# AN ALTERNATIVE APPROACH TO ESTIMATION IN THE FUNCTIONAL MEASUREMENT ERROR PROBLEM

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**Abstract:** A method is proposed which simplifies estimation of all parameters in the functional measurement error model with no error in the equation under the assumption that the error covariance for the explanatory variables is known and the model variance is unknown. The proposed method is shown to produce the same estimates given for the structural model in Schneeweiss (1976) and Fuller(1987) .

Key words: Errors-in-variables; Estimation; Maximum Likelihood; Measurement error.

## 1. Introduction

Measurement error models have been explored since the latter part of the 19th century when Adcock (1877, 1878) investigated estimation properties under somewhat restrictive but realistic assumptions in simple linear regression models. Since then much has been accomplished in the way of estimation and hypothesis testing in error in variables models, especially in the past 10 to 15 years.

Fuller (1987) represents the most comprehensive single source of information on errors in variables models here to date. Fuller's book covers the topic of errors in variables in simple linear regression models to multivariate linear regression models to nonlinear regression models. The book's emphasis is placed on estimation techniques, which includes estimating true values for the fixed model and predicting true values for the random model.

Let the measurement error model be defined as follows:

$$y_i = \underset{\sim}{x}_i \underset{\sim}{\beta}\,, \tag{1.1a}$$

$$(Y_i, \underset{\sim}{X}_i) = (y_i, \underset{\sim}{x}_i) + (e_i, \underset{\sim}{u}_i), \tag{1.1b}$$

for i=1, 2,..., n, where $\{\underset{\sim}{x}_i\}$ is a sequence of k-dimensional row vectors of true values and $\underset{\sim}{\epsilon}_i=(e_i, \underset{\sim}{u}_i)$ is the vector of measurement errors. The above formulation is a measurement error model with no error in the equation, *i.e.*, a perfect relationship between the true response and true explanatory variables is assumed.

The idea behind measurement error models is that instead of measuring the true value $\underset{\sim}{x}_i$, which is unobservable, one observes the sum

$$\underset{\sim}{X}_i = \underset{\sim}{x}_i + \underset{\sim}{u}_i ,$$

where $\underset{\sim}{u}_i$ is a random variable. The observed variable $\underset{\sim}{X}_i$ is sometimes called the ***manifest*** variable or the *indicator* variable. The unobserved variable $\underset{\sim}{x}_i$ is sometimes called the *latent* variable. If the $\underset{\sim}{x}_i$ are considered fixed, then the model is referred to as a *functional* model; if the $\underset{\sim}{x}_i$ are considered stochastic, then the model is referred to as a *structural* model. In this paper we consider only the functional measurement error model with no error in the equation.

Estimation in the functional measurement error model requires that we estimate the fixed but unobservable true explanatory variables, most often treating them as nuisance parameters. Under maximum likelihood estimation, consistent estimators of all parameters in the model do not exist since the number of parameters increases with increasing sample size, n (see Fuller (1987), p 104), which most often leads to an unbounded likelihood function.

Estimation for the functional measurement error model typically requires rather restrictive assumptions regarding the measurement error covariance matrix. Usually one assumes that the measurement error covariance matrix is either completely known or is known up to a scalar multiple. However, in many practical situations these assumptions cannot be met. For instance, consider the case of determining appropriate transformations for a response variable in the presence of measurement error. In many biostatistical applications, knowledge of the measurement error covariance matrix for the explanatory variables can obtained (usually via independent replicated measurements). Schneeweiss (1976) and Fuller (1980) examined estimation of regression parameters for similar error-in-variables models when an estimate of the error covariance matrix is known which requires using a measurement error model with an error in the equation.

In this paper an alternative method is proposed which simplifies estimation of parameters in the functional measurement error model with no error in the equation by selectively using asymptotic substitutions where appropriate. The error covariance for the explanatory variables is assumed known and the model variance is assumed unknown, which are very practical assumptions for many biostatistical applications.

## 2. Proposed Method of Estimation

### 2.1 Overview

The proposed method of estimation generally goes like this: Assume the model given by (1.1) under normality assumptions for the measurement errors with the error covariance for the explanatory variables known and the model variance unknown. For given $\underset{\sim}{x}$, derive the least-squares estimates of $\underset{\sim}{\beta}$

and the model variance, say $\sigma^2$, as functions of $\underset{\sim}{x}$. The least-squares estimates of $\underset{\sim}{\beta}$ and $\sigma^2$ are shown to converge in probability, under suitable regularity conditions, to $\underset{\sim}{\beta}$ and $\sigma^2$, respectively. It is proposed that one uses estimates of the probability limit of the least-squares estimates of $\underset{\sim}{\beta}$ and $\sigma^2$ as the final estimates of $\underset{\sim}{\beta}$ and $\sigma^2$.

After obtaining the least-squares estimates of $\underset{\sim}{\beta}$ and $\sigma^2$, substitute them back into the likelihood function and then maximize the likelihood with respect to $\underset{\sim}{x}$, using asymptotic expressions to simplify the derivation. The estimate of $\underset{\sim}{x}$ is shown to be unbiased by this method.

Schneeweiss (1976) provided estimates of $\underset{\sim}{\beta}$ and the model variance when the measurement error covariance is known for the independent variables only and under the assumption of the structural measurement error model while remarking that the estimates would also hold true for the functional measurement error model. Fuller (1980) proposed estimates of $\underset{\sim}{\beta}$ under various assumptions concerning the availability of an estimator of the entire measurement error covariance matrix. Both Schneeweiss and Fuller show that the estimator for $\underset{\sim}{\beta}$ has a limiting normal distribution. However, neither author provides estimators of the true explanatory variables in their respective papers. Of course, estimators of the true explanatory variables have been derived under other model assumptions (Fuller 1987).

**2.2 Method**

Let $\underset{\sim}{M}_{XX} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{X}_i' \underset{\sim}{X}_i$, $\underset{\sim}{M}_{XY} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{X}_i' Y_i$, and $m_{YY} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$. Assume the following regularity conditions:

$$
\begin{aligned}
&(a)\ \underset{n\to\infty}{\text{Plim}} \underset{\sim}{M}_{XX} = \underset{\sim}{\Gamma}_{XX},\\
&(b)\ \underset{n\to\infty}{\text{Plim}} \underset{\sim}{M}_{XY} = \underset{\sim}{\Gamma}_{XY},\\
&(c)\ \underset{n\to\infty}{\text{Plim}}\, m_{YY} = \sigma_{YY},\\
&(d)\ \underset{n\to\infty}{\text{Plim}} \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{u}_i' \underset{\sim}{u}_i = \underset{\sim}{\Gamma}_{uu},
\end{aligned}
\qquad (2.0)
$$

where Plim represents the probability limit.

Let $\underset{\sim}{\epsilon}_i \sim N(\underset{\sim}{0}, \underset{\sim}{\Sigma}_{\epsilon\epsilon})$, where $\underset{\sim}{\Sigma}_{\epsilon\epsilon} = \text{diag}(\sigma^2, \underset{\sim}{\Gamma}_{uu})$. $\underset{\sim}{\Gamma}_{uu}$ is the known error covariance matrix for the explanatory variables; $\sigma^2$ is the unknown model variance. Since the $\underset{\sim}{x}_i$ are fixed, the log-likelihood function can be given by

$$l = -\frac{n}{2}\log|2\pi\Gamma_{uu}| - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - x_i\beta)^2$$

$$- \frac{1}{2}\sum_{i=1}^{n}(X_i - x_i)\Gamma_{uu}^{-1}(X_i - x_i)'. \tag{2.1}$$

We first maximize (2.1) with respect to $\beta$ and $\sigma^2$ for a given $x$:

$$\frac{\partial l}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(Y_i - x_i\beta)^2, \tag{2.2}$$

$$\frac{\partial l}{\partial\beta} = \frac{1}{\sigma^2}\sum_{i=1}^{n}x_i'(Y_i - x_i\beta). \tag{2.3}$$

Setting (2.2) and (2.3) to zero and solving, we have the following least-squares estimators for $\sigma^2$ and $\beta$

$$\ddot{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - x_i\ddot{\beta})^2, \tag{2.4}$$

$$\ddot{\beta} = (\sum_{i=1}^{n}x_i'x_i)^{-1}\sum_{i=1}^{n}x_i'Y_i. \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.1), we have

$$l = -\frac{n}{2}\log|2\pi\Gamma_{uu}| - \frac{n}{2}\log\ddot{\sigma}^2 - \frac{n}{2} - \frac{1}{2}\sum_{i=1}^{n}(X_i - x_i)\Gamma_{uu}^{-1}(X_i - x_i)'. \tag{2.6}$$

Now, we maximize (2.6) with respect to $x$:

$$\frac{\partial l}{\partial x_i} = -\frac{n}{2\ddot{\sigma}^2}\frac{\partial\ddot{\sigma}^2}{\partial x_i} + (X_i - x_i)\Gamma_{uu}^{-1}. \tag{2.7}$$

Setting equation (2.7) to zero and solving for $x_i$ explicitly is rather prohibitive. Hence, we propose simplifying the process by generously substituting asymptotic expressions which make the solution much easier.

Since $x_i = X_i - u_i$, it is straight-forward to show that

$$\operatorname*{Plim}_{n\to\infty}\ddot{\sigma}^2 = \sigma_{YY} - \ddot{\beta}^{*\prime}\Gamma_{XY} = \ddot{\sigma}^{*2},$$

$$\operatorname*{Plim}_{n\to\infty}\ddot{\beta} = (\Gamma_{XX} - \Gamma_{uu})^{-1}\Gamma_{XY} = \ddot{\beta}^{*}.$$

In actuality, one can easily show that $\ddot{\beta}^* = \beta$ and $\ddot{\sigma}^{*2} = \sigma^2$. Hence, under the regularity assumptions made in this paper, the least-squares estimates of $\beta$ and $\sigma^2$ are consistent estimators (as functions of $x$) of $\beta$ and $\sigma^2$. To estimate $\beta$ and $\sigma^2$, simply replace $\Gamma_{XX}$, $\Gamma_{XY}$, and $\sigma_{YY}$ with $M_{XX}$, $M_{XY}$, and $m_{YY}$, respectively, to obtain $\hat{\beta}$ and $\hat{\sigma}^2$ (which are not functions of $\mathbf{x}$).

The estimates of $\beta$ and $\sigma^2$ obtained in this fashion are identical to those given in Schneeweiss (1976) and Fuller (1987, p 105) for the structural model. Schneeweiss (1976) shows that the estimator for $\beta$ has a limiting normal distribution for the structural model which is also true for this functional model. Schneeweiss claims that under the assumption of normality, the estimates are maximum likelihood estimates for the functional model. However, it can be shown that this assertion is not true.

Now, in (2.7), replace $\ddot{\beta}$ with $\ddot{\beta}^*$ in the partial derivative, $\partial\ddot{\sigma}^2/\partial\mathbf{x}_i$, i.e., replace

$$\frac{\partial\ddot{\sigma}^2}{\partial\mathbf{x}_i} = \frac{\partial}{\partial\mathbf{x}_i}[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mathbf{x}_i\ddot{\beta})^2]$$

with

$$\frac{\partial}{\partial\mathbf{x}_i}[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mathbf{x}_i\ddot{\beta}^*)^2].$$

This substitution yields the following

$$\frac{\partial l}{\partial\mathbf{x}_i} = -\frac{1}{\ddot{\sigma}^2}(Y_i - \mathbf{x}_i\ddot{\beta}^*)\ddot{\beta}^{*\prime} + (\mathbf{X}_i - \mathbf{x}_i)\Gamma_{uu}^{-1}.$$

Now, replacing $\ddot{\sigma}^2$ with $\ddot{\sigma}^{*2}$ in the above, setting the partial derivative equal to zero, and solving for $\mathbf{x}_i$ yields the following (which in form is the same as Fuller (1987, p 104):

$$\hat{\mathbf{x}}_i = (Y_i\hat{\beta}' + \hat{\sigma}^2\mathbf{X}_i\Gamma_{uu}^{-1})(\hat{\beta}\hat{\beta}' + \hat{\sigma}^2\Gamma_{uu}^{-1})^{-1}. \tag{2.8}$$

Now, $\hat{\mathbf{x}}_i$ converges in probability (and hence in distribution) to $\tilde{\mathbf{x}}_i = (Y_i\beta' + \sigma^2\mathbf{X}_i\Gamma_{uu}^{-1})(\beta\beta' + \sigma^2\Gamma_{uu}^{-1})^{-1}$. Observe that $\tilde{\mathbf{x}}_i$ is simply a linear combination of $(Y_i, \mathbf{X}_i) \sim N(\mu_i, \Sigma_{\epsilon\epsilon})$, where $\mu_i = (\mathbf{x}_i\beta, \mathbf{x}_i)$. Therefore, using well-known normal distribution theory results, it can be shown that

$$\tilde{\mathbf{x}}_i \sim N(\mu_i\mathbf{K}, \mathbf{K}'\Sigma_{\epsilon\epsilon}\mathbf{K}),$$

where

$$\mathbf{K} = \begin{bmatrix} \beta'(\beta\beta' + \sigma^2\Gamma_{uu}^{-1})^{-1} \\ \sigma^2\Gamma_{uu}^{-1}(\beta\beta' + \sigma^2\Gamma_{uu}^{-1})^{-1} \end{bmatrix}.$$

Using the result above, we can conclude that the estimator, $\hat{\mathbf{x}}_i$, converges in distribution to a random variable which is unbiased for $\mathbf{x}_i$ since

$$E(\hat{\mathbf{x}}_i) - \mathbf{x}_i = \mathbf{x}_i[(\boldsymbol{\beta}\boldsymbol{\beta}' + \sigma^2\boldsymbol{\Gamma}_{uu}^{-1})(\boldsymbol{\beta}\boldsymbol{\beta}' + \sigma^2\boldsymbol{\Gamma}_{uu}^{-1})^{-1} - \mathbf{I}] = 0.$$

## 3. Remarks

An alternative method of estimating parameters in a special case of the functional measurement error model has been proposed. Closed form expressions of the maximum likelihood estimators for this special case have proven to be too cumbersome to compute. The method proposed here simplifies estimation of all parameters in the model under very reasonable regularity conditions by substituting asymptotic expressions where appropriate. The estimates obtained for the functional model are shown to be identical to those of Schneeweiss (1976) for the structural model. We provide an estimator for the true, fixed , unobservable explanatory variables which he does not. The estimator is shown to converge in distribution to a random variable which is unbiased.

Fuller (1987, p 173) provides modified estimators for the regression coefficients in the functional measurement error model with no error in the equation under the assumption that the error covariance matrix is known or an unbiased estimate exists and under the assumption that the error covariance matrix is known up to a scalar multiple. In most practical situations, neither of these assumptions are particularly realistic.

Since the method proposed in this paper is based on asymptotic results, it is informative to compare the results of Fuller's modified estimators to results from the method proposed in this paper via simulated data. Under our model assumptions, Fuller's modified estimator for $\boldsymbol{\beta}$ is given by

$$\tilde{\boldsymbol{\beta}} = [\mathbf{M}_{XX} - (\hat{\lambda} - n^{-1}\alpha)\boldsymbol{\Gamma}_{uu}]^{-1}\mathbf{M}_{XY},$$

where $\hat{\lambda}$ is the smallest root of $|\mathbf{M}_{ZZ} - \lambda\mathbf{S}_{\epsilon\epsilon}| = 0$, $\mathbf{M}_{ZZ} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{Z}_i'\mathbf{Z}_i$ with $\mathbf{Z}_i = (Y_i,\ \mathbf{X}_i)$, $\mathbf{S}_{\epsilon\epsilon}$ is an unbiased estimator of $\boldsymbol{\Sigma}_{\epsilon\epsilon}$ distributed as a multiple of a Wishart with $d_f$ degrees of freedom, $d_f$ is proportional to n, and $\alpha > 0$ is a fixed number. In comparison, the estimator proposed in this paper is readily seen to be simpler.

We simulated 20 samples of 50, 100, 200, 500, and 1000 observations generated by the following model

$$Y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + e_i, \qquad \mathbf{X}_i = (x_{1i}, x_{2i}) + (u_{1i}, u_{2i}) = \mathbf{x}_i + \mathbf{u}_i,$$

where $(\mathbf{x}_i, e_i, \mathbf{u}_i) \sim \mathrm{NID}(\mathbf{0}, \sigma^2\mathbf{I})$, $\beta_1 = \beta_2 = 0.7$, $\sigma^2 = 1$. In Fuller's method we let $\alpha = 4$ since he states that the mean square error of $\tilde{\boldsymbol{\beta}}$ is smaller for $\alpha = 4$ than for any smaller $\alpha$ through terms of order $n^{-2}$.

Tables 1-5 show the results of the simulation. Using the Euclidean norm as a measure of distance, $||\tilde{\beta} - \beta|| = [(\tilde{\beta}_1 - \beta_1)^2 + (\tilde{\beta}_2 - \beta_2)^2]^{1/2}$, Fuller's method generally performs better than our proposed method for observations of size 50. For observations of size 100 or more, the method proposed in this paper is seen to perform generally better than Fuller's method. Indeed, for observations of size 500 and 1000, there is a marked difference between the performance of Fuller's method and the our method. In the simulation, Fuller's method usually underestimated the true parameter value for observations of size 100 or more. However, the method proposed here did result in negative estimates of the variance in some samples for observations of size 50, suggesting that the asymptotic approximations may not perform very well for small samples.

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## References

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**Table 1**

N=50

| SAMPLE | $\tilde{\beta}$ | $\hat{\beta}$ | $\lVert\tilde{\beta}-\beta\rVert$ | $\lVert\hat{\beta}-\beta\rVert$ |
|---|---|---|---|---|
| 1 | 0.281 | 0.429 | 0.419 | 1.889 |
| | 0.709 | 2.569 | | |
| 2 | 0.422 | 0.759 | 0.382 | 0.059 |
| | 0.438 | 0.707 | | |
| 3 | 0.198 | 0.368 | 0.562 | 0.347 |
| | 0.448 | 0.800 | | |
| 4 | 0.620 | 1.310 | 0.410 | 0.616 |
| | 0.298 | 0.787 | | |
| 5 | 0.491 | 0.934 | 0.396 | 0.311 |
| | 0.363 | 0.495 | | |
| 6 | 0.431 | 0.712 | 0.560 | 0.471 |
| | 0.209 | 0.229 | | |
| 7 | 0.822 | 1.682 | 0.277 | 1.161 |
| | 0.451 | 1.319 | | |
| 8 | 0.515 | -0.60 | 0.562 | 52.80 |
| | 0.169 | 53.48 | | |
| 9 | 0.529 | 0.863 | 0.459 | 0.275 |
| | 0.274 | 0.479 | | |
| 10 | 0.618 | 1.161 | 0.123 | 1.122 |
| | 0.609 | 1.723 | | |
| 11 | 0.280 | 0.500 | 0.447 | 0.226 |
| | 0.546 | 0.805 | | |
| 12 | 0.687 | 2.635 | 0.534 | 2.266 |
| | 0.166 | 1.878 | | |
| 13 | 0.554 | 1.087 | 0.618 | 0.854 |
| | 0.100 | -0.06 | | |
| 14 | 0.377 | 0.535 | 0.574 | 0.331 |
| | 0.226 | 0.413 | | |
| 15 | 0.568 | 1.141 | 0.247 | 0.547 |
| | 0.491 | 1.023 | | |
| 16 | 0.419 | 0.791 | 0.364 | 0.096 |
| | 0.469 | 0.731 | | |
| 17 | 0.494 | 0.759 | 0.459 | 0.238 |
| | 0.289 | 0.470 | | |
| 18 | 0.517 | 1.059 | 0.364 | 0.361 |
| | 0.385 | 0.732 | | |
| 19 | 0.090 | 0.315 | 0.670 | 0.399 |
| | 0.424 | 0.805 | | |
| 20 | 0.339 | 0.537 | 0.598 | 0.432 |
| | 0.224 | 0.300 | | |

**Table 2**

N=100

| SAMPLE | $\tilde{\beta}$ | $\hat{\beta}$ | $\|\|\tilde{\beta}-\beta\|\|$ | $\|\|\hat{\beta}-\beta\|\|$ |
|---|---|---|---|---|
| 1 | 0.419 | 1.115 | 0.375 | 0.499 |
| | 0.451 | 0.978 | | |
| 2 | 0.484 | 1.102 | 0.301 | 0.420 |
| | 0.491 | 0.821 | | |
| 3 | 0.575 | 1.277 | 0.209 | 0.774 |
| | 0.532 | 1.215 | | |
| 4 | 0.382 | 0.756 | 0.442 | 0.067 |
| | 0.392 | 0.736 | | |
| 5 | 0.491 | 0.855 | 0.345 | 0.155 |
| | 0.425 | 0.697 | | |
| 6 | 0.168 | 0.273 | 0.678 | 0.526 |
| | 0.281 | 0.394 | | |
| 7 | 0.238 | 0.351 | 0.501 | 0.415 |
| | 0.505 | 0.923 | | |
| 8 | 0.478 | 0.932 | 0.369 | 0.268 |
| | 0.405 | 0.566 | | |
| 9 | 0.471 | 0.819 | 0.322 | 0.215 |
| | 0.474 | 0.879 | | |
| 10 | 0.386 | 0.813 | 0.439 | 0.126 |
| | 0.393 | 0.755 | | |
| 11 | 0.480 | 0.654 | 0.385 | 0.083 |
| | 0.384 | 0.631 | | |
| 12 | 0.365 | 0.708 | 0.470 | 0.659 |
| | 0.371 | 1.359 | | |
| 13 | 0.366 | 0.651 | 0.334 | 0.754 |
| | 0.678 | 1.452 | | |
| 14 | 0.408 | 0.574 | 0.429 | 0.142 |
| | 0.385 | 0.767 | | |
| 15 | 0.591 | 1.078 | 0.540 | 0.578 |
| | 0.171 | 0.263 | | |
| 16 | 0.345 | 0.449 | 0.443 | 0.251 |
| | 0.436 | 0.698 | | |
| 17 | 0.374 | 0.896 | 0.447 | 0.197 |
| | 0.394 | 0.685 | | |
| 18 | 0.282 | 0.496 | 0.462 | 0.235 |
| | 0.503 | 0.816 | | |
| 19 | 0.360 | 0.638 | 0.431 | 0.092 |
| | 0.436 | 0.632 | | |
| 20 | 0.292 | 0.470 | 0.494 | 0.242 |
| | 0.421 | 0.626 | | |

Table 3

N=200

| SAMPLE | $\tilde{\beta}$ | $\hat{\beta}$ | $\|\|\tilde{\beta} - \beta\|\|$ | $\|\|\hat{\beta} - \beta\|\|$ |
|---|---|---|---|---|
| 1 | 0.441 | 0.753 | 0.418 | 0.102 |
| | 0.372 | 0.612 | | |
| 2 | 0.526 | 0.875 | 0.294 | 0.272 |
| | 0.464 | 0.907 | | |
| 3 | 0.356 | 0.576 | 0.450 | 0.131 |
| | 0.411 | 0.741 | | |
| 4 | 0.502 | 0.757 | 0.375 | 0.079 |
| | 0.382 | 0.645 | | |
| 5 | 0.394 | 0.685 | 0.414 | 0.252 |
| | 0.421 | 0.951 | | |
| 6 | 0.428 | 0.837 | 0.341 | 0.202 |
| | 0.494 | 0.848 | | |
| 7 | 0.444 | 0.799 | 0.379 | 0.121 |
| | 0.420 | 0.768 | | |
| 8 | 0.536 | 0.999 | 0.239 | 0.361 |
| | 0.526 | 0.903 | | |
| 9 | 0.447 | 0.789 | 0.313 | 0.277 |
| | 0.516 | 0.962 | | |
| 10 | 0.435 | 0.822 | 0.373 | 0.122 |
| | 0.437 | 0.689 | | |
| 11 | 0.365 | 0.589 | 0.429 | 0.123 |
| | 0.432 | 0.755 | | |
| 12 | 0.412 | 0.709 | 0.362 | 0.162 |
| | 0.481 | 0.861 | | |
| 13 | 0.493 | 0.829 | 0.392 | 0.148 |
| | 0.367 | 0.627 | | |
| 14 | 0.467 | 0.804 | 0.347 | 0.330 |
| | 0.443 | 1.013 | | |
| 15 | 0.390 | 0.696 | 0.479 | 0.129 |
| | 0.335 | 0.572 | | |
| 16 | 0.397 | 0.834 | 0.344 | 0.277 |
| | 0.538 | 0.943 | | |
| 17 | 0.409 | 0.696 | 0.446 | 0.117 |
| | 0.361 | 0.583 | | |
| 18 | 0.338 | 0.553 | 0.445 | 0.195 |
| | 0.442 | 0.828 | | |
| 19 | 0.564 | 1.034 | 0.234 | 0.494 |
| | 0.509 | 1.065 | | |
| 20 | 0.387 | 0.774 | 0.434 | 0.104 |
| | 0.399 | 0.773 | | |

**Table 4**

N=500

| SAMPLE | $\tilde{\beta}$ | $\hat{\beta}$ | $\|\|\tilde{\beta}-\beta\|\|$ | $\|\|\hat{\beta}-\beta\|\|$ |
|---|---|---|---|---|
| 1 | 0.359 | 0.593 | 0.444 | 0.107 |
| | 0.414 | 0.704 | | |
| 2 | 0.437 | 0.664 | 0.403 | 0.078 |
| | 0.395 | 0.631 | | |
| 3 | 0.479 | 0.916 | 0.371 | 0.218 |
| | 0.402 | 0.730 | | |
| 4 | 0.450 | 0.745 | 0.386 | 0.050 |
| | 0.406 | 0.678 | | |
| 5 | 0.367 | 0.575 | 0.463 | 0.142 |
| | 0.379 | 0.632 | | |
| 6 | 0.527 | 0.914 | 0.349 | 0.221 |
| | 0.397 | 0.646 | | |
| 7 | 0.446 | 0.719 | 0.314 | 0.165 |
| | 0.515 | 0.864 | | |
| 8 | 0.420 | 0.736 | 0.348 | 0.143 |
| | 0.493 | 0.839 | | |
| 9 | 0.442 | 0.742 | 0.404 | 0.104 |
| | 0.389 | 0.605 | | |
| 10 | 0.504 | 0.897 | 0.374 | 0.214 |
| | 0.381 | 0.617 | | |
| 11 | 0.349 | 0.574 | 0.478 | 0.141 |
| | 0.376 | 0.636 | | |
| 12 | 0.297 | 0.526 | 0.495 | 0.181 |
| | 0.413 | 0.649 | | |
| 13 | 0.375 | 0.583 | 0.449 | 0.131 |
| | 0.389 | 0.641 | | |
| 14 | 0.379 | 0.665 | 0.406 | 0.046 |
| | 0.452 | 0.730 | | |
| 15 | 0.386 | 0.712 | 0.392 | 0.082 |
| | 0.465 | 0.781 | | |
| 16 | 0.433 | 0.752 | 0.396 | 0.052 |
| | 0.408 | 0.708 | | |
| 17 | 0.482 | 0.825 | 0.390 | 0.134 |
| | 0.376 | 0.653 | | |
| 18 | 0.393 | 0.596 | 0.370 | 0.173 |
| | 0.493 | 0.838 | | |
| 19 | 0.363 | 0.593 | 0.441 | 0.107 |
| | 0.415 | 0.700 | | |
| 20 | 0.440 | 0.708 | 0.422 | 0.155 |
| | 0.368 | 0.545 | | |

**Table 5**

N=1000

| SAMPLE | $\tilde{\beta}$ | $\hat{\beta}$ | $\|\|\tilde{\beta} - \beta\|\|$ | $\|\|\hat{\beta} - \beta\|\|$ |
|---|---|---|---|---|
| 1 | 0.407 | 0.637 | 0.401 | 0.063 |
| | 0.426 | 0.701 | | |
| 2 | 0.407 | 0.625 | 0.441 | 0.118 |
| | 0.371 | 0.609 | | |
| 3 | 0.404 | 0.711 | 0.394 | 0.012 |
| | 0.440 | 0.704 | | |
| 4 | 0.438 | 0.792 | 0.344 | 0.168 |
| | 0.477 | 0.841 | | |
| 5 | 0.424 | 0.718 | 0.359 | 0.092 |
| | 0.471 | 0.790 | | |
| 6 | 0.451 | 0.813 | 0.349 | 0.136 |
| | 0.456 | 0.776 | | |
| 7 | 0.453 | 0.796 | 0.342 | 0.119 |
| | 0.464 | 0.770 | | |
| 8 | 0.371 | 0.615 | 0.439 | 0.085 |
| | 0.409 | 0.694 | | |
| 9 | 0.352 | 0.560 | 0.413 | 0.178 |
| | 0.478 | 0.809 | | |
| 10 | 0.465 | 0.779 | 0.343 | 0.106 |
| | 0.451 | 0.771 | | |
| 11 | 0.453 | 0.753 | 0.376 | 0.060 |
| | 0.416 | 0.728 | | |
| 12 | 0.389 | 0.643 | 0.399 | 0.113 |
| | 0.450 | 0.798 | | |
| 13 | 0.446 | 0.758 | 0.355 | 0.080 |
| | 0.453 | 0.755 | | |
| 14 | 0.454 | 0.745 | 0.348 | 0.049 |
| | 0.455 | 0.719 | | |
| 15 | 0.424 | 0.723 | 0.387 | 0.023 |
| | 0.429 | 0.700 | | |
| 16 | 0.432 | 0.698 | 0.408 | 0.070 |
| | 0.393 | 0.630 | | |
| 17 | 0.443 | 0.739 | 0.355 | 0.046 |
| | 0.454 | 0.723 | | |
| 18 | 0.487 | 0.784 | 0.370 | 0.089 |
| | 0.397 | 0.670 | | |
| 19 | 0.368 | 0.596 | 0.460 | 0.149 |
| | 0.381 | 0.594 | | |
| 20 | 0.466 | 0.831 | 0.357 | 0.133 |
| | 0.430 | 0.722 | | |