}\,\Gamma(½(n-r))}{2^{n+r}\,\Gamma(½n)}\left[\left(\frac{a_1}{a_2}\right)^{½r} + 1 + \sum_{k=1}^{n-1}\frac{\binom{n}{k}}{B(½k,\ ½(n-k))}\cdot\right.$$
$$\left.\{B_{1-b}(½(n-k),\ ½(k-r)) + \left(\frac{a_1}{a_2}\right)^{½r} B_b(½k,\ ½(n-k-r))\}\right]. \tag{14}$$

where $B_v(u_1,u_2) = \int_0^v y^{u_1-1}(1-y)^{u_2-1}\,dy$ and $B(u_1,u_2) = B_1(u_1,u_2)$.

In the next section, we will present numerical values for the mean and variance of $\hat{C}_{jkp}/C_{jkp}$ for certain values of the parameters, and comment thereon.

NUMERICAL RESULTS

Table 1 presents numerical values of

$$E[\hat{C}_{jkp}/C_{jkp}] \quad \text{and} \quad Var(\hat{C}_{jkp}/C_{jkp})$$

for several values of (USL-T)/d, and n = 10, 20, 30.

Table 1. Values of $E=E[\hat{C}_{jkp}/C_{jkp}]$ and $V=Var(\hat{C}_{jkp}/C_{jkp})$ when $\mu=T$

n	$\frac{USL-T}{d}$	1.0	0.9	0.8	0.7	0.6	0.5	0.4
10	E	0.9210	1.0075	1.0814	1.1420	1.1900	1.2269	1.2550
	V	0.0615	0.0782	0.1054	0.1440	0.1930	0.2504	0.3154
20	E	0.9191	0.9989	1.0536	1.0864	1.1036	1.1116	1.1151
	V	0.0290	0.0387	0.0561	0.0775	0.0974	0.1128	0.1232
30	E	0.9245	1.0001	1.0431	1.0623	1.0691	1.0711	1.0715
	V	0.0194	0.0264	0.0396	0.0519	0.0597	0.0634	0.0648

It is instructive to compare these values with similar quantities for $\hat{C}_{pk}/C_{pk}$ and $\hat{C}_{pm}/C_{pm}$, in Kotz and Johnson (1992) and Pearn et al. (1992) respectively. (Values available for the second author, on request.)

As to Table 1 itself, we have the following comments. In general the estimator $\hat{C}_{jkp}$ is biased. The bias is negative when $T = \frac{1}{2}(USL+LSL)$ but increases as (USL-T)/d decreases. The increase is quite substantial when (USL-T)/d is as small as 0.4. As might be expected the variance of $\hat{C}_{jkp}$ decreases as n increases - it increases as (USL-T)/d decreases, as the target value gets nearer to the upper specification limit. The bias, also, decreases as n increases; this effect is particularly noticeable for smaller values of (USL-T)/d.

EXTENSIONS

Since the index C_{jkp} is intended to make allowance for asymmetry in the distribution of the process characteristic X it would be of interest to learn something of the distribution of the estimator $\hat{C}_{jkp}$ under such conditions. The analysis of the preceding section can be extended to certain kinds of asymmetry of distribution of X, although they are not, unfortunately, of what may be regarded as common kinds of asymmetry.

We note two ways in which this may be done.

(i) If the population density function of X is

$$\tfrac{1}{2}[g(x;T,\sigma_1) + g(-x;-T,\sigma_2)] \tag{15}$$

where

$$g(x;T,\sigma) = \begin{cases} 2(\sigma\sqrt{2\pi})^{-1} \exp\{-\tfrac{1}{2}(\frac{x-T}{\sigma})^2] & \text{for } x \geq T \\ 0 & \text{for } x < T, \end{cases}$$

then the distribution of $\hat{D}$ is as in (10) with a_j replaced by $a_j(\sigma/\sigma_j)^2$ $(j = 1,2)$.

(ii) The number of X_i's exceeding T can have a binomial distribution with parameters n, p with $0 < p < 1$.

If (i) and (ii) are combined, the density function of X will be

$$p\ g(x;T,\sigma_1) + (1-p)\ g(-x;-T,\sigma_2). \tag{16}$$

For either (i) or (ii) the expected value of X is not, in general, T for density functions (15) or (16). (The distribution of X is a mixture of a half-normal and a negative half-normal.) However, the conditional distributions of $\Sigma_{X_i>T}(X_i-T)^2/\sigma_1^2$ and $\Sigma_{X_i<T}(X_i-T)^2/\sigma_2^2$, given K are still χ_K^2 and χ_{n-K}^2, respectively.

BIBLIOGRAPHY

Beazley, C.E. and Marcucci, M.O. (1988) Capability indices: Process performance indicators, Annual Tech. Conf., ASQC, Dallas, Texas (pp. 516-523).

Bissell, A.F. (1990) How reliable is your capability index? Appl. Statist., 39, 331-340.

Boyles, R.A. (1991) The Taguchi capability index, J. Qual. Technol., 23, 1-26.

Chan, L.K., Cheng, S.W. and Spiring, F.A. (1988) The robustness of the process capability index, C_p, to departures from normality. In Statistical Theory and Data Analysis II (K. Matusita, ed.) North Holland: Amsterdam, pp. 223-239.

Chou, Y.M. and Owen, D.B. (1989) On the distribution of the estimated process capability indices, Commun. Statist. - Theor. Meth., 18, 4549-4560.

Kitska, D. (1991) C_p and C_{pk} indices should be discontinued, Quality Progress, 24(3), 8.

Kotz, S. and Johnson, N.L. (1992) Some process capability indicators are more reliable than one might think (A comment on Bissell (1990)). Appl. Statist. (to appear).

Pearn, W.L., Kotz, S. and Johnson, N.L. (1992) Distributional and inferential properties of process capability indices, old and new. Journal of Quality Technology (submitted).

Spiring, F.A. (1991) Assessing process capability in the presence of systematic assignable cause, J. Qual. Technol., 23, 125-134.

Zhang, N.F., Steinbeck, G.A. and Wardrop, D.M. (1990) Interval estimation of process capability index C_{pk}. Commun. Statist. - Theor. Meth., 19, 4455-4470.

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