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DEVIATIONS BETWEEN SAMPLE QUANTILES AND EMPIRICAL PROCESSES UNDER ABSOLUTE REGULAR PROPERTIES

by

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It is shown that Glivenko-Cantelli type theorems are held for empirical and quantile processes, if the sequence of random variables satisfies the absolute regular condition. We also show that the quantile process can be approximated by a Gaussian process if absolute regular coefficients are of order n^{-q}, for $q > 8$. This approximation result is used to obtain the model-free prediction intervals; their performance is demonstrated by simulation.

KEY WORDS: absolute regular, empirical process, quantile process, prediction interval.

Abb. title Prediction Interval for Absolute Regular Process

1. INTRODUCTION

The main aim of this study is to establish a strong approximation result for the quantile process, assuming that the observations satisfy the absolute regular property. Further, we apply this result to demonstrate how to construct model free one-step ahead prediction intervals for future observations. The literature on the strong approximation for the quantile process under independently and identically distributed random variables is extensive, with prominent contributions from Bahadur (1966), Kiefer (1970), Csáki (1977), Csörgo and Révèsz (1978), among others. Their results were attained by relating empirical processes with those of the quantile. Our attention here focuses on obtaining the asymptotic representation theory of quantile processes via empirical processes under absolutely regular property and by assuming that the distribution function of the observations satisfies some regularity conditions. This work deals exclusively with the classical approach to the study of quantile processes based on absolute regular observations. The results obtained are similar to those obtained for the i.i.d. case. The theory here is more or less self-contained, however, it does use a number of known results.

To be explicit, some notations are introduced. Throughout this work, we suppose that $\{X_i; i \in \mathbf{Z}\}$ ($\mathbf{Z}$ is the set of integers) is a sequence of strictly stationary random variables on a probability space $\mathcal{L}_2(\Omega, \mathrm{F}, \mathrm{P})$, with $EX_i = 0$, $EX_i^2 < \infty$ and common distribution function $\mathbf{F}(x)$. The empirical distribution function at stage $n \in \mathbf{N}$ is defined by $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x]}$, $x \in R$. Define the n^{th} sample quantile function $F_n^{-1}(\cdot)$ of $\mathbf{F}^{-1}(t) = \inf\{x: \mathbf{F}(x) \geq t\}$ by

$$F_n^{-1}(t) = \begin{cases} \inf\{x: F_n(x) \geq t\} & \text{for } t > 0 \\ F_n^{-1}(0+) & \text{for } t = 0. \end{cases}$$

This is the left continuous inverse of the right continuous $F_n(\cdot)$. For a random sample $X_i, \ldots, X_n$ of strictly stationary r.v.'s, define the n^{th} sample quantile process $V_n(t)$ by

$$\{V_n(t): t \in (0,1)\} = \{n^{\frac{1}{2}}(F_n^{-1}(t) - F^{-1}(t)); t \in (0,1)\}.$$

Csörgo and Révèsz (1978) have shown that if $\{X_i; i \in N\}$ is a sequence of i.i.d. r.v.'s with a continuous d.f. $F(\cdot)$, which satisfies some regularity conditions, then one can introduce a Brownian bridge $\{B_n(t); t \in [0, 1], n \in N\}$ such that

(1.1) $$\lim \sup_{n\uparrow\infty} \sup_{t\in(0,1)} |f(F^{-1}(t))V_n(t) - B_n(t)| \stackrel{a.s.}{=} 0.$$

where $f(\cdot)$ is the probability density of X's.

The objective of this study is to obtain similar results to (1.1) for a sequence of r.v.s satisfying both the strictly stationary and absolute regular property. Further, it is intended to exploit these results in showing how to construct prediction intervals for future observations. The approach utilized to obtain these results will bear considerable resemblance to the theory developed for the i.i.d. case; however, there are some non-trivial technical problems to overcome on the way. The symbol "c" denotes a generic positive constant not necessarily the same at each appearance, while c_i, $i = 1, 2, \ldots$ denotes particular versions of c. Define $a_n = 0(b_n)$ $(b_n \geq 0)$, if $\exists c > 0$ such that

$|a_n| \le cb_n$ for all n. For $x \in \mathbf{R}$, we let $[x] = 1$, if $x \le 1$ and $[x] = \max\{n \in \mathbf{N}: n \le x\}$. It is well worth noting that the class of strictly stationary absolute regular random variables is very broad. For example, this class includes one-sided linear models, as long as the following conditions are satisfied: (a) the probability distribution (p.d.) of the white noise is absolutely continuous, like Gaussian, Cauchy, exponential or uniform (on some interval) or even a p.d. belonging to a much wider class, and (b) the coefficient of the linear model decays at an exponential rate or even at a polynomial rate (Bradley, 1986).

2. STATEMENT OF RESULTS

For $a \le b$, $a, b \in \mathbf{Z}$, let $\mathbf{F}_a^b$ denote the σ-algebra generated by $X_a, X_{a+1}, \ldots X_b$. The sequence $\{X_i, i \in \mathbf{Z}\}$ of random variables satisfies the absolute regular condition if

$$(2.1)\quad \beta(n) = E\sup_{A \in F_n^\infty} |P(A|F_{-\infty}^0) - P(A)| \downarrow 0 \text{ as } n \uparrow \infty.$$

Since some of the preceding results will be related to a strong mixing property, we will also state the strong mixing coefficients to be

$$(2.2)\quad \alpha(n) = \sup_{B \in F_{-\infty}^0} \sup_{A \in F_n^\infty} |P(A \cap B) - P(A)P(B)|.$$

It can be shown that $\alpha(n) = \frac{1}{2}\sup_{A \in F_n^\infty} E|P(A|F_{-\infty}^0) - P(A)| \le \frac{1}{2}\beta(n)$ (Eithier and Kurtz 1986, p. 345). This follows from Fatou's Lemma, which concludes the known result that every absolute regular sequence is strong mixing.

For convenient reference, the basic conditions on $F(\cdot)$, from which the various results are obtained, are gathered together here.

F_1: $F(x)$ is twice differentiable on (a, b), where $-\infty \leq a = \sup\{x: F(x) = 0\}$ and $\infty \geq b = \inf\{x: F(x) = 1\}$.

F_2: $\inf_{0<s<1} f(F^{-1}(s)) > 0$.

F_3: $\sup_{0<s<1} | f'(F^{-1}(s)) | < \infty$.

F_4: $F(x)$ is strictly increasing.

F_5: $f(x)$ is unimodal.

To study the difference between empirical and quantile process, it is natural to look at the following weighted difference

$$R_n(t) = V_n(t) + \frac{1}{f(F^{-1}(t))} U_n(t) \text{ on } t \in (0,1), \tag{2.3}$$

where $U_n(t) = n^{1/2}(F_n(F^{-1}(t)) - t)$ is the uniform empirical process. The close tie between $V_n(t)$ and $U_n(t)$ can easily be seen by a simple Taylor's expansion of $F_n^{-1}(t) - F^{-1}(t)$.

Bahadur (1966) showed that if $X_1, \ldots, X_n$ are i.i.d. r.v.'s with the distribution function satisfying certain fairly mild regularity conditions, then

$$|R_n(t)| = O(n^{-1/4} (\log n)^{1/2} (\log\log n)^{1/4}), \tag{2.4}$$

with probability one. Similar properties are obtained in the following theorem under absolute regular and strictly stationary conditions.

THEOREM 1. Under F_1, F_2 and F_3 and absolute regular coefficient, satisfying

$$\beta(n) = 0(n^{-(4+2\beta)}), \text{ for some } \beta > 2,$$

there exists a constant $c_1 > 0$, such that

$$\lim_{n\uparrow\infty}\sup_{0<t<1}\frac{n^{\frac{1}{2}}}{(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{4}}}|f(F^{-1}(t))R_n(t)| \le c_1,$$

with probability one.

The second result which will be obtained is a strong approximation for the quantile processes. To accomplish this, some further notations are required. In exactly the same way as in the i.i.d. case, we shall redefine the space for which the sequence $\{X_i; i \in \mathbf{Z}\}$ was generated to a space which is rich enough in the sense that a separable Gaussian process can be defined on it. Now, the separable Gaussian process will be called Brownian bridge $\{B(t); t \in [0, 1]\}$ if $B(1) = B(0) = 0$, $EB(t) = 0$, and has covariance function $EB(t)B(t') = \Gamma(t, t')$, for $0 \le t, t' \le 1$. To define $\Gamma(t, t')$, we write

$$g_n(t) = I_{[0,t]}(U_n) - t, \quad n \ge 1,$$

where $\{U_n; n \in \mathbf{Z}\}$ is a uniform, on [0, 1], strictly stationary absolute regular sequence of random variables. Then, for $0 \le t, t' \le 1$,

$$(2.5) \quad \Gamma(t,t') = Eg_1(t)g_1(t') + \sum_{n=2}^{\infty}(Eg_1(t)g_n(t') + Eg_n(t)g_1(t')),$$

such that the series on the right-hand side of $\Gamma(t, t')$ is absolutely continuous.

THEOREM 2. Let $\{X_i; i \in Z\}$ be a strictly stationary real valued sequence of random variables satisfying the absolute regular condition with absolute regular coefficient

$$\beta(n) = 0(n^{-(4+2\beta)}), \text{ for some } \beta > 2.$$

Suppose that F_1, F_2 and F_3 are satisfied, then there exists a Brownian bridge defined on the same probability space as the above sequence with $EB_n(t)B_n(t') = \Gamma(t, t')$, for $0 \leq t$, $t' \leq 1$, such that with probability one,

$$\lim \text{sum}_{n\uparrow\infty} \sup_{0<t<1} | f(F^{-1}(t)) V_n(t) + B_n(t) | = 0.$$

The final result obtained here, which is linked to Theorems 1 and 2, is the randomly located tolerance interval. The first to suggest this method was Butler (1982), who assumed that the observations were i.i.d. Cho and Miller (1987) produced some of these results under a uniform mixing sequence, and Fotopoulos, et al. (1992) have shown that this is also true under strong mixing sequences.

The class of the $100\gamma\%$ prediction interval is

(2.6) $\{I(\delta) = [F^{-1}(\delta), F^{-1}(\delta + \gamma)]; \quad 0 \leq \delta \leq 1-\gamma\}.$

For a random sample $X_1, \ldots, X_n$, with F known, the tolerance interval for X_{n+1} is the one which supports the smallest trimmed variance (see, e.g., Cho and Miller, 1987). This occurs at δ^*, the value of δ in which the minimum is attained. If F is unknown, estimators of both δ^* and F^{-1} are required. It is shown that under absolute regular and strictly stationary conditions, the following result is true; the proof is omitted, since this can be done in exactly the same fashion as in i.i.d. case, Butler (1982); see also Cho and Miller (1987).

THEOREM 3. Let $\{X_i; i \in \mathbf{Z}\}$ be an absolute regular strictly stationary sequence of real valued random variables with absolute regular coefficient

$$\beta(n) = 0(n^{-(4+2\beta)}) \text{ for some } \beta > 2.$$

Suppose that F_1 - F_5 are satisfied, then

i) $\hat{\delta}^* \to \delta^*$ in probability and

ii) $P_n(\gamma) = P(F_n^{-1}(\hat{\delta}^*) \leq X_{n+1} \leq F_n^{-1}(\hat{\delta}^* + \gamma)) \to \gamma$ in probability.

Numerical results of the coverage probability $P_n(\gamma)$ are discussed in the next section.

3. APPLICATIONS IN PREDICTION

Cho and Miller (1987) showed that the result of Theorems 2 and 3 hold for the process with $\alpha(n) = 0(e^{-\theta n})$ for some $0 < \theta < 1$. Using this approximation result, they obtained the model-free prediction interval (P.I.) for the first order autoregressive Gaussian process. While the model-free P.I. is simple to apply, it is also especially useful for non-Gaussian processes models, see e.g., Smith (1986) for related discussion.

To see how this model-free P.I. works, for absolute regular strictly stationary processes, we performed the simulation by generating observations from ARMA(1, 1)

$$X_t = \phi_1 X_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$$

where ϕ and θ are the autoregressive and moving average parameters and ε_t is an innovation from a standard Cauchy distribution. We note that X_t is a non-Gaussian process. The parameter values range from -0.9 to 0.9 for ϕ and from -0.8 to 0.8 for θ.

Following a similar procedure as in Cho and Miller (1987), we generate the 151 observations and discarded first 50 observations to minimize the effect of starting values. Using the 100 observations, we construct the 90% P.I using the RNCHY routine of IMSL and use the last observation to check the coverage. The coverage percentages are obtained from 400 replications and are reported in Table 1. It is observed that most of the coverage percentages are ranged from 87% to 92%, which is similar to the result reported in Cho and Miller. Even though we did the simulation only for ARMA(1,1) process with an innovation from a standard Cauchy distribution, considering that most of the time series data can be approximated by the low order ARMA models, e.g., ARMA(p,q), $p \leq 2$ and $q \leq 2$, we observed the same conclusions as did Cho and Miller for ARMA(l,q) and $q \geq 0$ with a Gaussian innovation.

4. PROOFS

The proofs of the type of theorems stated above will run as follows:

i) We introduce the nth uniform empirical process version, and we show that this process satisfies the law of iterated logarithms.

ii) Using the quantile transformation, it is then shown that the law of iterated logarithm is satisfied for the nth uniform sample quantile process.

iii) The study of a strong invariance statement Bahadur's process follows. This will prove Theorem 1.

and iv) The proof of Theorem 2 is highly dependent upon the exponentiality property, which is provided by Yoshihara (1978).

The following result will play an important role in resolving the problems stated in this study.

LEMMA 1. Let $q \geq 0$, $n \geq 1$ be integers and let $1 \leq R \leq n$. Suppose that $d = |t-s| \geq n^{-1/4}$ for $0 \leq t, s < 1$ and $\beta(n) = 0(n^{-(4+2\beta)})$ for some $\beta > 2$. Then, for $a \geq 1/4$

$$P(|\sum_{i=q+1}^{q+n}(g_i(t) - g_i(s))| > c_1 R d^a (n \log_2 n)^{1/2})$$
$$\leq c_2 \exp(-c_3 R d^{-a} \log_2 n) + c_4 R^{-2} n^{-(\beta-1)}.$$

PROOF. It is clear that, for $n = 1, 2, \ldots,$

i) $|g_n(t) - g_n(s)| \leq 1$ for $0 < t \leq s \leq 1$,

ii) $E(g_n(t) - g_n(s)) = 0$,

and iii) $E(g_n(t) - g_n(s))^2 = d(1 - d)$.

Since the sequence $\{U_i; i \in \mathbf{Z}\}$ is strictly stationary, it is enough to show Lemma 1, only for $q = 0$. In conjunction with Theorem 4 of Yoshihara (1978), it is observed that

$m_0 = 1$, and $\sigma_0^2 = d(1 - d) \leq d$, for d dependent upon n.

Set $r = c_1 R d^a (\log \log n)^{1/2}$ for some $a \geq 1/4$ and $m = [n^{2a}(\log n)^{-(2+\delta)}]$ for some $\delta > 0$, it follows that

$$\frac{r}{\sigma_0^2 (nm)^{1/2}} = c_2 R d^{-a} (\log n)^{(1+\delta/2)} (\log_2 n)^{1/2} d^{-(1-2a)} n^{-a}.$$

We let $d^{-(1-2a)}n^{-a} = 0(1)$, i.e., $n^{-(a/(1-2a))} = 0(d)$, for some $a \geq 1/4$, which agrees with the restriction stated in the lemma. The final remark concludes that

$$\frac{r}{\sigma_0^2(nm)^{1/2}} = 0(d^{-a}R(\log n)^{(1+\delta/2)}(\log_2 n)^{1/2}.$$

Via Theorem 4 (Yoshihara 1978), and the fact that $\beta(n) = 0(n^{-(4+2\beta)})$, and $\beta > 2$, the proof of Lemma 1 is now completed. ■

We continue by showing that the law of iterated logarithms is true for the uniform empirical process under strictly stationary absolute regular conditions.

LEMMA 2. Under the conditions of Lemma 1, it follows that

$$\limsup_{n\uparrow\infty}\sup_{0\leq s\leq 1}\frac{|\sum_{i\leq n}g_i(s)|}{(n\log\log n)^{1/2}} \leq c_5 \quad \text{a.s.}$$

PROOF. To show this result, it is only required to prove that $\forall\ \varepsilon > 0$, $\exists c > 0$ and $n_0(\varepsilon)$: with probability one,

(4.1) $$\frac{|\sum_{i\leq n}(g_i(t) - g_i(s))|}{(n\log\log n)^{1/2}} \leq cd^a + \varepsilon, \quad \text{for all } n \geq n_0,\ \text{with } d = |t-s| \text{ and } a \geq 1/4 .$$

However, this result is shown in Philipp (1977) under milder conditions than those in Lemma 1. ■

For the next result, we define the uniform quantile process by

$$\{A_n(s);\ 0 \leq s \leq 1\} = \{n^{1/2}(E_n^{-1}(s) - s);\ 0 \leq s \leq 1\},$$

where $E_n(s)$ and $E_n^{-1}(s)$ are the uniform analog of $F_n(s)$ and $F_n^{-1}(s)$, respectively.

LEMMA 3. Under the conditions of Lemma 1, the following result is true.

$$\lim\sup_{n\uparrow\infty}\sup_{0<s<1}\frac{|A_n(s)|}{(\log\log n)^{1/2}} \le c_6 \quad \text{a.s.}$$

The proof of Lemma 3 can be found in Babu and Singh (1978).

To show Theorem 1, we shall first establish that the uniform analog of Bahadur's process is satisfied, and then we shall relate this to a general case.

It turns out that the uniform analog of (2.3) is given by

$$(4.2)\qquad R_n^*(s) = n^{-1/2}\sum_{i\le n} g_i(s) + A_n(s).$$

LEMMA 4. Under the conditions of Lemma 1, the following result is true.

$$\lim\sup_{n\uparrow\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4}\sup_{0<s<1}|R_n^*(s)| \le c_7 \quad \text{a.s.}$$

PROOF. In conjunction with (2.12) (Babu and Singh, 1978), it yields that

$$(4.3)\qquad n^{-1/2}|R_n^*(t)| \le 3\sup_{|x|\le 2c\lambda_n}|E_n(x) - E_n(t) - d|,$$

where $\lambda_n = n^{-1/2}(\log\log n)^{1/2}$. By taking the supremum with respect to 't' in both sides of (4.3), it follows that

$$(4.4)\qquad \begin{aligned} n^{-1/2}\sup_{0<t<1}|R_n^*(t)| &\le 3\max_{|j|\le w_n}\max_{k<v_n}|E_n((k+j)b_n) - E_n(kb_n) - jb_n| + 2b_n \\ &= W_n + 2b_n, \end{aligned}$$

where $b_n = n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}$, $v_n = [1/b_n] + 1$ and $w_n = [2c\lambda_n/b_n] + 1$. It remains to show that $\lim\sup_{n\uparrow\infty} b_n W_n \leq c$ a.s.

From Bonferroni's inequality, it follows that, for $a = 1/4$,

$$(4.5)\quad P(W_n > c_8 b_n) \leq c_9 n \sup_{0<s<1} \sup_{|t|<2c\lambda_n} P(|\textstyle\sum_{1\leq n}(g_i(s+t) - g_i(s))| > c_8 n^{-3/4}(\log n)^{1/2}(\log_2 n)^{1/4}).$$

Via Lemma 1, and by setting $q = [n^{2a}(\log n)^{-(2+\delta)}]$, $a = 1/4$ and $|d| \leq 2c\lambda_n$, we obtain that

$$(4.6)\quad P(|\textstyle\sum_{i\leq n}(g_i(s+t) - g_i(s))| > c_8 n^{-3/4}(\log n^{1/2}(\log\log n)^{1/4})$$

$$\leq c_{10} n^{1/2}(\log n)^{-(2+\delta)}\exp(-c_{11}\frac{n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}}{dn^{1/4}(\log n)^{-(1+\delta/2)}}) + 4n\beta(q)$$

$$\leq c_{12} n^{1/2}(\log n)^{-(2+\delta)}\exp(-c_{12}(\log n)^{(3+\delta)/2}(\log\log n)^{-1/4}) + 4n\beta(q)$$

$$= 0(n^{-(2+\beta)}) \quad \text{for some } \beta > 2.$$

The last statement follows, since $\beta(n) = 0(n^{-(4+2\beta)})$, for some $\beta > 2$. The remaining proof is obtained from the fact that $P(W_n > c_8 b_n) = 0(n^{-(1+\beta)})$, $\beta > 2$, and Borel-Cantelli's Lemma. ∎

With this information in hand, we are now ready to obtain the proof of the results stated in Section 2.

PROOF OF THEOREM 1. From the mean value theorem, it is obvious that under F_1-F_3

$$(4.7)\quad V_n(t) = n^{\frac{1}{2}}(F^{-1}(E_n^{-1}(t)) - F^{-1}(t))$$
$$= A_n(t)/f(F^{-1}(v_{t,n})$$
$$= A_n(t)/f(F^{-1}(t)) + A_n(t)\{\frac{f(F^{-1}(t))}{f(F^{-1}(v_{t,n}))} - 1\}/f(F^{-1}(t))$$
$$= A_n(t)/f(F^{-1}(t)) + A_n(t)\frac{(t-v_{t,n})f'(F^{-1}(\delta_{t,n}))}{f(F^{-1}(\delta_{t,n}))f(F^{-1}(v_{t,n}))f(F^{-1}(t))}$$
$$= (A_n(t) + n^{-\frac{1}{2}}\epsilon_n(t))/f(F^{-1}(t)),$$

where $v_{t,n} = (E_n^{-1}(t) \wedge t, E_n^{-1}(t) \vee t)$ and $\delta_{t,n} = (v_{t,n} \wedge t, v_{t,n} \vee t)$. Here, the conventions $\wedge$ and $\vee$ used above mean $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$, respectively.

Calling upon (2.3) and (4.7), it is seen that

$$R_n(t) = A_n(t)/f(F^{-1}(v_{t,n})) + U_n(t)/f(F^{-1}(t))$$
$$= (R_n^*(t) + n^{-\frac{1}{2}}\epsilon_n(t))/f(F^{-1}(t)).$$

Via Lemma 3, it can be seen that $n^{-\frac{1}{2}} \mid \epsilon_n(t) \mid \overset{a.s.}{=} n^{-\frac{1}{2}}0(U_n^2(t)) \overset{a.s.}{=} 0(\log\log n/n^{\frac{1}{2}})$. Also, from Lemma 4, we have that $\sup_{0<t<1} \mid R_n^*(t) \mid \overset{a.s.}{=} 0(n^{-\frac{1}{4}}(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{4}})$. Thus, by considering that F_1 - F_3 are valid, the proof of Theorem 1 is now completed. ■

PROOF OF THEOREM 2. Philipp (1982) has shown that if $\{U_i, i \in Z\}$ is a strictly stationary strong mixing sequence of uniform, on [0, 1], random variables, and if the strong mixing coefficient $\alpha(n)$ is of order n^{-6}, then there exists a Brownian bridge $B_n(s)$

with covariance $\Gamma(s, s')$ for $0 \leq s, s' \leq 1$ and a constant λ such that with probability one,

$$\text{(4.9)} \quad \sup_{0 \leq s \leq 1} |U_n(s) - B_n(s)| = 0((\log n)^{-\lambda}).$$

Reading through this result, one concludes that (4.9) is also true if $\alpha(n) \asymp n^{-(4+2\beta)}$, $\beta > 2$ ($f(n) \asymp g(n)$ means that $f(n)/g(n)$ is bounded away from zero and infinity). In Theorem 2, we have assumed that $\beta(n) = 0(n^{-(4+2\beta)})$; since $\alpha(n) \leq \beta(n)$, we conclude that (4.9) is also true under absolute regular sequences of uniform random variables.

Calling upon (2.3) and (4.8), it can be seen that, for $t \in [0, 1]$,

$$\text{(4.10)} \quad f(F^{-1}(t))V_n(t) + B_n(t) = -U_n(t) + B_n(t) + R_n^*(t) + n^{-\frac{1}{2}}\epsilon_n(t).$$

By taking the supremum in both sides of (4.10), it follows that

$$\text{(4.11)} \quad \begin{aligned} &\sup_{0<t<1} |f(F^{-1}(t))V_n(t) + B_n(t)| \\ &\qquad \leq \sup_{0<t<1} |U_n(t) - B_n(t)| + \sup_{0<t<1} |R_n^*(t)| + \sup_{0<t<1} |n^{-\frac{1}{2}}\epsilon_n(t)|. \end{aligned}$$

Combining Lemma 4 and some of the steps of the proof of Theorem 1, the proof of Theorem 2 is now completed. ∎

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References

Babu, G.J. and Singh, K. (1978). On deviations between empirical and quantile processes for mixing random variables. J. Mult Anal., 8, 532-549.

Bahadur, R.R. (1966). A note on quantiles in large samples. Ann. Math Stat., 37, 577-580.

Bradley, R.C. (1986). Basic properties of strong mixing conditions. Birkhäuser, Boston.

Butler, R.C. (1982). Nonparametric interval and point prediction using data trimmed by Cruggs-type outlier rule. Ann. Statist., 10, 197-204.

Cho, S. and Miller, R.B. (1987). Model-free one-step-ahead prediction intervals: Asymptotic theory and small sample simulations. Ann. Statist., 15, 1064-1078.

Csáki, E. (1977). The law of iterated logarithm for normalizing empirical distribution function. Z. Wahrs. V. Geb., 38, 147-167.

Csörgo, M. and Révèsz, P. (1978). Strong approximation of the quantile process, Ann. Statist., 6, 882-894.

Ethier, S.N. and Kurtz, T.G. (1986). Markov Processes. John Wiley and Sons: New York, Toronto.

Fotopoulos, S.B., Ahn, S.K. and Cho, S. (1992). Strong approximation of quantile processes and its application under strong mixing properties. Technical Report, Washington State University.

Kiefer, J. (1970). Deviations Between the Sample Quantile Process and the Sample d.f. in Non-parametric Techniques in Statistical Inference (M.L. Puri, ed.), Cambridge Univer. Press: London, New York.

Philipp, W. (1977). A functional law of the iterated logarithm for empirical distributions functions of nearly dependent variables, Ann. Prob., 5, 319-350.

Philipp, W. (1982). Invariance principle for sums of mixing random elements and the multivariate empirical process, Colloquia Motle Soc. Jan. Bol. Limit Theorems in Prob. and Stat. (Hung.).

Smith, R.L. (1986). Maximum likelihood estimator for the NEAR(2) model, J.R. Statist. Soc. A, 48, 251-257.

Yoshihara, K. (1978). Probability inequalities of sums of absolutely regular processes and their applications. Z. Wahr V. Geb., 43, 319-329.

Table 1

Coverage Percentage of One-Step Ahead P.I. for ARMA (1, 1)

	θ								
ϕ	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8
-0.9	0.870	0.915	0.910	0.887	0.897	0.880	0.910	0.890	0.885
-0.8	0.882	0.897	0.870	0.872	0.915	0.890	0.895	0.862	0.900
-0.7	0.902	0.882	0.907	0.917	0.877	0.897	0.877	0.900	0.895
-0.6	0.912	0.875	0.902	0.875	0.852	0.862	0.892	0.900	0.900
-0.5	0.885	0.887	0.902	0.910	0.890	0.895	0.907	0.892	0.910
-0.4	0.867	0.905	0.910	0.885	0.920	0.897	0.867	0.877	0.897
-0.3	0.860	0.910	0.882	0.900	0.902	0.895	0.905	0.897	0.885
-0.2	0.880	0.895	0.870	0.915	0.892	0.902	0.922	0.902	0.892
-0.1	0.862	0.885	0.912	0.890	0.895	0.895	0.927	0.882	0.892
0.0	0.892	0.887	0.887	0.915	0.892	0.892	0.897	0.905	0.877
0.1	0.877	0.895	0.880	0.887	0.865	0.925	0.892	0.927	0.900
0.2	0.887	0.882	0.912	0.882	0.880	0.900	0.870	0.885	0.915
0.3	0.897	0.890	0.862	0.900	0.880	0.887	0.875	0.870	0.887
0.4	0.885	0.902	0.900	0.892	0.907	0.902	0.900	0.882	0.902
0.5	0.870	0.865	0.877	0.900	0.867	0.897	0.887	0.895	0.900
0.6	0.900	0.890	0.882	0.892	0.897	0.920	0.850	0.885	0.890
0.7	0.882	0.905	0.870	0.895	0.872	0.892	0.875	0.895	0.887
0.8	0.887	0.910	0.905	0.887	0.890	0.862	0.890	0.882	0.902
0.9	0.832	0.890	0.902	0.880	0.865	0.900	0.872	0.875	0.877