SUBGROUP SIZE DESIGN AND SOME COMPARISONS OF Q($\overline{X}$) CHARTS WITH CLASSICAL $\overline{X}$ CHARTS

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In this work we give results that provide a guide in selecting sample size for either classical $\overline{X}$ charts or for the $Q(\overline{X})$ chart for the case when the process parameters are assumed known. Comparisons of the performance of the $Q(\overline{X})$ chart and the comparable $\overline{X}$ chart are given for the case when the parameters are not assumed known in advance. The results also support the assertion in Quesenberry (1991) that the Q-statistics for this case are approximetely independent standard normal statistics, even for very short runs.

Introduction

Quesenberry (1991), Q91, gave a class of Shewhart-type charts called Q-charts that can be used in a number of situations for charting to control a normal process mean when either the mean or variance are either known or unknown. If the mean and variance are not known, then the natural procedure is to use the "unknown parameters" or "short runs" charts given in that paper until sufficient data are available to estimate the parameters and switch to the "parameters known" case. Quesenberry (1992), Q92, gave results on the accuracy of classical 3-sigma charts with estimated control limits to help decide when this switch can be safely made. In the present paper we give further results to aid in the design and implementation of these Q-charts.

ARL, ATRL and SDRL in Chart Design, μ and σ Known

The Q-charts for the case when μ and σ are both known have performance properties the same as classical 3-sigma charts, so we will discuss this case in terms of those familiar charts. Suppose we take samples of size n from a normal $N(\mu, \sigma^2)$ process with known parameters μ and σ, and plot the sample means $\overline{X}_i$ on a classical 3-sigma chart with control limits

$$\begin{aligned} LCL &= \mu - 3\frac{\sigma}{\sqrt{n}} \\ UCL &= \mu + 3\frac{\sigma}{\sqrt{n}} \end{aligned} \tag{1}$$

Further, suppose that μ shifts to a new value $\mu + \delta\sigma$ at some point between samples. Then, if we consider a signal when a point falls outside the control limits, the probability of no signal, the operating characteristic, OC, function is given by

$$OC(n, \delta) = \Phi(3 - |\delta|\sqrt{n}) - \Phi(-3 - |\delta|\sqrt{n}) \tag{2}$$

for Φ the standard normal distribution function.

Now, if Y denotes the number of points plotted until the first signal, then Y is a geometric random variable and has mean, ARL, and standard deviation, SDRL, given by

$$\text{ARL}(n,\ \delta) = \frac{1}{1 - \text{OC}(n,\ \delta)},\quad \text{SDRL}(n,\ \delta) = \sqrt{[\text{ARL}(n,\ \delta)][\text{ARL}(n,\ \delta) - 1]} \tag{3}$$

For this case if ARL(n, δ) is very large we have ARL(n, δ) $\simeq$ SDRL(n, δ). See Q92 for a discussion of the dependence of the relationship in (3) upon the independence of the signaling events.

We can use the relations (2) and (3) to compute a table of values of ARL as given in Table 1. The ARL for the stable case, i.e. for $\delta = 0$, is 370 for all values of n. However, for $|\delta| > 0$ the ARL decreases as either n or $|\delta|$ increase. For small values of $|\delta|$ and n of the frequently used size of 5, the ARL is rather large and, on average, a large number of production units will be made before a signal is given. For example, for n = 5 and δ = 1 an average of 4.5 samples or a total of (4.5)(5) = 22.5 production units will be made before a signal. For practical purposes a quantity that is often of more interest than the average number of samples before a signal is the average number of production units made before a signal. That is, the average number of actual parts made after a mean shift before a signal is given. We call this quantity the *average total run length*, ATRL, and it is given by

Table 1: ARL Function for a 3-Sigma $\overline{X}$-Chart or Q($\overline{X}$)-Chart with Known Parameters

	$\lvert\delta\rvert$											
n	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
1	370	161.0	44.0	15.0	6.3	3.2	2.0	1.4	1.2	1.1	1.0	1.0
2	370	91.5	17.7	5.3	2.3	1.4	1.1	1.0	1.0	1.0	1.0	1.0
3	370	60.9	9.8	9.2	1.5	1.1	1.0	1.0	1.0	1.0	1.0	1.0
4	370	44.0	6.3	2.0	1.2	1.0	1.0	1.0	1.0	1.0	1.0	1.0
5	370	33.4	4.5	1.6	1.1	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	$\lvert\delta\rvert$											
n	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
6	370	165.7	46.1	15.9	6.7	3.4	2.1	1.5	1.2	1.1	1.0	1.0
8	370	134.0	32.4	10.4	4.3	2.3	1.5	1.2	1.1	1.0	1.0	1.0
10	370	111.7	24.2	7.4	3.1	1.8	1.3	1.1	1.0	1.0	1.0	1.0
12	370	95.0	18.8	5.6	2.4	1.5	1.1	1.0	1.0	1.0	1.0	1.0
14	370	82.2	15.1	4.4	2.0	1.3	1.1	1.0	1.0	1.0	1.0	1.0
16	370	71.9	12.4	3.6	1.7	1.2	1.0	1.0	1.0	1.0	1.0	1.0
18	370	63.6	10.4	3.1	1.5	1.1	1.0	1.0	1.0	1.0	1.0	1.0
20	370	56.8	8.9	2.7	1.4	1.1	1.0	1.0	1.0	1.0	1.0	1.0
22	370	51.0	7.7	2.3	1.3	1.0	1.0	1.0	1.0	1.0	1.0	1.0
24	370	46.1	6.7	2.1	1.2	1.0	1.0	1.0	1.0	1.0	1.0	1.0
26	370	41.9	5.9	1.9	1.2	1.0	1.0	1.0	1.0	1.0	1.0	1.0
28	370	38.3	5.3	1.8	1.1	1.0	1.0	1.0	1.0	1.0	1.0	1.0
30	370	34.2	4.8	1.6	1.1	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Table 2: Best Sample Size n and Minimum ATRL for a 3-Sigma $\overline{X}$-Chart or $Q(\overline{X})$-Chart with Known Parameters

$\|\delta\|$:	0	.1	.2	.3	.4	.5	.6	.7
ATRL:	(370.4)n	1762.1	440.5	195.8	110.1	70.5	48.9	36.0
n:	n	1108	227	123	69	44	31	23
$\|\delta\|$:	.8	.9	1.0	1.1	1.2	1.3	1.4	1.5
ATRL:	27.5	21.8	17.6	14.6	12.2	10.5	9.0	7.8
n:	17	14	11	9	8	7	6	5
$\|\delta\|$:	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3
ATRL:	6.9	6.1	5.5	4.9	4.4	4.1	3.7	3.3
n:	4	4	3	3	3	3	2	2
$\|\delta\|$:	2.4	2.5	2.6	2.7	2.8	2.9	3.0	3.1
ATRL:	3.1	2.8	2.7	2.5	2.4	2.2	2.0	1.9
n:	2	2	2	2	1	1	1	1

$$\text{ATRL} = n(\text{ARL}) \tag{4}$$

Now, ATRL has a property which makes it a particularly relevant quantity to consider in the context of choosing sample size n. While ARL is a strictly decreasing function of n for a fixed value of $|\delta|$ (>0), for this case the ATRL will take a minimum value at some integer n. We have computed these values of sample size n and the corresponding value of ATRL for a range of values of δ and these results are given in Table 2. A number of points can be observed by studying Table 2. Note that the customary sample size of n = 5 is actually ideal for detecting a shift of $|\delta| = 1.5$ standard deviations in the process mean and n = 4 is ideal for detecting a shift of 1.6 or 1.7. A sample of size n = 2 is best for detecting shifts in the interval (2.2, 2.7) standard deviations, and a sample of size 1 (an individual observations chart) is best to quickly detect shifts of 2.8 or more in the process mean. To detect smaller shifts the best sample size increases rapidly as $|\delta|$ decreases. The values of ATRL and best sample sizes in Table 2 are an envelope representing the limits of the best performance we can expect to achieve on a 3-sigma $\overline{X}$-chart that signals for a point outside the control limits. The major point to be observed here is that clearly we cannot expect to improve the performance of classical $\overline{X}$ charts by simply increasing the sample size.

Designing and Implementing Q-charts, μ and σ Unknown

We next consider designing Q-charts for an unknown process mean when the process variance is also unknown. The main issue is the choice of sample size n, when sample size is at our disposal as a design parameter. The individual measurements chart of Q91 is essentially the chart for batches of size n = 1. Now, the principle advantage of Q-charts for all cases is that for the stable case when the parameters are constant, i.e. $\delta = 0$, the plotted points are known to be distributed as independent standard normal random variables, therefore the expected pattern of points for a stable process is well understood, and so we can readily recognize anomalous patterns. As an aid in designing these "short runs" Q-charts -- i.e. charts for unknown parameters -- we shall consider the sensitivity of the charts to detect a shift in the process mean from μ to $\mu + \delta\sigma$ by having a point fall outside the control limits. For the stable case of $\delta = 0$, as the number of samples m increases the unknown parameters Q-charts will in all cases have plot points that converge to the values they would have if the parameters were known. Therefore Table 2 that relates δ, average total run length (ATRL) and the best sample size can serve as a guide to select n. From this we note that as a general principle we would choose n large to detect a small shift δ quickly, and to detect a large shift quickly we would choose small values of batch size n.

To give more guidance in selecting batch size n, we have performed a simulation study of the ability of charts with n = 1, 2, and 5 to detect shifts of sizes δ = 1, 2, 3, 4, 5, and 6. The results for samples of size n = 2 are given in Table 3. A shift of δ in the process mean was made after m samples of size n = 2, and then we checked to see if the Q-chart issues at least one signal by having a point outside control limits on at least one of the next 5 samples. This was repeated 10,000 times for each table entry. The proportion that would be expected if the parameters were known is given in the last column for m = ∞. Note that the accuracy of the simulated values can be assessed by using the value obtained as a binomial proportion p in the standard deviation formula

$$\text{SD(Table entry)} = \sqrt{\frac{p(1 - p)}{10000}}$$

For the δ = 0 row we should use the limiting value of p = 0.0134. Then for this row

$$\text{SD(Table entry)} = \sqrt{\frac{(0.0134)(0.9866)}{10000}} = 0.0011$$

This provides evidence in support of the statement in Q91 that the Q-statistics for batch data are approximately independent, even for the first few points. The first entry in the δ = 0 row of Table 3 is 0.0116, which is the most distant value from the limiting nominal value of 0.0134, and this is only (0.0134 − 0.0116)/ 0.0011 = 1.64 standard deviation units.

Table 4 gives comparable information for samples of size n = 5, except that in this table the signal rates given are on the next two samples. Thus the rates in both tables are for the next ten actual observations. The values of m in the tables are chosen so that -- except for the first column -- the entries are comparable for the two tables since they give signal rates on the next ten observations after the same total number of observations have been observed. For example, Table 3 for m = 5 and

Table 3: Proportion of Signals on Next 5 Samples After Shift, n = 2

$\|\delta\|$	Number of Samples before Shift, m								
	m:	2	5	10	15	25	50	100	∞
	2m:	4	10	20	30	50	100	200	
0		0.0116	0.0136	0.0147	0.0136	0.0136	0.0122	0.0139	0.0134
1		0.024	0.052	0.093	0.119	0.151	0.190	0.228	0.252
2		0.071	0.220	0.448	0.582	0.726	0.842	0.897	0.941
3		0.147	0.519	0.858	0.951	0.988	0.999	1.000	1.000
4		0.245	0.794	0.988	0.999	1.000	1.000	1.000	1.000
5		0.388	0.941	1.000	1.000	1.000	1.000	1.000	1.000
6		0.529	0.989	1.000	1.000	1.000	1.000	1.000	1.000

Table 4: Proportion of Signals on Next 2 Samples After Shift, n = 5

$\|\delta\|$	Number of Samples before Shift, m								
	m:	1	2	4	6	10	20	40	∞
	5m:	5	10	20	30	50	100	200	
0		0.0052	0.0060	0.0058	0.0058	0.0070	0.0057	0.0044	0.0054
1		0.047	0.095	0.166	0.210	0.266	0.324	0.349	0.395
2		0.287	0.572	0.821	0.906	0.959	0.980	0.990	0.995
3		0.706	0.952	0.998	1.000	1.000	1.000	1.000	1.000
4		0.943	0.999	1.000	1.000	1.000	1.000	1.000	1.000
5		0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000
6		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 5: Proportion of Signals on Next Ten Observations after Shift

$\|\delta\|$	Number of Observations Before Mean Shift								
	m:	5	10	20	30	50	100	200	∞
0		0.027	0.027	0.027	0.027	0.027	0.027	0.027	0.027
1		0.028	0.046	0.079	0.102	0.131	0.158	0.169	0.206
2		0.035	0.112	0.241	0.334	0.473	0.617	0.724	0.822
3		0.078	0.247	0.520	0.689	0.856	0.963	0.992	0.999
4		0.150	0.490	0.816	0.937	0.989	1.000	1.000	1.000
5		0.268	0.735	0.961	0.995	1.000	1.000	1.000	1.000
6		0.415	0.906	0.997	1.000	1.000	1.000	1.000	1.000

$\delta = 2$ gives an observed signal rate of 0.220, and Table 4 gives a comparable signal rate for m = 2 and $\delta = 2$ of 0.572.

Table 5 gives the observed signal rates from charting the Q(X) statistics for individual observations of Q91. For this case the distribution theory given in Q91 is exact for $\delta = 0$ and it was not necessary to simulate the constant rate of 0.027. Comparing these three tables shows that if we wish to have a low false signal probability and a high signal probability on the next ten observations after a mean shift then we should use the $Q(\overline{X})$-chart with n = 5. However, this chart has another weakness in comparison with the n = 2 or n = 1 charts. It has no ability to signal a shift on fewer than 5 observations after a shift, whereas the n = 2 chart can signal on the second observation after a shift, and the n = 1 chart can signal on the very next observation, and for some cases often will for large values of δ.

Therefore we make the following recommendations for designing a Q-chart. If concern is primarily to detect relatively small shifts of δ of perhaps less than 2 to 2.5 then we would use n = 5; for δ of 2.5 to perhaps 4 we could use n = 2 and for δ of 4 or more perhaps use n = 1.

There are, of course, many other points that will often be important in designing the charts, and many of these are the same as, or similar to, those for designing classical charts. For example, there is often much interest in detecting patterns of individual observations, and then we must use the n = 1 chart. Grouping individual observations data into samples has a masking effect on the behavior of individual observations.

Q-Charts *vs* the Classical $\overline{X}$-Chart

The $Q(\overline{X})$-chart for the case when the parameters are known is equivalent to the classical 3-sigma $\overline{X}$ chart made by plotting sample means on a chart with control limits given by (1). The probabilities of signals from the next ten measurements after a mean shift for either of these charts are given in the last columns of Tables 3, 4 and 5.

When parameters are unknown, the usual approach to making an $\overline{X}$ chart is as follows. A calibration data set of m samples of size n each are first observed, and estimates of μ and σ are computed from this data. These estimates of μ and σ are plugged into the control limits formulas of (1), and then the means of these or further samples are plotted on these charts and the estimated control limits are treated as though they are the actual correct control limits of (1). Frequently, estimates $\overline{\overline{X}}$ and $\overline{S}/c_4$ from these data are used to estimate μ and σ and the estimated control limits are then

$$\begin{aligned} \overline{UCL} &= \overline{\overline{X}} + \frac{3\,\overline{S}}{c_4\sqrt{n}} \\ \overline{LCL} &= \overline{\overline{X}} - \frac{3\,\overline{S}}{c_4\sqrt{n}} \end{aligned} \qquad (5)$$

Table 6: Sensitivities of $\overline{X}$ Charts Based on m Calibration Samples of Size n = 5

$\lvert\delta\rvert$	m:	1	2	4	6	10	20	30	40	∞
	5m:	5	10	20	30	50	100	150	200	
0		0.1383	0.0614	0.0294	0.0199	0.0114	0.0074	0.0076	0.0056	0.0054
1		0.468	0.444	0.421	0.423	0.417	0.395	0.400	0.394	0.395
2		0.879	0.935	0.966	0.978	0.985	0.991	0.992	0.992	0.995
3		0.993	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
4		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

It is of interest to consider data like that in Tables 3, 4, and 5 for this classical chart. Table 6 gives results of a simulation study for this chart, assuming a normal process, and that the process parameters μ and σ are constant for the period while the first m samples are being taken. For each entry in the table a set of m samples of size n = 5 each was generated from a N(0, 1) distribution, and these data used to compute the control limits of (5). Two additional samples were then generated from a N(0, δ) distribution and checked to see if at least one sample mean was outside the control limits. This procedure was repeated 10,000 times for each entry in Table 6 and the proportion of signals given are reported in the table. The results given in Tables 4 and 6 can be compared, however, it should be borne in mind that, as shown in Q92, the events of signals on points for the chart of Table 6 are dependent, and this creates difficulties in interpreting point patterns. The probabilities of false signals from Table 6 are larger than the nominal value of 0.0054 for m < 40 and much larger for m as small as 10. For $|\delta| > 0$, and especially for small values of $|\delta|$, the probability of detecting a shift is much larger for Table 6 than for Table 4. However, since the chart of Table 6 does not control the false alarm rate at the required value of 0.0054 these values are not really comparable.

In many applications the parameters μ and σ may shift while the calibration data set is being taken and it is of interest to compare the performance of the chart with control limits (5) with the corresponding Q-Chart for this unstable case. Of course, the parameters can shift in many different ways and we can consider only a few of the possibilities.

Suppose that when we take m samples of size n each that the first m_1 samples are from a distribution with mean μ_0 and standard deviation σ_0, and then the last $m - m_1$ samples are from a process distribution with mean $\mu_0 + \delta\sigma_0$ and standard deviation σ_0. Then the first m_1 sample means themselves have distribution mean μ_0 and the last $m - m_1$ sample means have distribution mean $\mu_0 + \delta\sigma_0$ and standard deviation σ_0. However, the statistic $\overline{\overline{X}}$ has mean $E(\overline{\overline{X}}) = \mu_0 + (m - m_1)\,\delta\sigma_0/m$, and standard deviation σ_0/mn. In particular, if m = 30 and m_1 = 15 so that the shift occurs after observed value 15, then $\overline{\overline{X}}$ is an unbiased estimate of $\mu_0 + \frac{\delta\sigma_0}{2}$. This means that the sample means $\overline{X}_1, \cdots$ obtained before the shift and those obtained after the shift should tend to fall on opposite sides of the chart center line. To illustrate this behavior we have generated $m_1 = 15$ samples of size $n = 5$ from a N(5, 1) distribution and then 15 more samples of size 5 from a N(5.75, 1) distribution. From these values we obtained $\overline{\overline{X}} = 5.37$ and $\hat{\sigma} = 1.02$. The $Q(\overline{X})$-chart of these data for the parameters unknown case is shown in Figure 1. The $Q(\overline{X})$-chart using these estimated parameter values is shown in Figure 2, and the point patterns and signals for this chart are exactly the same as those for the $\overline{X}$ chart with limits given by (5).

The $Q(\overline{X})$ chart for the unknown parameters case gives a clear indication of the upward shift of the mean on about the 16th sample. The 18th mean is above the upper 3-sigma limit and 13 of the last 15 points are above the center line. We also note that the phenomenon discussed by Quesenberry (1991) of the tendency of the chart to eventually settle back to indicate a stable process is also evident. The $Q(\overline{X})$ graph using the estimated values of μ and σ shows a similar general pattern of plotted points. However, the pattern of points has been shifted down on the graph because $\overline{\overline{X}}$ now estimates an average of the value of the process mean before (μ_0) and after ($\mu_0 + \delta\ \sigma_0$) the shift. As a result of this estimated value of the process mean, no points are outside control limits and none of the usual tests for patterns indicate a mean shift. It also should be kept in mind that each point of Figure 1 can be plotted as soon as the sample is observed and trends can be detected as soon as they occur, whereas Figure 2 must be plotted retrospectively after the 30 subgroups have been observed.

Of course, this example is only illustrative but it is representative of the behavior that can be expected when the mean shifts during the period while a calibration sample is being taken. If the variance should shift during the period while the calibration sample is being taken, then corresponding problems are caused with $\overline{X}$, S and R charts. In other words, if an assignable cause disrupts the process while a calibration sample is being taken, then the traditional approach of plotting charts retrospectively using estimates from the calibration sample can be ineffective in detecting its presence.

Figure 1: The Q($\overline{X}$) with Unknown Parameters

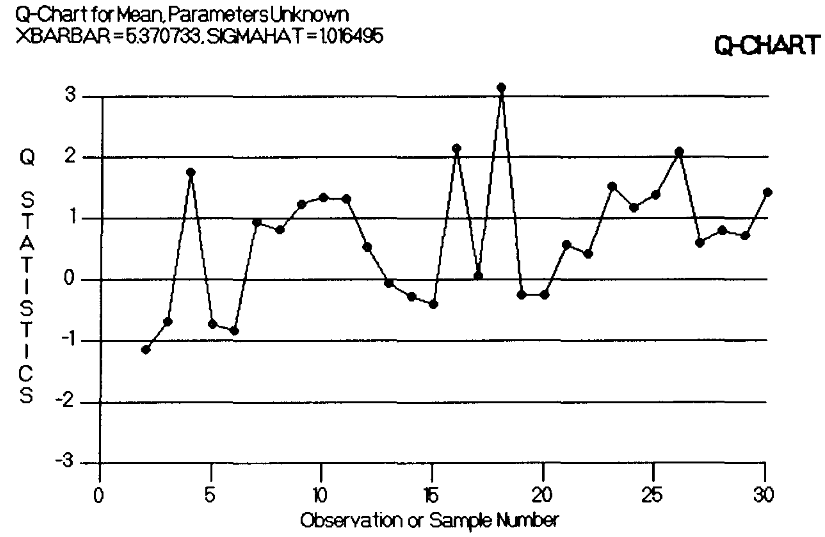
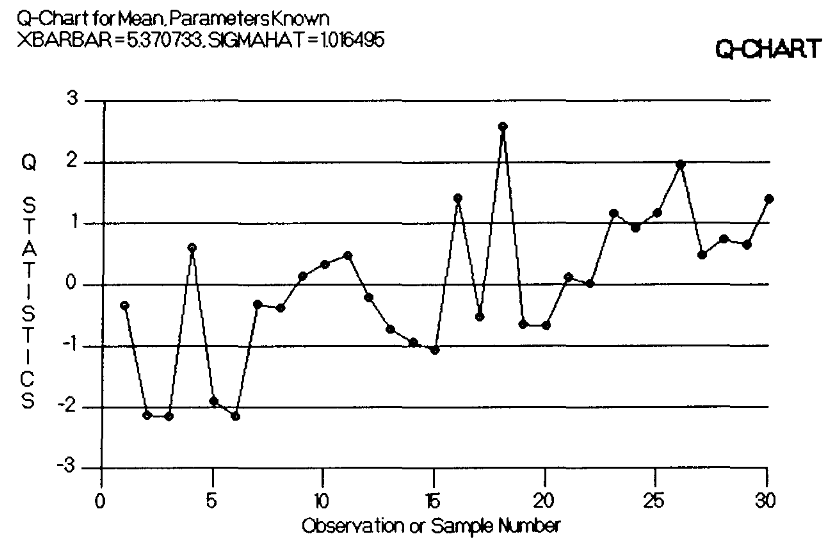


Figure 2: The the Known Parameters Q($\overline{X}$) Using $\overline{\overline{X}} = 5.37$ for μ and $S_p = 1.02$ for σ



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