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THE EXACT APPROXIMATION ORDER OF THE RANDOM MAXIMUM OF CUMULATIVE SUMS OF INDEPENDENT VARIABLES

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ABSTRACT

We investigate the asymptotic behavior of the uniform distance between the distributions of the random maximum of cumulative sums and $\sup_{t\in[0,1]}W_t$, where W_t is the Wiener process. It is assumed here that the variates are independent and identically distributed. We show that, under some weak conditions on the random index of the maximum, the approximation order of the uniform distance is as sharp as in the Berry-Essen Inequality. The main tools of achieving this are the use of the Hausdorff-metric and some probabilistic arguments.

INTRODUCTION AND NOTATIONS

The main scope of this study is to assess the asymptotic performance order for the uniform distance between the distribution of the random maximum of cumulative sums (CUSUM) of independent random variables and $G(x) = 2\Phi(x)-1$, $x > 0$, where $\Phi(x)$ denotes the standard normal distribution. The literature on the maxima of CUSUM is extensive, with prominent contributions from Erdös and Kac (1946), Donsker (1951) and Billingsley (1968), among others. Applications of the theory have been used on a variety of problems, including Markov chains, random walks, renewal theory, quality control, queuing theory and others. However, in most of these applications the index of the maxima of CUSUM was assumed to be fixed, and for those applications in which the index was random, it was assumed that the index over a known function tends "with" or "in" probability to a constant value. Our attention here focuses on obtaining the rate of convergence in the functional central limit theorem, for more general, yet realistic, stopping rules.

Throughout this paper, we suppose that $\{X, X_i; i \in \mathbf{N}\}$ {N is the set of natural numbers) is a sequence of real-valued independently and identically distributed random variables (i.i.d.r.v.'s) defined on $L_3(\Omega, \mathbf{F}, \mathbf{P}, \mathbf{R})$, the system of all random variables $X : \Omega \rightarrow \mathbf{R}$ ($\mathbf{R}$ is the set of real numbers) with $EX = 0$, $EX^2 = 1$ and $E|X|^3 < \infty$. Let $S_o = 0$, $S_n = X_1 + \ldots + X_n$ and let $M_{k,n} = \max_{k \leq j \leq n}(S_j - S_k)$, for $k = 0, 1, 2, \ldots, n$. Instead of $M_{o,n}$, we shall simply use M_n. It is clear that $M_{k,n} \geq 0$, for all $0 \leq k \leq n$, $n \in \mathbf{N}$, and it is a non-decreasing random variable. Let $T_n : \Omega \rightarrow \mathbf{N}$, for $n \in \mathbf{N}$ and $T : \Omega \rightarrow [d, \infty)$, be $\mathbf{F}$-measurable with $d > 0$ and fixed. The symbol "c" denotes a generic positive constant, not necessarily the same at each appearance, while c_i, $i = 1, 2, \ldots$ denotes particular versions of c. Define $a_n = 0(b_n)$, if $\exists\ c >$

0, such that $|a_n| \leq cb_n$ for all n. For $x \in \mathbf{R}$, we let $[x] = 1$, if $x \leq 1$ and $[x] = \max\{n \in N; n \leq x\}$, if $x > 1$. Let I(E) represent the indicator function of the event E.

Shreehari (1968), motivated by the work of Anscombe (1952), Renyi (1960) and Blum et al. (1963), has shown that if $T_n : \Omega \rightarrow N$, $n \in N$ and $T : \rightarrow (0, \infty)$ are F-measurable, such that for all $\delta > 0$,

$$(1.1) \quad \lim_{n\rightarrow\infty} P(|n^{-1}T_n(\omega) - T(\omega)| > \delta) = 0,$$

and, if $EX_i = 0$ and $Var(X_i) = 1$, then

$$(1.2) \quad \lim_{n\rightarrow\infty} P(M_{T_n} \leq xT_n^{1/2}) = G(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2\Phi(x) - 1 & \text{if } x \geq 0. \end{cases}$$

The importance of studying (1.2) is well established in various applications of stochastic processes; for example, renewal and queuing theory. Further, it is equally desirable for both application and theoretical purposes to know how early the distribution of M_{T_n} converges to G(x).

The purpose of this paper is to obtain the sharpest possible order of approximation of $H(T_n, T) = \sup_{x \in [0,\infty)} |P(M_{T_n} \leq xT_n^{1/2}) - G(x)|$ under some weak conditions on T_n and T. In establishing this, we are aided by using some ideas found in Landers and Rogge (1988). Here, T is a random variable which might depend upon the sequence $\{X_i, i \in N\}$. The approach, introduced here, requires the assumption of a condition on T that characterizes its dependence on the process.

It turns out that the one-sided Hausdorff-metric between the σ-fields $\sigma(T)$ and $\mathbf{F}_n \equiv \sigma(X_1, \ldots, X_n)$ allows one to formulate such a condition.

The condition needs the following preliminary definitions: Let $\mathbf{F}_{(1)}$ and $\mathbf{F}_{(2)}$ be σ-fields and define

$$(1.3)\quad d(A, \mathbf{F}_{(2)}) = \inf_{B \in \mathbf{F}_{(2)}} P(A \triangle B) \text{ and}$$
$$\rho(\mathbf{F}_{(1)}, \mathbf{F}_{(2)}) = \sup_{A \in \mathbf{F}_{(1)}} d(A, \mathbf{F}_{(2)}).$$

Then $\rho(\mathbf{F}_{(1)}\ \mathbf{F}_{(2)}) + \rho(\mathbf{F}_{(2)}, \mathbf{F}_{(1)})$ is the Hausdorff-metric between $\mathbf{F}_{(1)}$ and $\mathbf{F}_{(2)}$, if the σ-fields are completed; otherwise, we have only pseudometric in general. The use of Hausdorff distances is adopted here.

Let $0 < \alpha \leq ½$, $\beta \in \mathbf{R}$. Further, let $T_n : \Omega \rightarrow \mathbf{N}$; $T : \Omega \rightarrow [d, \infty)$ be F-measurable with $d > 0$; $\{\varepsilon_i;\ i \in \mathbf{N}\}$ be such that $\varepsilon_n \geq 1/n$ and ε_n tends to zero as n tends to infinity. Finally, let

$$(1.4)\quad P\left(\left| \frac{T_n}{[nT]} - 1 \right| > \varepsilon_n\right) = 0(\varepsilon_n^{½}), \text{ and}$$

$$(1.5)\quad \rho(\sigma(T), \mathbf{F}_n) = 0(n^{-\alpha}(\log n)^{\beta}).$$

Then, we shall show that

$$(1.6)\quad H(T_n, T) = \sup_{x \in [0,\infty)} \left| P(M_{T_n} \leq xT_n^{½}) - G(x) \right| = 0(\varepsilon_n^{½}) + 0(\delta_n),$$

where δ_n depends upon the order of $\rho(\sigma(T), \mathbf{F}_n)$; this dependence will be described subsequently.

The layout of the remainder of the paper is organized as follows: Section 2 contains the main result of the investigation. Section 3 contains the proof of the main result stated in Section 2; some remarks are presented in Section 4; the proof of certain auxiliary lemmas are delayed until Section 5.

THE RESULT

In this section, we first exhibit sufficient conditions for obtaining the rate of convergence of the uniform distance of the distribution of M_{T_n} and G(x).

The sequence of constants $\{\varepsilon_i;\ i \in N\}$ is such that $\varepsilon_n \geq 1/n$ and converges to zero as n tends to infinity. The list of conditions are:

C1: The sequence $\{X, X_i;\ i \in N\} \in L_3(\Omega, F, P, R)$, i.e., $EX = 0$, $EX^2 = 1$ and $E|X|^3 < \infty$.

Let $T_n : \Omega \to N$, $n \in N$ and $T : \Omega \to [d, \infty)$ be F-measurable with $d > 0$, then

C2: $$P\left(\left|\frac{T_n}{[nT]} - 1\right| > \varepsilon_n\right) = 0(\varepsilon_n^{1/2}),$$

and C3: $\rho(\sigma(T), F_n) = 0(n^{-\alpha}(\log n)^{\beta})$, for $\alpha \in (0, 1/2]$, and $\beta \in R$.

Our main result is now summarized in the following Theorem.

Theorem. If Conditions C1-C3 are satisfied, then

$$(2.1)\quad H(T_n, T) = \sup_{x\in[0,\infty)} |P(M_{T_n} \leq xT_n^{1/2}) - G(x)| = 0(\varepsilon_n^{1/2}) + 0(\delta_n),$$

where

$$\delta_n = \delta_n(\alpha,\beta) = \begin{cases} n^{-1/2} & \alpha = 1/2,\ \beta < -3/2 \\ n^{-1/2} \log \log n & \alpha = 1/2,\ \beta = -3/2 \\ n^{-1/2}(\log n)^{\beta+3/2} & \alpha = 1/2,\ \beta > -3/2 \\ n^{-\alpha}(\log n)^{\beta+\alpha} & \alpha \in (0, 1/2),\ \beta \in \mathbf{R}. \end{cases}$$

DISCUSSION

The following remarks add further insight into the results obtained here.

i. If we replace C2 by the strongest conditions, i.e.,

$$P(|([nT])^{-1}T_n - 1| > \varepsilon_n) = 0$$

then, following the proof of the Theorem, we observe from (4.3) that $H(T_n, T) \leq I + II$, only. From (4.4) $I = 0(\delta_n)$ and from (4.16), $II \leq \sup_{x\in[0,\infty)}\{P(M_{[nT]} > x[nT]^{1/2}) - P(M_{[nT](1-\varepsilon_n)]} > x[nT]^{1/2}) = 0(\varepsilon_n^{1/2}) + 0(\delta_n)$. Hence a better approximation order than $0(\varepsilon_n^{1/2}) + 0(\delta_n)$ for (2.1) cannot be obtained. On the other hand, replacing it by a weaker condition

$$P(|([nT])^{-1}T_n - 1| > \varepsilon_n) = 0(\varepsilon_n^{1/2}\alpha_n),$$

with α_n tending to infinity, we can no longer obtain approximation order $0(\varepsilon_n^{1/2}) + 0(\delta_n)$ for (2.1) as stated in the Theorem;

ii. If T is a constant, then $\rho(\sigma(T), F_n) \equiv 0$. In this case, arguing essentially the same way as above, it follows from (4.4) that $I = 0(n^{-1/2})$ Nagaev (1970). Similarly, from (4.16) - (4.22) it follows that $II = 0(\varepsilon_n^{1/2})$. Consequently, the approximation order of $H(T_n, T)$ cannot be better than $0(\varepsilon_n^{1/2})$;

iii. If $T_n = n$, and $T = 1$, then it follows that $H(n, 1) = 0(n^{-1/2})$, Nagaev (1970), which, of course, agrees with (2.1), since $\varepsilon_n \geq 1/n$ (i.e., we apply our result for $\varepsilon_n = 1/n$).

iv. If, instead of $E \mid X \mid^3$, we have $E \mid X \mid^{2+\delta}$, for $0 < \delta \leq 1$, then, the analysis remains the same, but some modifications need to be made. In particular, ε_n can be chosen such that $\varepsilon_n \geq n^{-\delta}$, and ε_n tends to zero as n tends to infinity. For δ_n, everything remains the same, except instead of $\alpha = 1/2$ or $\alpha \in (0, 1/2)$, we set $\alpha = \delta/2$ or $\alpha \in (0, \delta/2)$, respectively.

v. Moreover, if $E \mid X \mid^r < \infty$, for $r \geq 4$, then one may obtain a better approximation (the remaining term will not possibly exceed the order of $n^{-r/2}$); but this might be done at the expense of algebraic simplicity, since one should look at the second, third, or even larger order of approximation.

A possible answer to this question (v) may be formulated as follows:

Let

$$H'(T_n, T) = \sup_{x \in [0,\infty)} \mid P(S_{T_n} \leq T_n^{1/2} x) - R(x) \mid ,$$

where $R(x) = G(x) + g(x)\Sigma_{j=1}^{r} Q_j(x)\ E[nT]^{-j/2}$, $g(x)$ is the $\frac{dG(x)}{dx}$, and $Q_j(x)$ is a polynomial of degree 3j-1 (different to those in partial sums in terms of the coefficients), and their coefficients are functions of the cumulants of the X's up to j+2 order. Suppose that

$$E\,|\,X\,|^{\,r+2+\delta} < \infty, \text{ for some } \delta \in (0,\ 1];$$

$$P(\,|\,\frac{T_n}{[nT]} - 1\,|\, > \epsilon_n) = 0(\epsilon_n^{1/2}), \text{ for some } \epsilon_n \geq n^{-(r+\delta)} \text{ and } \epsilon_n \downarrow 0;$$

$$\rho(\sigma(T),F_n) = 0(n^{-\alpha}(\log n)^{\beta}), \text{ for } \alpha \in (\frac{r}{2},\frac{r+\delta}{2}], \text{ and } \beta \text{ as above;}$$

and $\limsup_{|t|\uparrow\infty} |\,f(t)\,| < 1$, Cramer's Condition,

where f(t) is the characteristic function of X. Then, one may have that

$$H(T_n,\ T) = 0(n^{-(r+\delta)/2}).$$

This corresponds to the Chebyshev-Cramer expansion.

The verification and validity of this result is not attempted here.

Application. Suppose that $\{X_i : i \in N\}$ is an i.i.d. sequence with mean $\mu \geq 0$ and standard deviation σ, both of which are unknown and exist. Define the following stopping rule

$$T_c = \inf\{n \geq 1 : S_n \leq -c\hat{\sigma}_n\}, \text{ for some } c > 0,$$

where $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$.

It is known that

$$\frac{\mu T_c}{c\hat{\sigma}_{T_c}} \to 1 \quad \text{a.s.}$$

This, in turn, shows that a sequence $\{\varepsilon_n : n \in N\}$ of constants exists such that

$$P\left(\left| \mu\frac{T_c}{c\hat{\sigma}_{T_c}} - 1 \right| > \varepsilon_{[c]}\right) = 0(\varepsilon_{[c]}^{1/2})$$

for some $\varepsilon_n \geq 1/n$.

Since $\hat{\sigma}_n \xrightarrow{a.s.} \sigma$ and $T_c \xrightarrow{a.s.} \infty$, as $c \uparrow \infty$, it follows that $\hat{\sigma}_{T_n} \xrightarrow{a.s.} \sigma$. If, on the other hand, $E|X|^p < \infty$, for some $p > 2$, it is true that $\hat{\sigma}_{T_c} \leq \sqrt{\sup_{n\geq 2}\hat{\sigma}_n^2}$. Thus, using the maximal ergodic lemma, it can be seen that $E\hat{\sigma}_{T_c} < \infty$.

Furthermore, since the random variable $\hat{\sigma}_{T_c} > 0$ fulfills the condition $E\hat{\sigma}_{T_c} < \infty$, it can also satisfy C3, this is because

$$\begin{aligned}\rho(\sigma(\hat{\sigma}_{T_c}^2), \sigma(X_1, X_2, \ldots, X_{[c]})) &\leq \sup_B P(\hat{\sigma}_{T_c}^2 \in B \,\Delta\, (\hat{\sigma}_{T_c}^2 \in B \bigcap [X_1, \ldots, X_{[c]}]) \\ &\leq P(\hat{\sigma}_{T_c}^2 > [c]) \leq [c]^{-1/2} E\sigma_{T_c}.\end{aligned}$$

Thus, one can conclude from this, that the stopping rule T_c is one of the class presented in the Theorem.

In quality control, it is of interest to know the position of change (out of control) of a particular process, so an action needs to be taken, Page (1954). Therefore, determining the probability that the process is below a certain level before it hits, for the first time, the boundary $-c\hat{\sigma}T_c$ (which is one of the lines that the process is considered to be out of control if it is passed), is of great importance. However, this is nothing else than finding the best approximation of the probability

$$P(\max_{0 \le j \le T_c} (S_j - j\mu) \le xT_c^{1/2}), \quad x > 0.$$

Hence the following corollary is in order.

Corollary 1. If $\{X_n : n \in N\}$ is an i.i.d. sequence of r.v.'s with mean $\mu \in (0, \infty)$ and finite variance σ^2, estimated by $\hat{\sigma}_n^{1/2} = \frac{1}{n} \sum_{i=1}^{n}(X_i - \bar{X})^2$, and if

$$T_c = \inf\{n \ge 1 : S_n < -c\hat{\sigma}_n\}, \ c > 0,$$

then, for $x > 0$,

$$\sup_{x \in [0, \infty)} | P(\max_{0 \le j \le T_c}(S_j - j\mu) < x\sigma T_c^{1/2}) - G(x) | = 0(\varepsilon_{[c]}^{1/2}) + 0([c]^{-1/4}),$$

where $\{\varepsilon_n : n \in N\}$ is defined above.

Arguing exactly as above, it can be seen that if $\{X_n : n \in N\}$ is an i.i.d. sequence with $EX = \mu \in (0, \infty)$ and $T_c = \inf\{n \ge 1 : S_n < -cn^{\alpha}\}$, $c > 0$, $0 < \alpha < 1$, then

$$\frac{\mu T_c^{1-\alpha}}{c\hat{\sigma}_n} \xrightarrow{a.s.} \quad \text{as } c \uparrow \infty.$$

Hence the following result is also true.

Corollary 2. If $\{X_n : n \in N\}$ is an i.i.d. sequence of r.v.s with mean $\mu \in (0, \infty)$ and finite variance σ^2, and if

$$T_c = \inf\{n \geq 1 : S_n < -cn^{\alpha}\hat{\sigma}_n\},\ c > 0,\ 0 < \alpha < 1,$$

then, for $x > 0$,

$$\sup_{x\in[0,\infty)} |\ P(\max_{0\leq j\leq T_c}(S_j - j\mu) < x\sigma T_c^{\frac{1}{2}(1-\alpha)}) - G(x)\ | = 0(\varepsilon_{[c]}^{\frac{1}{2}}) + 0([c]^{-\frac{1}{2}}),$$

where $\{\varepsilon_n : n \in N\}$ is given above.

PROOF

This segment of our work bears considerable resemblance to the work of Landers-Rogge (1988), where similar results are proven for the random sums. For reasons of convenience, we operate with the same notations used by Landers-Rogge (1988). As mentioned earlier, the proof of some useful auxiliary Lemmas are presented in Section 4.

It is easy to see that

$$(4.1)\quad H(T_n, T) = \sup_{x\in[0,\infty)} \left| P(M_{T_n} \leq xT_n^{\frac{1}{2}}) - G(x) \right|$$
$$= \sup_{x\in[0,\infty)} \left| P(\xi_n M_{T_n} \leq x[nT]^{\frac{1}{2}}) - G(x) \right|,$$

where $\xi_n = \dfrac{[nT]^{\frac{1}{2}}}{T_n}$.

Then, since $P(\,|\,\frac{[nT]^{1/2}}{T_n^{1/2}} - 1\,|\, > (2\varepsilon_n)^{1/2}) = 0(\varepsilon_n^{1/2})$ (see, e.g., Landers-Rogge, 1988), it follows from Lemma 1 that

$$(4.2)\quad H(T_n, T) = \sup_{x\in[0,\infty)} |P(M_{T_n} \le x[nT]^{1/2}) - G(x)| + 0(\varepsilon_n^{1/2}).$$

Also, using the triangle inequality, (4.2) can be bounded above as follows.

$$(4.3)\quad H(T_n, T) \le \sup_{x\in[0,\infty)} |P(M_{[nT]} \le x[nT]^{1/2}) - G(x)|$$
$$+ \sup_{x\in[0,\infty)} |P(M_{T_n} \le x[nT]^{1/2}) - P(M_{[nT]} \le x[nT]^{1/2})| + 0(\varepsilon_n^{1/2})$$
$$= I + II + 0(\varepsilon_n^{1/2}).$$

In order to show the Theorem, it is sufficient to show that

$$(4.4)\quad I = \sup_{x\in[0,\infty)} |P(M_{[nT]} \le x[nT]^{1/2}) - G(x)| = 0(\delta_n) \text{ and}$$

$$(4.5)\quad II = \sup_{x\in[0,\infty)} |P(M_{T_n} \le x[nT]^{1/2}) - P(M_{[nT]} \le x[nT]^{1/2})| = 0(\varepsilon_n^{1/2}) + 0(\delta_n).$$

<u>Proof of (4.4)</u>. Let $N_1 = \{2^i;\ i \in N\}$ and $N_n = \{\nu \in N_1;\ \nu \le [\frac{n}{\log n}]\}$. Set $j(n) = \max N_n$. Let $B_m \in F$, $m \in N$, and $B_m(\nu) = \{P(B_m | F_\nu) > 1/2\}$, $B_m(0) = \phi$ and $B_m(1/2) = 0$. It is obvious that $B_m(\nu) \in F_\nu$. Next, we partition the event B_m as follows.

$$(4.6)\quad I(B_m) = I(B_m) - I(B_m(j(m))) + \sum_{\nu\in N_m} \{I(B_m(\nu)) - I(B_m(\nu/2))\} + I(B_m(1)).$$

In showing (4.4), it is first necessary to define D_ν for $\nu = 1, 2, \ldots, j(m)$,

$$D_\nu = \sup_{x\in[0,\infty)} |E\{I(M_m \le xm^{1/2}) - G(x)\}\{I(B_m(\nu)) - I(B_m(\nu/2))\}|.$$

It is now natural to extend the above definition by introducing the sequence $\{\Delta_m,\ m \geq nd\}$, which is related to D_ν, $\nu = 1, 2, \ldots, j(m)$, as follows:

$$(4.7)\quad \Delta_m = \sup_{x\in[0,\infty)} \left| P(M_m \leq xm^{1/2}, B_m) - G(x)P(B_m) \right|$$
$$\leq E\left| I(B_m) - I(B_m(j(m))) \right| + \sum_{\nu\in N_m \cup\{1\}} D_\nu$$
$$= d(B_m, F_{j(m)}) + \sum_{\nu\in N_m \cup\{1\}} D_\nu.$$

The last equality in (4.7) follows from Lemma 2(ii), and the way we constructed $B_m(\nu)$. (See, also (12) in Landers-Rogge, 1988).

Thus, in finding a manageable bound for (4.7), it is necessary to obtain an inequality for D_ν. Calling upon Lemma 3 and the fact that $\nu \leq [\frac{m}{\log m}]$, we have that

$$(4.8)\quad D_\nu = \sup_{x\in[0,\infty)} \left| \int [P(M_m \leq x^{1/2} m \mid F_\nu) - G(x)][I(B_m(\nu)) - I(B_m(\nu/2))]dP \right|$$
$$\leq c_2 \frac{1}{m^{1/2}} \{\nu^{1/2} P(B_m(\nu)\Delta B_m(\nu/2)) + \int_{B_m(\nu)\Delta B_m(\nu/2)} \{ | S_\nu | + M_\nu\} dP\},$$

and

$$(4.9)\quad D_1 \leq c_3 \frac{1}{m^{1/2}} \{P(B_m(1)) + \int_{B_m(1)} |X| dP\}.$$

And now, combining (4.8) and (4.9), the following bound for Δ_m is obtained,

$$(4.10)\quad \Delta_m \leq d(B_m, F_{j(m)}) + c_3 \frac{1}{m^{1/2}} \{P(B_m(1)) + \int_{B_m(1)} |X| dP\}$$

$$+ c_2 \frac{1}{m^{1/2}} \sum_{\nu \in N_m} \{\nu^{1/2} P(B_m(\nu) \Delta B_m(\tfrac{\nu}{2})) + \int_{B_m(\nu)\Delta B_m(\nu/2)} \{ | S_\nu | + M_\nu\} dP\}.$$

$$\leq d(B_m, F_{j(m)}) + c_3 \frac{1}{m^{1/2}} \{P(B_m(1)) + \int_{B_m(1)} |X| dP\}$$

$$+ c_4 \frac{1}{m^{1/2}} \sum_{\nu \in N_m} \{\nu^{1/2} d(B_m, F_{\nu/2}) + \int_{B_m(\nu)\Delta B_m(\nu/2)} \{ | S_\nu | + M_\nu\} dP\}.$$

The final statement follows from the fact that

$$P(B_m(\nu)\Delta B_m(\nu/2)) \leq 2d(B_m, F_{\nu/2}).$$

The next step of the proof of (4.4) is to express the left hand side of (4.4) with respect to (4.10). To do this, we shall first define the event B_m to be $\{[nT] = m\}$. It is clear that $B_m \in \sigma(T)$, the σ-algebra generated by T. For reasons of convenience, we assume that $nd \geq 3$, where d is the lower bound of the domain of T.

Thus,

$$(4.11) \quad I = \sup_{x\in[0,\infty)} | P(M_{[nT]} \leq x[nT]^{1/2}) - G(x) |$$

$$= \sup_{x\in[0,\infty)} | \sum_{m\geq nd-1} \{P(M_m \leq xm^{1/2}, B_m) - G(x)P(B_m)\} |$$

$$\leq \sum_{m\geq nd-1} \sup_{x\in[0,\infty)} | P(M_m \leq xm^{1/2}, B_m) - G(x)P(B_m) | .$$

Set K_n to be the event $\{M_n > (2n\log n)^{1/2}\}$. Then, by (4.10), (4.11) can be bounded above as follows:

$$(4.12) \quad I \leq \sum_{j=1}^{5} R_j(n), \text{ where}$$

$$R_1(n) = \sum_{m \geq dn-1} d(B_m, \mathbf{F}_{j(m)})$$

$$R_2(n) = c_5 \sum_{m \geq dn-1} \frac{1}{m^{1/2}} \{P(B_m(1)) + \int_{B_m(1)} | \; |X| dP\}$$

$$R_3(n) = c_6 \sum_{\substack{m \geq dn-1 \\ \nu \in N_m}} \frac{1}{m^{1/2}} (2\nu \log \nu)^{1/2} d(B_m, \mathbf{F}_{\nu/2}).$$

$$R_4(n) = c_7 \sum_{m \geq dn-1} \frac{1}{m^{1/2}} \int_{K_\nu \cup B_m(\nu) \Delta B_m(\nu/2)} |S_\nu| dP$$

and $$R_5(n) = c_7 \sum_{m \geq dn-1} \frac{1}{m^{1/2}} \int_{K_\nu \cup B_m(\nu) \Delta B_m(\nu/2)} M_\nu dP.$$

The rest of the proof of (4.4) relies on proving that each $R_i(n) = 0(\delta_n)$, $i = 1, 2, 3, 4$ and 5.

Landers-Rogge (1988) have shown that $R_i(m)$ for $i = 1, 2, 3$ and 4 are of order $0(\delta_n)$, so it remains to show that $R_5(n)$ is also of the same order. In conjunction with Landers-Rogge's (1985) arguments (see e.g., expressions (29)-(32)), it follows that

(4.13) $$R_5(n) = c_7 \sum_{\substack{n \geq dn-1 \\ \nu \in N_m}} \frac{1}{m^{1/2}} \int_{K_\nu \cap B_m(\nu) \Delta B_m(\nu/2)} M_\nu dP$$

$$\leq c_8 \frac{1}{n^{1/2}} \sum_{\nu \in N_1} \int_{K_\nu} M_\nu dP$$

$$= c_8 \frac{1}{n^{1/2}} \sum_{\nu \in N_1} (\log \nu)^{1/2} \int \frac{M_\nu/\nu^{1/2}}{(\log \nu)^{1/2}} I\left(\frac{M_\nu/\nu^{1/2}}{(\log \nu)^{1/2}} > 1\right) dP$$

$$\leq c_8 \frac{1}{n^{1/2}} \sum_{\nu \in N_1} (\log \nu)^{1/2} \sum_{k \in N} P(M_\nu > k(\nu \log \nu)^{1/2}).$$

In conjunction with Lemma 4, it follows that, for $c > 1$,

$$\text{(4.14)}\quad R_5(n) \leq c_8 \frac{1}{n^{1/2}} \sum_{\nu \in N_1} \nu^{-1/2}(c\log \nu)^{-1} \sum_{k \in N} k^{-3}$$

$$\leq c_9 \frac{1}{n^{1/2}} \leq c_9 \delta_n.$$

This completes the proof that $I = 0(\delta_n)$.

Proof of (4.5). Since M_n is a non-decreasing sequence and $M_n \geq 0$, for all $n \in N$, it follows that

$$\text{(4.15)}\quad P(M_{T_n} \leq x[nT]^{1/2}) \leq P(M_{T_n} \leq x[nT]^{1/2}, \left|\frac{T_n}{[nT]} - 1\right| \leq \varepsilon_n) + 0(\varepsilon_n^{1/2})$$

$$\leq P(M_{[[nT](1-\varepsilon_n)]} \leq x[nT]^{1/2}) + 0(\varepsilon_n^{1/2}).$$

Using (4.15), the left-hand side of (4.5) can now be bounded above as shown in the following inequality.

$$\text{(4.16)}\quad II \leq \sup_{x \in [0,\infty)}\{P(M_{[nT]} > x[nT]^{1/2}) - P(M_{[[nT](1-\varepsilon_n)]} > x[nT]^{1/2})\} + 0(\varepsilon_n^{1/2}).$$

Further, a decomposition of the event B_m is needed. Let A_m be the event $\{P(B_m | F_{j(n)}) > 1/2\}$. The events A_m, for $m \geq nd$, are disjoint and $A_m \in F_{j(n)}$. Also, because of Lemma 2(ii), $P(B_m \Delta A_m) = d(B_m, F_{j(n)})$.

Hence

$$\text{(4.17)}\quad II \leq \sup_{x \in [0,\infty)}\{P(M_{[nT]} > x[nT]^{1/2}) - P(M_{[[nT](1-\varepsilon_n)]} > x[nT]^{1/2})\}$$

$$= \sup_{x \in [0,\infty)}\{\sum_{m \geq dn-1} \{P(M_m > xm^{1/2}, B_m) - P(M_{[m(1-\varepsilon_n)]} > xm^{1/2}, B_m)\}\}$$

$$\leq 2 \sum_{m \geq dn-1} d(B_m, F_{j(n)}) + \sum_{m \geq dn-1} \sup_{x \in [0,\infty)} \{P(A_m, M_m > xm^{1/2})$$

$$- P(A_m, M_{[m(1-\varepsilon_n)]} > xm^{1/2})\}.$$

To obtain the desired result in (4.5), a close study of the following difference is required. Since $M_n \leq M_k + M_{k,n}$ for $0 \leq k \leq n$, it follows that

$$(4.18)\quad R_{m,n} = P(A_m, M_m > xm^{1/2}) - P(A_m, M_{[m(1-\varepsilon_n)]} > xm^{1/2})$$

$$\leq P(A_m, M_{[m(1-\varepsilon_n)]} + M_{[m(1-\varepsilon_n)]}, m > xm^{1/2}) - P(A_m, M_{[m(1-\varepsilon_n)]} > xm^{1/2}).$$

Now, since $A_m \in F_{j(n)} \equiv \sigma(X_1, \ldots, X_{j(n)})$, $j(n) = [\frac{n}{\log n}]$,

$M_{[m(1-\varepsilon_n)],m} \in \sigma(X_{[m(1-\varepsilon_n)]+1}, \ldots, X_m)$, it follows that A_m and $M_{[m(1-\varepsilon_n)]}$ are independent.

Thus, (4.18) can be written as follows.

$$(4.19)\quad R_{m,n} = \int P(A_m, m^{1/2}x - h < M_{[m(1-\varepsilon_n)]} \leq m^{1/2}x) dP(M_{[m(1-\varepsilon_n)],m} \leq h).$$

Then, applying some of the steps in the proof of Lemma 3, (4.19) is deduced that

$$(4.20)\quad R_{m,n} = \int_{A_m} \{ \int P(m^{1/2}x - h < M_{[m(1-\varepsilon_n)]} \leq xm^{1/2} \mid F_{j(n)}) dP(M_{[m(1-\varepsilon_n)],m} \leq h)\} dP$$

$$\leq \frac{c_{10} P(A_m)}{([m(1-\varepsilon_n)] - j(n))^{1/2}}$$

$$+ \int_{A_m} \int \left| \Phi \left(\frac{m^{1/2}x - M_{j(n)}}{([m(1-\varepsilon_n)] - j(n))^{1/2}} \right) \right.$$

$$- \Phi \left(\frac{m^{1/2}x - M_{j(n)} - h}{([m(1-\varepsilon_n)] - j(n))^{1/2}} \right) \Bigg| dP(M_{[m(1-\varepsilon_n)],m} \le h)dP.$$

Therefore, in exactly the same way as in Lemma 3, and since $m \ge nd$ and $j(n) = [\frac{n}{\log n}]$, the integral part in the second term of (4.20) is bounded above by $c_{11} \frac{|h|}{m^{1/2}}$, which implies that

$$(4.21) \quad R_{m,n} \le c_{12} P(A_m) \left\{\frac{1}{m^{1/2}} + \frac{E|M_{[m(1-\varepsilon_n)],m}|}{m^{1/2}}\right\}.$$

Since, $EX = 0$, and $EX^2 = 1$, it is known that, for $r > 1$,

$$E|S_m - S_{[m(1-\varepsilon_n)]}|^{2r} \le c(\varepsilon_n m)^r.$$

Hence, by Erdös-Stèckin result (see Lemma A in Moricz, 1976), it follows that,

$$E|M_{[m(1-\varepsilon_n)],m}| \le (E|M_{[m(1-\varepsilon_n)],m}|^{2r})^{1/(2r)} \le c^* \varepsilon_n^{1/2} m^{1/2}.$$

For the sake of completeness, Erdös-Stèckin result is stated at the end of Section 4 as Lemma 5. In our case, $a_{[m(1-\varepsilon_n)]+1} = \ldots = a_m = 1$. Thus, (4.21) can be expressed as follows,

$$(4.22) \quad R_{m,n} \le c_{13} P(A_m) \left\{\frac{1}{m^{1/2}} + \varepsilon_n^{1/2}\right\}.$$

Inserting (4.22) into (4.18) and the result of this substitution into (4.17), the proof that $II = 0(\delta_n) + 0(\varepsilon_n^{1/2})$ is now completed, since $\sum_{m \ge nd} P(A_m) \le 1$ and $\sum_{m \ge nd} d(B_m, F_{j(n)}) \le 4\rho(\sigma(T), F_{j(n)}) = 0(\delta_n)$.

AUXILIARY RESULTS

In this section, we collect all the Lemmas which were used in the proof of the Theorem.

The proof of the following Lemma is simple and, hence, is omitted.

Lemma 1. Let $\{\xi_n; n \in N\}$ and $\{Y_n; n \in N\}$ be sequences of r.v.'s and let $W_n = \xi_n Y_n$. Suppose that $\{\alpha_n; n \in N\}$ and $\{\beta_n; n \in N\}$ are two sequences of positive constants which tend to zero as n tends to infinity. If

i. $\sup_{x \in [0,\infty)} |P(Y_n \leq x) - G(x)| = 0(\beta_n)$

and ii. $P(|\xi_n - 1| > \alpha_n) = 0(\beta_n)$,

then

$$\sup_{x \in [0,\infty)} |P(W_n \leq x) - G(x)| = 0(\alpha_n) + 0(\beta_n).$$

Lemma 2. (Landers-Rogge, 1986) Let $\mathbf{F}_{(1)}$ and $\mathbf{F}_{(2)}$ be two subfields of $\mathbf{F}$, then

i. If $\{B_n; n \in N\}$ is a sequence of disjoint events of $\mathbf{F}_{(1)}$, then the following result is true:

$$\sum_{n \in N} d(B_n, \mathbf{F}_{(2)}) \leq 4\rho(\mathbf{F}_{(1)}, \mathbf{F}_{(2)}),$$

and ii. If $A \in \mathbf{F}$, then if $B = \{P(A | \mathbf{F}_{(1)}) > ½\}$, we have that

$$P(A \Delta B) = d(A, \mathbf{F}_{(1)}).$$

The following result is the conditional version of Nagaev's result.

Lemma 3. Let $\{X_i;\ i \in N\}$ be a sequence of i.i.d.r.v.'s with $EX_i = 0$, $EX_i^2 = 1$ and $E|X_i|^3 = b < \infty$. Then, for all $k < [\frac{n}{\log n}]$, the following inequality is satisfied P - a.e.

$$\sup_{x\in[0,\infty)}|P(M_n \le xn^{1/2}|F_k) - G(x)| \le \frac{c_1}{n^{1/2}}\{k^{1/2} + M_k + |S_k|\}.$$

Proof. Since $X_1, X_2, \ldots$ are i.i.d., then the variables M_k and $M_{k,n}$ are independent and M_{n-k} and $M_{k,n}$ are equiprobable. Further, it is easy to see that, for all $k \le h \le n$, $S_h - S_k \le \max_{k\le h\le n} S_h + |S_k|$, which implies that $M_{k,n} \le \max_{k\le h\le n} S_h + |S_k|$, and for all $0 \le h \le k$, $S_h \le \max_{0\le h\le k} S_h$, which again implies that $M_k \le \max_{0\le h<k} S_h$. By adding these two inequalities it follows that for $0\le k\le n$, $M_n \ge M_k + M_{k,n} - |S_k|$. Hence,

$$\begin{aligned}
(5.1)\quad \Delta_n(F_k) &= \sup_{x\in[0,\infty)}|P(M_n \le xn^{1/2}|F_k) - G(x)| \\
&\le \sup_{x\in[0,\infty)}|P(M_k + M_{k,n} - |S_k| \le xn^{1/2}|F_k) - G(x)| \\
&= \sup_{x\in[0,\infty)}\left|P\left(\frac{M_{k,n}}{(n-k)^{1/2}} \le x\left(\frac{n}{n-k}\right)^{1/2} - \frac{M_k - |S_k|}{(n-k)^{1/2}}\,\middle|\,F_k\right) - G(x)\right| \\
&\le \sup_{x\in[0,\infty)}\left|P\left(\frac{M_{k,n}}{(n-k)^{1/2}} \le x\left(\frac{n}{n-k}\right)^{1/2} - \frac{M_k - |S_k|}{(n-k)^{1/2}}\,\middle|\,F_k\right)\right. \\
&\quad \left. - G\left(x\left(\frac{n}{n-k}\right)^{1/2} - \frac{M_k - |S_k|}{(n-k)^{1/2}}\right)\right| \\
&\quad + \sup_{x\in[0,\infty)}\left|G\left(x\left(\frac{n}{n-k}\right)^{1/2} - \frac{M_k - |S_k|}{(n-k)^{1/2}}\right) - G(x)\right|
\end{aligned}$$

$$\leq \sup_{x\in[0,\infty)} |P(M_{n-k} \leq x(n-k)^{½}) - G(x)|$$

$$+ 2\sup_{x\in[0,\infty)} |\Phi(x(\frac{n}{n-k})^{½} - \frac{M_k - |S_k|}{(n-k)^{½}}) - \Phi(x(\frac{n}{n-k})^{½})|$$

$$+ 2\sup_{x\in[0,\infty)} |\Phi(x(\frac{n}{n-k})^{½}) - \Phi(x)|$$

$$= I_1 + I_2 + I_3 \text{ say.}$$

Since $k \leq [\frac{n}{\log n}]$, it is clear (Petrov, p. 114, 1974) that

$$(5.2)\quad I_2 \leq \sqrt{\frac{2}{\pi}} \frac{|M_k - |S_k||}{(n-k)^{½}} \leq c_1 \frac{M_k + |S_k|}{n^{½}} \text{ a.s., and}$$

$$(5.3)\quad I_3 \leq \sqrt{\frac{2}{\pi e}} \{(\frac{n}{n-k})^{½} - 1\} \leq c_2 \frac{k}{n-k} \leq c_1 \frac{k^{½}}{2n^{½}} \text{ a.s.}$$

As far as I_1 is concerned, it follows from Nagaev (1970) that

$$(5.4)\quad I_1 \leq c_3 b^2 \frac{1}{(n-k)^{½}} \leq c_3 \frac{k^{½}}{2n^{½}}.$$

Hence, by substituting (5.2), (5.3) and (5.4) into (5.1), the result follows immediately.

<u>Lemma 4</u>. Let $\{X_i;\ i \in N\}$ be a sequence of i.i.d.r.v.'s with $EX_i = 0$, $EX_i^2 = 1$ and $E|X_i|^3 = b < \infty$. Then, for all $x \geq (2\log n)^{½}$

$$(5.5)\quad P(M_n > xn^{½}) \leq c\{x^{-6}n^{-½} + nP(|X_1| > \frac{1}{3}(x-2)n^{½})\}.$$

Proof. It is known, Révèsz (1968), that

(5.6) $P(M_n > xn^{1/2}) \leq \frac{4}{3} P(S_n > n^{1/2}(x-2))$.

Now, to check the validity of (4.5), it is sufficient to prove that

(5.7) $P(S_n > xn^{1/2}) \leq c \frac{1}{x^6 n^{1/2}} + 2nP(|X_1| > \frac{1}{3} xn^{1/2})$.

The proof of (5.7) can be seen in Lemma 1, Landers-Rogge (1984). This completes the proof of Lemma 4.

The following result is also stated in Moricz (1976) and it is true for either independent or dependent random variables.

Lemma 5. (Erdös-Stèckin) Let $\gamma > 2$ and let $\{a_k\}$ be a sequence of numbers such that, for $n \geq b + 1 \geq 1$,

$$E|S_n - S_b|^\gamma \leq c_\gamma \left(\sum_{k=b+1}^{n} a_k^2\right)^{\gamma/2}.$$

Then,

$$E|M_{b,n}|^\gamma \leq c_\varepsilon c_\gamma \left(\sum_{k=b+1}^{n} a_k^2\right)^{\gamma/2},$$

where c_ε does not depend on γ for $\gamma \geq 2 + \varepsilon$, $\varepsilon \geq 0$.

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