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A NOTE ON THE SIMPLE RANDOM WALK IN THE PLANE

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A NOTE ON THE SIMPLE RANDOM WALK IN THE PLANE

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Abstract: Downham and Fotopoulos (1988) derive bounds for six properties of the simple two-dimensional random walk on the vertices of a rectangular lattice. New bounds for two of the properties are derived that are algebraically simple, numerically close and of the known asymptotic form.

Key Words: inequalities, random walks

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1. Introduction

Let S_n be the position of the simple random walk on a two-dimensional lattice after n steps and let $S_o = 0$. Consider the model in which each vertex of the lattice has four nearest neighbors and

$$Pr(S_{n+1} = y \mid S_n = x) = \begin{cases} \frac{1}{4} & \text{if x and y are nearest neighbors} \\ 0 & \text{otherwise.} \end{cases}$$

Downham and Fotopoulos (1988) obtain bounds for the probability that the origin is revisited at the nth step, u_n; for the probability that the origin has not been revisited prior to the nth step, r_n; and for the expected number of distinct vertices visited by the nth step, h_n. These bounds are of the correct asymptotic forms (see, for example, Spitzer, 1964, and Kelly, 1977), and are sufficiently close for any envisaged applications.

If the time between successive steps is an exponential random variable with unit mean, then the probability that the origin is being revisited at time t is given by

$$u(t) = \sum_0^\infty \frac{u_{2n} t^{2n} e^{-t}}{(2n)!},$$

the probability that the walk has not passed through the origin by time t is given by

$$r(t) = \sum_0^\infty \frac{r_{n+1} t^n e^{-t}}{n!}, \tag{1}$$

and the expected number of vertices visited by time t is given by

$$h(t) = \sum_0^\infty \frac{h_n t^n e^{-t}}{n!}. \tag{2}$$

Downham and Fotopoulos (1988) obtain bounds for u(t), r(t) and h(t) that are of the correct asymptotic form. Improved bounds for r(t) and h(t) are derived here that are algebraically simple, of the known asymptotic form, and sufficiently close for most applications.

2. Bounds for r(t) and h(t)

The bounds given in Theorem 2 of Downham and Fotopoulos (1988) –

$$\frac{\pi}{A + \log n} > r_{2n} > \frac{\pi}{C + \log n} \qquad (n = 2, 3, 4 \ldots),$$

$$\frac{2\pi n}{A - 1 + \log n} > h_{2n-1} > \frac{2\pi n}{C - 1 + \log n} \qquad (n = 7, 8, 9, \ldots), \tag{3}$$

where A = 1.066π and C = 1.16π – are modified to simplify the proofs of Theorems 1 and 2.

It follows from $r_{2n-1} = r_{2n}$, and the above inequalities for r_{2n}, that

$$\frac{\pi}{A - \log 2 + \log n} > r_n > \frac{\pi}{C - \log 2 + \log(n+1)}, \tag{4}$$

for n = 3, 4, 5, ... The lower bound holds for all positive integers n and the upper bound holds for n = 1, but not for n = 2.

Substituting the appropriate bounds in (3) into $h_{2n} = h_{2n+1} - r_{2(n+1)}$,

$$h_{2n} > \frac{\pi(2n + 1)}{C - 1 - \log 2 + \log(2n+2)},$$

for n = 4, 5, 6, ... To account for all values of n, it is easily shown that

$$h_n > \frac{\pi(n+0.84)}{C - 1 - \log 2 + \log(n+2)}. \tag{5}$$

Substituting the upper bounds for h_{2n-1} and h_{2n+1} into $2h_{2n} = h_{2n-1} + h_{2n+1}$

$$h_{2n} < \frac{\pi(2n+1)}{A - 1 - \log 2 + \log(2n+1)},$$

which follows from consideration of the first and second differentials of the function $f(x) = x/(A\text{-}1\text{-}\log 2 + \log x)$ for $x > 1$. It follows that

$$h_n < \frac{\pi(n+1)}{A - 1 - \log 2 + \log(n+1)}, \tag{6}$$

for $n = 1, 2, 3, \ldots$. Bound (6) also holds for $n = 0$.

<u>Lemma</u>. For any $a > 0$ and $b \geq 1$,

$$\frac{1}{a + \log b} = \int_0^\infty \phi_a(u)\, e^{-bu} du,$$

where $\phi_a(u) = \int_0^\infty \frac{u^{x-1}e^{-ax}}{\Gamma(x)}\, dx.$

<u>Proof</u>. For $a > 0$ and $b \geq 1$

$$\begin{aligned} \frac{1}{a + \log b} &= \int_0^\infty (be^a)^{-x} dx \\ &= \int_0^\infty\int_0^\infty e^{-ax}\frac{u^{x-1}e^{-bu}}{\Gamma(x)} du\, dx \\ &= \int_0^\infty \phi_a(u)\, e^{-bu} du \end{aligned}$$

from Fubini's Theorem.

Theorem 1. For t > 1,

$$\frac{\pi}{A - \log 2 + \log(t+1)} + \frac{0.46}{(t-1)} > r(t) > \frac{\pi}{C - \log 2 + \log(t+2)}. \qquad (7)$$

Proof. Let β = C - log 2 and γ = A - log 2. Substituting the lower bound for r_n in (4) into (1) and using the Lemma,

$$r(t) > \pi \sum_{n=0}^{\infty} \int_0^{\infty} \phi_\beta(u) e^{-(n+2)u} \frac{t^n e^{-t}}{n!} du$$

$$= \pi \int_0^{\infty} \phi_\beta(u) \exp(-t-2u+te^{-u}) du.$$

Noting that $1-e^{-u} < u$, for all $u > 0$,

$$r(t) > \pi \int_0^{\infty} \phi_\beta(u) e^{-u(2+t)} du$$

and the lower bound follows from the Lemma.

Substituting the upper bound for r_n in (4) into (1), using the Lemma and summing as for the lower bound,

$$r(t) < \pi \int_0^{\infty} \phi_\gamma(u) g(u) du + (1 - \frac{\pi}{A}) te^{-t}. \qquad (8)$$

where $g(u) = \exp(-t+te^{-u})-e^{-ut}$. If $t > 1$, g(u) is positive, has one maximum at u_1, say, and tends to e^{-t} as $u \to \infty$. An algebraic expression for u_1 is not readily available, but for $t > 1$, $g(u_1) = \exp(-t+te^{-u_1}) - \exp(-u_1 t) < e^{-1}/(t-1)$. From (8) and applying the Lemma,

$$r(t) < \frac{\pi}{\gamma + \log(t+1)} + \frac{\pi e^{-1}}{\gamma(t-1)} + (1 - \frac{\pi}{A}) te^{-t}.$$

Noting that $\pi(e\gamma)^{-1} + (1 - \pi/A)t(t-1)e^{-t} < 0.46$, the upper bound in (7) follows.

Theorem 2. For $t > 1$,

$$\frac{\pi(t+1)}{A-1-\log 2+\log(t+1)} + 0.5 + \frac{1.2}{t-1} \geq h(t) \geq \frac{\pi(t+0.84)}{C-1-\log 2+\log(t+3)}. \tag{9}$$

Proof. Substituting lower bound (5) for h_n into (2) and using the Lemma,

$$h(t) > \pi\sum_{n=0}^{\infty}\int_0^{\infty}\phi_{\beta-1}(u)\,(n+0.84)\,\frac{t^n}{n!}e^{-t}e^{-(n+2)u}du$$

$$> \pi\int_0^{\infty}\phi_{\beta-1}(u)\,(t+0.84)\,e^{-u(3+t)}du,$$

and the lower bound in (9) follows from the Lemma.

Substituting upper bound (6) for h_n into (2) and using the Lemma,

$$h(t) < \pi\sum_{n=0}^{\infty}\int_0^{\infty}\phi_{\gamma-1}(u)\,(n+1)\,\frac{t^n}{n!}e^{-t}e^{-(n+1)u}du$$

$$= \pi\int_0^{\infty}\phi_{\gamma-1}(u)\,(t+e^{u})\exp(-t-2u+te^{-u})\,du.$$

Using the same argument as in the derivation of the upper bound for $r(t)$, for $t > 1$,

$$h(t) < \frac{\pi t}{\gamma-1+\log(2+t)} + \frac{\pi}{\gamma-1+\log(t+1)} + \frac{\pi e^{-1}}{(\gamma-1)(t-1)} + \frac{\pi t e^{-1}}{(t-1)(\gamma-1+\log 2)}.$$

The upper bound is obtained by straightforward algebraic manipulation.

The asymptotic behavior of $r(t)$ and $h(t)$ can be investigated from the bounds in (7) and (9):

$$r(t) = \frac{\pi}{\log t} + O((\log t)^{-2})$$

and

$$h(t) = \frac{\pi t}{\log t} + O(t(\log t)^{-2}),$$

where $0((\log t)^{-2})$ and $0(t(\log t)^{-2})$ are both negative. The values of the asymptotic forms for r(t) and h(t), together with the values of the bounds in (7) and (9) are given in Table 1 for some values of t. The bounds are seen to be close.

TABLE 1
Values of bounds (7) and (9) and the asymptotic forms for r(t) and h(t).
(All values are rounded to three significant figures.)

	r(t)			h(t)		
t	Lower	Upper	$\pi/\log t$	Lower	Upper	$\pi t/\log t$
20	0.520	0.575	1.05	12.9	14.65	21.0
200	0.380	0.397	0.593	86.9	91.6	119
1000	0.298	0.307	0.413	658	680	827
2×10^{2}	0.180	0.183	0.217	0.382×10^{6}	0.389×10^{6}	0.433×10^{6}
2×10^{10}	0.118	0.119	0.133	0.245×10^{10}	0.248×10^{10}	0.265×10^{10}

3. Discussion

Kelly (1977) shows that the simple random walk in two dimensions can be applied to the spread of an abnormal clone in the basal layer of the epithelium, when the normal and abnormal cells divide at the same rate. The asymptotic forms are inaccurate even for large values of t (Table 1), but bounds (7) and (9) are close.

Three of the four bounds are both algebraically simpler and numerically better than those given in Downham and Fotopoulos (1988). Closer bounds for r(t) and h(t) can be derived, but their expressions are algebraically less simple than those in (7) and (9). The bounds given here are a compromise between numerical accuracy and algebraic simplicity.

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