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by

Gordon Simons

and

Yi-Ching Yao

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Asymptotic Sine Laws Arising from Alternating Random Permutations and Sequences

by Gordon Simons and Yi–Ching Yao

Universities of North Carolina at Chapel Hill and Colorado State University

Summary

In the last century, Désiré André obtained many remarkable properties of the numbers of alternating permutations, linking them to trigonometric functions among other things. By considering the probability that a random permutation is alternating and that a random sequence (from a uniform distribution) is alternating, and by conditioning on the first element of the sequence, his results are extended and illuminated. In particular, several "asymptotic sine laws" are obtained, some with exponential rates of convergence.

<u>1. Introduction.</u> A finite sequence $x_1,x_2,\dots,x_n$ of distinct real numbers is said to be *alternating* if either

(1.1) $$x_1 > x_2 < x_3 > \cdots < (>)\, x_n \quad \text{or} \quad x_1 < x_2 > x_3 < \cdots > (<)\, x_n.$$

A permutation π on $\{1,2,\dots,n\}$ is *alternating* if the sequence $\pi(1),\pi(2),\dots,\pi(n)$ is alternating. Denote the number of alternating permutations by $2M_n$; the inequality $\pi(1) > \pi(2)$ holds for exactly M_n of these. Some sample values are:

$$M_2 = 1,\ M_3 = 2,\ M_4 = 5,\ M_5 = 16,\ M_6 = 61,\ M_7 = 272,\ M_8 = 1385,\ M_9 = 7936.$$

Let $X_1,X_2,\dots,X_n$ be iid uniformly distributed random variables on $[0,1]$, let A_n denote the event that $X_1,X_2,\dots,X_n$ is alternating with $X_1 > X_2$, and set

$$P_n := P(A_n).$$

It is readily apparent that

$$P_n = \frac{M_n}{n\,!},$$

i.e., P_n is the probability that a random permutation π on $\{1,2,\dots,n\}$ is alternating with $\pi(1) > \pi(2)$.

A great deal is known about the sequence of values P_n, due in large measure to the extensive study of the numbers M_n by Désiré André (1879, 1881, 1883, 1894, 1895). We shall be content with a brief history, *described here in terms of the* P_n*'s.* In 1879, he established a remarkable link with the tangent and secant functions:

$$\tan x + \sec x = P_0 + P_1 x + P_2 x^2 + P_3 x^3 + \cdots \quad (P_0 \equiv P_1 \equiv 1). \tag{1.2}$$

The terms with odd powers of x generate tan x, the others sec x. (For this reason, the M_n's with odd index are sometimes called *tangent numbers*, and with even index, *secant numbers.*) This was supplemented in (1881) by many other trigonometric relationships such as

$$\cot x = x^{-1} - (2^2-1)^{-1} P_1 x - (2^4-1)^{-1} P_3 x^3 - \cdots$$

and

$$\sec^2 x = P_1 + 3 P_3 x^2 + 5 P_5 x^4 + \cdots .$$

Moreover, he described various expansions of P_n, for fixed $n \geq 1$, such as

$$\begin{aligned} P_n &= 2 \left(\tfrac{2}{\pi}\right)^{n+1} \left\{ \frac{1}{1^{n+1}} + \frac{(-1)^{n+1}}{3^{n+1}} + \frac{1}{5^{n+1}} + \frac{(-1)^{n+1}}{7^{n+1}} + \cdots \right\} \\ &= 2 \left(\tfrac{2}{\pi}\right)^{n+1} \sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^{n+1}} . \end{aligned} \tag{1.3}$$

(This shows that the numbers P_n are, for some reason, linked to the well known Riemann-zeta function.) In (1883), he concluded from (1.3) (or a similar result) that the ratio $P_n / \{2(2/\pi)^{n+1}\} \to 1$ as $n \to \infty$. In fact, it follows from (1.3) that the convergence is exponentially fast:

$$\left| \frac{P_n}{2 \left(\frac{2}{\pi}\right)^{n+1}} - 1 \right| \leq \frac{1}{3^n}, \quad n \geq 1. \tag{1.4}$$

To motivate the present note, we ask the question: How is the probability of A_n influenced once X_1 is observed? Clearly, A_n should be more likely when X_1 is large, and less likely when X_1 is small. A related question: How is the probability of A_n influenced after one is told the rank of X_1 (among $X_1, X_2, \ldots, X_n$)? Again, the probability of A_n should be positively related to the rank of X_1. Very precise answers to these questions are possible when n is large — answers which expose a pair of "asymptotic sine laws."

Let $p_n(u)$ denote the conditional probability of A_n given $X_1 = u$. Thus,

$$\int_0^1 p_n(w)\, dw = P_n = \frac{M_n}{n!}. \tag{1.5}$$

Starting with

$$p_2(u) = P(X_2 < u) = u \tag{1.6}$$

and the recursion

$$p_n(u) = \int_0^u p_{n-1}(1-w)\, dw, \tag{1.7}$$

one obtains

$$p_3(u) = u - \frac{u^2}{2},\ p_4(u) = \frac{u}{2} - \frac{u^3}{6},\ \ p_5(u) = \frac{u}{3} - \frac{u^3}{6} + \frac{u^4}{24},\ \ p_6(u) = \frac{5u}{24} - \frac{u^3}{12} + \frac{u^5}{120},$$

etc. While no clear pattern is revealed, these functions are closely related, as the following analog of (1.3) shows:

$$p_n(u) = 2\,(2/\pi)^n \sum_{k=-\infty}^{\infty} \frac{\sin\{(4k+1)\frac{\pi}{2}u\}}{(4k+1)^n},\ 0 \le u \le 1,\ n \ge 2. \tag{1.8}$$

Of course, André's formula (1.3) is implied by (1.5) and (1.8). In fact, the occurrence of the numbers $\frac{2}{\pi\,(4k+1)}$ in (1.3) is "explained" by (1.8): These are just the eigenvalues of the kernel function $K(u,v) := 1_{\{u+v > 1\}}$ arising in the derivation of (1.8).

What makes (1.8) particularly interesting, and useful, is that it immediately exposes an excellent inequality, our first "(asymptotic) sine law":

$$\left| \frac{p_n(u)}{2\,(\frac{2}{\pi})^n} - \sin\{\tfrac{\pi}{2}u\} \right| \le \frac{1}{3^{n-1}},\quad 0 \le u \le 1,\ n \ge 2. \tag{1.9}$$

Observe that the ratio

$$f_n(u) := p_n(u)/P_n \tag{1.10}$$

is just the conditional density of X_1 given the event A_n occurs. Combining (1.4), (1.9) and (1.10), we see that this first sine law can be expressed as:

(1.11) $$f_n(u) \to \frac{\pi}{2}\sin\frac{\pi}{2}u \quad \text{as } n\to\infty, \text{ uniformly in } u\ (0 \le u \le 1).$$

The convergence is exponentially fast.

Now, let p_{nr} denote the conditional probability of A_n given the rank of X_1 is r (among $X_1,X_2,\ldots,X_n$) ($1 \le r \le n$). Is there an analog of (1.8) for the p_{nr}'s ? While we can't answer this question, we can obtain, by another approach, a somewhat weaker analog of (1.9), our second sine law:

(1.12) $$\frac{p_{nr}}{2\ (\frac{2}{\pi})^n} = \sin\{\tfrac{\pi}{2}u_{nr}\} + O(\tfrac{1}{n}) \quad \text{as } n\to\infty, \text{ uniformly in } r\ (r = 1,\ldots,n),$$

whenever u_{nr} is in the interval $[\frac{r-1}{n}, \frac{r}{n}]$.

Observe that the ratio

(1.13) $$f_{nr} := \frac{p_{nr}}{nP_n}$$

is the conditional probability that X_1 has rank r (among $X_1,X_2,\ldots,X_n$) given the event A_n occurs. Combining (1.4), (1.12) and (1.13), we see that this second sine law can be expressed as

(1.14) $$f_{nr} = \frac{\pi}{2n}\sin\{\tfrac{\pi}{2}\tfrac{2r-1}{2n}\} + O(\tfrac{1}{n^2}) \quad \text{as } n\to\infty, \text{ uniformly in } r\ (r = 1,\ldots,n).$$

Notice that (1.9) and (1.12) together indicate there is very little difference between being told "X_1 has rank r", and being told, for any u in the interval $[\frac{r-1}{n}, \frac{r}{n}]$, that "$X_1 = u$".

It is tempting to speculate whether important roles exist for the $p_n(u)$'s and p_{nr}'s that mirror the impressive role played by the P_n's in formula (1.2). Not that we know of. Nevertheless, Aubrey Kempner (1933) and R.C. Entringer (1966) have evidenced the *mathematical usefulness* of the integers

$$m_n(r) := (n-1)!\cdot p_{nr}.$$

(It follows from the definition of p_{nr} that $m_n(r)$ is just the number of alternating permutations π on $\{1,2,\ldots,n\}$ for which $\pi(1) > \pi(2)$ and $\pi(1) = r$.) Kempner suggested the linear recursion

(1.15) $$m_n(r) = \sum_{k=1}^{r-1} m_{n-1}(n-k)$$

and the obvious formula

(1.16) $$M_n = \sum_{r=1}^{n} m_n(r)$$

as a simple way of evaluating the integers M_n; he needed these for his qualitative description of a certain class of polynomial functions. Entringer formally proved (1.15) and used it to establish an identity, involving a bivariate generating function, which, in turn, he used to link Euler and Bernoulli numbers to the M_n's.

Section 2 establishes (1.8) and, hence, the sine law shown in (1.9). A generalization of (1.11) for the ratio $P(A_n \mid X_{k_i} = u_i, i = 1,\dots,r)/P(A_n)$ is also obtained, where the k_i's (depending on n) are far apart from one another.

Section 3 establishes the sine law described in (1.12), and correction terms of orders $1/n$ and $1/n^2$ are found which improve the accuracy in (1.12) by two orders of magnitude.

We conclude this introduction with the observation that the distributional assumption imposed on the (iid) X_i's is merely a convenience; any *continuous* distribution yields similar but (according to a standard argument) *equivalent* results.

2. The continuous case. In this section, we shall (a) discuss the mathematics leading up to equation (1.8), and (b) discuss a generalization under which attention is focused upon the ratio $P(A_n \mid X_{k_i} = u_i, i = 1,\dots,r)/P(A_n)$ for large n with the k_i's far apart. It will be recalled that equation (1.8) provides the basis for our first "sine law" described in (1.9). Included, is a quick review of relevant notation, with some amplifications.

Let $X_1, X_2, \dots$ be iid uniform random variables on $[0,1]$. Denote by A_n the event that

$$X_1 > X_2 < X_3 > X_4 < \cdots < (>)\, X_n,$$

where the sense of the last inequality is either $<$ or $>$ according as n is odd or even, and denote the probability $P(A_n)$ by P_n, $n \geq 2$. Also, let B_n be the event that

$$X_1 < X_2 > X_3 < X_4 > \cdots > (<)\, X_n,$$

with the sense of every inequality reversed. For $n \geq 2$ and $0 \leq u \leq 1$, define the functions

$$p_n(u) := P(u > X_2 < X_3 > X_4 < \cdots < (>)\, X_n) \quad (= P(A_n | X_1 = u))$$

and

$$q_n(u) := P(u < X_2 > X_3 < X_4 > \cdots > (<)\, X_n) \quad (= P(B_n | X_1 = u)),$$

which are the only *continuous* versions of the indicated conditional probabilities, *the versions we shall use* throughout the paper.

By considering the transform $Y_i = 1 - X_i$, $i = 1,\ldots,n$, it is easy to see that

$$q_n(u) = p_n(1-u),$$

so that

$$p_n(u) = \int_0^u q_{n-1}(w)\, dw = \int_0^u p_{n-1}(1-w)\, dw,$$

which establishes (1.7).

<u>THEOREM 1.</u> *The functions* $p_n(u)$, $n \geq 2$, *defined on* $[0,1]$, *are expandable as shown in* (1.8).

We remark that the infinite sum in (1.8) clearly converges absolutely and uniformly in u when $n \geq 2$, and hence pointwise. The reader should interpret future infinite sums from this perspective *unless otherwise stated.* In contrast, it is also possible to interpret (1.8) in an L^2 sense:

$$\int_0^1 \left\{ p_n(u) - 2\,(2/\pi)^n \sum_{k=-T}^{T} \frac{\sin\{(4k+1)\frac{\pi}{2}u\}}{(4k+1)^n} \right\}^2 du \to 0 \text{ as } T \to \infty. \tag{2.1}$$

In fact, (1.8) is valid in this sense for $n = 1$, with $p_1(u) := 1$.

<u>PROOF OF THEOREM 1.</u> Providing (1.8) holds for $n = 2$, a simple induction step completes the proof:

$$\begin{aligned} p_{n+1}(u) &= \int_0^u p_n(1-w)\, dw \\ &= \int_0^u \left\{ 2\,(2/\pi)^n \sum_{k=-\infty}^{\infty} \frac{\sin\{(4k+1)\frac{\pi}{2}(1-w)\}}{(4k+1)^n} \right\} dw \\ &= 2\,(2/\pi)^{n+1} \sum_{k=-\infty}^{\infty} \frac{\sin\{(4k+1)\frac{\pi}{2}u\}}{(4k+1)^{n+1}}. \end{aligned}$$

Establishing (1.8) for $n = 2$ seems to require an L^2 approach. To motivate this, observe that

$$p_{n+1}(u) = \int_{1-u}^{1} p_n(w)\,dw = \int_0^1 K(u,w)\,p_n(w)\,dw,$$

where K is a self adjoint kernel given by

$$K(u,w) = 1_{\{u+w > 1\}}, \quad 0 \le u,w \le 1.$$

This kernel has eigenvalues

$$\lambda_k = \frac{2}{\pi\,(4k+1)} \quad (k = 0,\pm 1,\pm 2,\ldots),$$

with corresponding eigenfunctions

$$\phi_k(u) = \sin\{\tfrac{\pi}{2}(4k+1)\,u\}.$$

I.e.,

$$\int_0^1 K(u,w)\sin\{\tfrac{\pi}{2}(4k+1)\,w\}\,dw = \left\{\frac{2}{\pi\,(4k+1)}\right\}\sin\{\tfrac{\pi}{2}(4k+1)\,u\},$$

for all integers k. Further, observe that (1.8) can be expressed as

$$p_n(u) = 2\sum_{k=-\infty}^{\infty} (\lambda_k)^n\,\phi_k(u).$$

It is easy to show that

$$K(u,v) = 2\sum_{k=-\infty}^{\infty} \lambda_k\,\phi_k(u)\,\phi_k(v)$$

as elements of $L^2([0,1]\times[0,1])$, and that K has no eigenvalues equal to zero. Consequently, by a result of Hochstadt (1973, p. 61), the collection of functions

$$\left\{\sqrt{2}\,\sin\{\tfrac{\pi}{2}(4k+1)\,u\} : k = 0,\pm 1,\pm 2,\ldots\right\}$$

forms a *complete* orthonormal set for $L^2[0,1]$. Thus from (1.6),

$$\int_0^1 p_2(u)\left\{\sqrt{2}\,\sin\{\tfrac{\pi}{2}(4k+1)\,u\}\right\}du = \sqrt{2}\int_0^1 u\,\sin\{\tfrac{\pi}{2}(4k+1)\,u\}\,du$$

$$= \frac{\sqrt{2}\;2^2}{\pi^2(4k+1)^2},$$

so that

$$p_2(u) = \sum_{k=-\infty}^{\infty} \frac{\sqrt{2}\;2^2}{\pi^2(4k+1)^2}\left\{\sqrt{2}\,\sin\{\tfrac{\pi}{2}(4k+1)\,u\}\right\},$$

which, for the current L^2 context, has the interpretation shown in (2.1). This can easily be extended to the pointwise interpretation envisioned in the statement of the theorem. □

The variant of the first sine law shown in (1.11) can be expressed more precisely as

$$\max_{0 \le u \le 1} |f_n(u) - \tfrac{\pi}{2} \sin \tfrac{\pi}{2} u| = |f_n(1) - \tfrac{\pi}{2}| = \frac{2\pi}{3^{n+1}} + O(\frac{1}{5^n}) \text{ as } n \to \infty. \tag{2.2}$$

The latter equality follows directly from (1.3), (1.8) and (1.10) while the first can be shown analytically for $n \ge 2$. Briefly, this analytical argument uses

$$f_n(u) - \tfrac{\pi}{2} \sin \tfrac{\pi}{2} u = \frac{2(\frac{2}{\pi})^n}{P_n} \left\{ \frac{\sin \{\frac{-3\pi}{2} u\}}{(-3)^n} - \frac{\sin \{\frac{\pi}{2} u\}}{(-3)^{n+1}} + R_n(u) \right\}, \tag{2.3}$$

where

$$R_n(u) := \sum_{k \ne 0,-1} \left\{ \frac{\sin \{(4k+1) \frac{\pi}{2} u\}}{(4k+1)^n} - \frac{\sin \{\frac{\pi}{2} u\}}{(4k+1)^{n+1}} \right\}.$$

The absolute value of the right side of (2.3), *without the remainder term* $R_n(u)$, is maximized at $u = 1$. This remainder term is negligible enough to rule out the smaller u as a maximizing argument, and its derivative is negligible enough to rule out the larger $u < 1$. The details of this argument are omitted.

Let

$$p_n(u,v) := P(A_n | X_1 = u, X_n = v) \tag{2.4}$$

(convenient notation that should not be confused with $p_n(u)$), to be interpreted here as

$$P(u > X_2 < X_3 > \cdots < (>) X_{n-1} > (<) v). \tag{2.5}$$

The sense of the last inequality is either < or > according as n is odd or even. Note that this is the only continuous version of (2.4) for $n \ge 3$. For example,

$$p_3(u,v) = P(u > X_2 < v) = \min(u,v). \tag{2.6}$$

While a continuous version for $n = 2$ does not exist, the version

$$p_2(u,v) = 1_{\{u > v\}} \tag{2.7}$$

seems a reasonable interpretation of (2.4).

Note that

$$P(B_n | X_1 = u,\ X_n = v) = p_n(1-u,1-v).$$

The analog of (1.7) is

$$p_n(u,v) = \int_0^u p_{n-1}(1-w,1-v)\,dw = \int_{1-u}^1 p_{n-1}(w,1-v)\,dw = \int_0^1 K(u,w)\,p_{n-1}(w,1-v)\,dw.$$

Thus it follows that

$$(2.8) \qquad p_{n+2}(u,v) = \int_0^1 K^*(u,w)\,p_n(w,v)\,dw,\ \ n \geq 2,$$

where

$$(2.9) \qquad K^*(u,w) = \int_0^1 K(u,s)\,K(s,w)\,ds = \min(u,w),\ \ 0 \leq u,w \leq 1.$$

The initial cases, n = 2 and n = 3, are given in (2.7) and (2.6), respectively. With equation (2.8), the unique continuous versions of the conditional probabilities in (2.4) can be found recursively for $n \geq 4$, versions that are in full agreement with (2.5). For example,

$$p_4(u,v) = u - uv - \tfrac{1}{2}(u-v)^2\,1_{\{u > v\}},\quad p_5(u,v) = uv - \tfrac{1}{2}uv\max(u,v) - \tfrac{1}{6}\{\min(u,v)\}^3,$$

etc.

<u>THEOREM 2.</u> *For odd* $n \geq 3$,

$$(2.10) \qquad p_n(u,v) = 2\,(2/\pi)^{n-1} \sum_{k=-\infty}^{\infty} \frac{\sin\{(4k+1)\frac{\pi}{2}u\}\ \sin\{(4k+1)\frac{\pi}{2}v\}}{(4k+1)^{n-1}},\ 0 \leq u,v \leq 1.$$

For even $n \geq 4$,

$$(2.11) \qquad p_n(u,v) = 2\,(2/\pi)^{n-1} \sum_{k=-\infty}^{\infty} \frac{\sin\{(4k+1)\frac{\pi}{2}u\}\ \cos\{(4k+1)\frac{\pi}{2}v\}}{(4k+1)^{n-1}},\ 0 \leq u,v \leq 1.$$

For $n = 2$*, the same formula holds pointwise (with respect to symmetric partial sums) except on the line* $u = v$*, and, for all* v *in* [0,1]*, as an* L^2 *limit in the variable* u*. I.e., for each* v *in the unit interval,*

$$\int_0^1 \left\{ p_2(u,v) - (4/\pi) \sum_{k=-T}^{T} \frac{\sin\{(4k+1)\frac{\pi}{2}u\}\ \cos\{(4k+1)\frac{\pi}{2}v\}}{4k+1} \right\}^2 du \to 0 \text{ as } T \to \infty.$$

PROOF. The same induction argument used in Theorem 1 is applicable here, based on (2.8) rather than (1.7). The initial cases, n = 2 and n = 3, need to be argued separately.

The case n = 3 is straightforward: According to (2.6) and (2.9), $p_3(u,v) = K^*(u,v)$. Thus,

$$\int_0^1 p_3(u,v) \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ u\}\right\} du = \int_0^1 K^*(u,v) \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ u\}\right\} du$$

$$= (\lambda_k)^2 \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ v\}\right\} = \left\{\frac{2}{\pi\ (4k+1)}\right\}^2 \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ v\}\right\},$$

so that

$$p_3(u,v) = \left\{\frac{2}{\pi\ (4k+1)}\right\}^2 \sum_{k=-\infty}^{\infty} \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ u\}\right\} \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ v\}\right\}$$

$$= 2\ (2/\pi)^2 \sum_{k=-\infty}^{\infty} \frac{\sin\{(4k+1)\frac{\pi}{2}u\}\ \ \sin\{(4k+1)\frac{\pi}{2}v\}}{(4k+1)^2}\ .$$

This L^2 limit easily assumes the asserted pointwise interpretation as well.

According to (2.7), $p_2(u,v) = 1_{\{u > v\}} = K(u,1-v)$. Thus

$$\int_0^1 p_2(u,v) \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ u\}\right\} du = \int_v^1 K(u,1-v) \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ u\}\right\} du$$

$$= \lambda_k \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ (1-v)\}\right\} = \left\{\frac{2}{\pi\ (4k+1)}\right\} \left\{\sqrt{2}\ \cos\{\tfrac{\pi}{2}(4k+1)\ v\}\right\},$$

so that

$$p_2(u,v) = \sum_{k=-\infty}^{\infty} \left\{\frac{2}{\pi\ (4k+1)}\right\} \left\{\sqrt{2}\ \sin\{\tfrac{\pi}{2}(4k+1)\ u\}\right\} \left\{\sqrt{2}\ \cos\{\tfrac{\pi}{2}(4k+1)\ v\}\right\}$$

$$= 2\ (2/\pi) \sum_{k=-\infty}^{\infty} \frac{\sin\{(4k+1)\frac{\pi}{2}u\}\ \ \cos\{(4k+1)\frac{\pi}{2}v\}}{(4k+1)},$$

the asserted form for $p_2(u,v)$ *as an* L^2 *limit.*

It can be shown for n = 2 that the right side of (2.11) equals 1 if u > v; equals 0 if u < v; equals $\frac{1}{2}$ if $0 < u = v < 1$; equals 0 if u = v = 0,1. The details will be omitted. □

Theorem 2 yields excellent approximations, especially for large n:

$$\left| \frac{p_n(u,v)}{2\ (\frac{2}{\pi})^{n-1}} - \sin\{\tfrac{\pi}{2}u\}\cos\{\tfrac{\pi}{2}v\} \right| \le \frac{1}{3^{n-2}}, \quad 0 \le u,v \le 1,\ \text{even } n \ge 2, \tag{2.12}$$

and

$$\left| \frac{p_n(u,v)}{2\,(\frac{2}{\pi})^{n-1}} - \sin\{\tfrac{\pi}{2}u\}\sin\{\tfrac{\pi}{2}v\} \right| \le \frac{1}{3^{n-2}}, \quad 0 \le u,v \le 1, \text{ odd } n \ge 3. \tag{2.13}$$

We shall now derive a sine law for the conditional density of X_1 at u given A_n and $X_n = v$. A bicontinuous version of this density is described in the next lemma for $n \ge 4$; at most one such version is possible, and it is necessarily regular.

<u>LEMMA 1.</u> *For odd* $n \ge 5$, *the function*

$$f_n(u|v) := \begin{cases} \dfrac{p_n(u,v)}{\int_0^1 p_n(w,v)\,dw} & \text{for } 0 < v \le 1 \\ f_{n-2}(u) & \text{for } v = 0 \end{cases}, \quad 0 \le u \le 1, \tag{2.14}$$

and for even $n \ge 4$, *the function*

$$f_n(u|v) := \begin{cases} \dfrac{p_n(u,v)}{\int_0^1 p_n(w,v)\,dw} & \text{for } 0 \le v < 1 \\ f_{n-2}(u) & \text{for } v = 1 \end{cases}, \quad 0 \le u \le 1, \tag{2.15}$$

is a regular version of the conditional density of X_1 *given* A_n *and* $X_n = v$. *Both are bicontinuous in the closed unit square.*

It is easily seen that the integrals appearing in (2.14) and (2.15) can be expressed in terms of $p_n(\cdot)$:

$$\int_0^1 p_n(w,v)\,dw = \begin{cases} p_n(v) & \text{for odd } n, \\ p_n(1-v) & \text{for even } n. \end{cases} \tag{2.16}$$

<u>PROOF OF LEMMA 1.</u> Observe that the upper and lower parts on the right sides of (2.14) and (2.15) are density functions in u on the unit interval. Moreover, the upper parts are just special cases of Bayes theorem (since X_1 and X_n are independent and the unconditional density of X_1 is identically one on the unit interval). The asserted regularity follows from standard theorems.

Now, these upper parts are bicontinuous, *where defined*, since $p_n(u,v)$ is bicontinuous. But they assume the indeterminate form $\frac{0}{0}$ when $v = 0$ and $v = 1$, respectively. Bicontinuity can be extended to the entire closed unit square by replacing these upper parts by the conditional density $f_{n-2}(u)$ (referred to in (1.10) and (1.11)), which is just the uniform limit of $f_n(u|v)$ as v approaches 0 and 1, respectively. This claim follows from (1.8) and (2.10) and (2.11); it depends on the observation that $p_{n-2}(u)$ is the (one-sided) partial derivative of $p_n(u,v)$ with respect to v at v = 0 and v = 1, respectively. □

The form shown in (2.14) describes a *regular* version for n = 3 if we set $f_1(u) \equiv 1$. But a *bicontinuous* version does not exist: When n = 3, the upper part on the right side of (2.14) assumes the form $\min(u,v)/(v - v^2/2)$ $(0 < v \leq 1)$, and this misbehaves as (u,v) approaches the origin.

THEOREM 3.

$$| f_n(u|v) - \tfrac{\pi}{2}\sin\{\tfrac{\pi}{2}u\} | \leq \frac{1}{3^{n-4}}, \quad n \geq 4. \tag{2.17}$$

uniformly for (u,v) *within the closed unit square.*

PROOF. We shall argue this for odd n by using (2.10). The proof for even n requires (2.11) and a similar argument. From (2.10), we obtain

$$\frac{p_n(u,v)}{2\ (\frac{2}{\pi})^{n-1}} = \sin\{\tfrac{\pi}{2}u\}\sin\{\tfrac{\pi}{2}v\} + I_n(u,v) \tag{2.18}$$

and

$$\frac{\int_0^1 p_n(w,v)\ dw}{2\ (\frac{2}{\pi})^n} = \sin\{\tfrac{\pi}{2}v\} + J_n(v) \tag{2.19}$$

where

$$I_n(u,v) := \sum_{k \neq 0} \frac{\sin\{(4k+1)\frac{\pi}{2}u\}\ \ \sin\{(4k+1)\frac{\pi}{2}v\}}{(4k+1)^{n-1}}$$

and

$$J_n(v) := \sum_{k \neq 0} \frac{\sin\{(4k+1)\frac{\pi}{2}v\}}{(4k+1)^n},$$

where the indicated sums are over all nonzero integers k. It is easily seen for integers r that

$$|\sin\{r\tfrac{\pi}{2}v\}| \le |r| \sin\{\tfrac{\pi}{2}v\}, \quad 0 \le v \le 1.$$

Thus

(2.20) $$|I_n(u,v)| \le \sin\{\tfrac{\pi}{2}v\} \sum_{k \ne 0} \frac{1}{(4k+1)^{n-2}}$$

and

(2.21) $$|J_n(v)| \le \sin\{\tfrac{\pi}{2}v\} \sum_{k \ne 0} \frac{1}{(4k+1)^{n-1}}.$$

Since

$$\frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \cdots < \frac{\frac{\pi^2}{8} - 1}{3^{n-2}},$$

the desired conclusion follows from (the upper form in) (2.14), (2.18), (2.19), (2.20) and (2.21). (Because of the bicontinuity described in Lemma 1, it is enough to restrict attentions to $v > 0$, as the upper form in (2.14) does.) □

An improvement in (2.17) akin to (2.2) seems feasible, but will not be pursued here.

For integers $s \ge 1$ and $t \ge s+2$, let

$$p_{st}(u,v) := 2\,(2/\pi)^{(t-s)} \sum_{k=-\infty}^{\infty} \frac{\xi_s((4k+1)\,\frac{\pi}{2}\,u)\;\xi_t((4k+1)\,\frac{\pi}{2}\,v)}{(4k+1)^{(t-s)}}, \; 0 \le u,v \le 1,$$

where

$$\xi_i(w) := \begin{cases} \sin w & \text{for odd integers } i \\ \cos w & \text{for even integers } i \end{cases}.$$

This generalizes $p_n(u,v)$ (see Theorem 2), which now can be written as $p_{1n}(u,v)$.

Now let $1 = s_0 < s_1 < \cdots < s_r = n$ with $s_i \ge s_{i-1} + 2$, and consider the conditional probability

$$p_{s_0 s_1 \ldots s_r}(u_0,u_1,\ldots,u_r) := P(A_n \mid X_{s_i} = u_i,\ 0 \le i \le r).$$

Then Theorem 2 has the following generalization: For all $u_0,u_1,\ldots,u_r$,

$$p_{s_0 s_1 \ldots s_r}(u_0,u_1,\ldots,u_r) = \prod_{i=1}^{r} p_{s_{i-1},s_i}(u_{i-1},u_i).$$

This leads to analogues of (2.12) and (2.13): For all $u_0,u_1,\ldots,u_r$,

$$\left|\frac{p_{s_0 s_1 \ldots s_r}(u_0,u_1,\ldots,u_r)}{2^r\,(\frac{2}{\pi})^{n-1}} - \xi_1(\tfrac{\pi}{2}u_0)\,\xi_n(\tfrac{\pi}{2}u_r)\prod_{i=1}^{r-1}\left\{\xi_{s_i}(\tfrac{\pi}{2}u_i)\right\}^2\right|$$

$$\leq \left\{1 + \frac{\frac{\pi^2}{8} - 1}{3^{\Delta_n} - 2}\right\}^r - 1,$$

where

$$\Delta_n := \min\{s_i - s_{i-1} : 1 \leq i \leq r\}.$$

While this bound is not best possible, it has the advantage of being simple. Clearly, even when r increases with n, the bound can be kept small as long as Δ_n is sufficiently large.

<u>3. The discrete case.</u> In this section, we shall (a) derive the sine law described in (1.12), and (b) discuss second and third order terms. Included, is a quick review of relevant notation.

The iid uniformly distributed random variables $X_1, X_2, \ldots, X_n$ provide a natural and convenient means of describing a random permutation π on $\{1,2,\ldots,n\}$:

$$\pi(i) := \text{the rank of } X_i \text{ among } X_1,\ldots,X_n \quad (1 \leq i \leq n).$$

In particular, the event A_n, that $X_1,\ldots,X_n$ is alternating with $X_1 > X_2$, can be described in terms of ranks as the event that $\pi(1),\ldots,\pi(n)$ is alternating with $\pi(1) > \pi(2)$. Then

$M_n :=$ number of these random permutations that are alternating with $\pi(1) > \pi(2)$,

$m_n(r) :=$ number of these that are alternating with $\pi(1) > \pi(2)$ and $\pi(1) = r$,

and

$$p_{nr} := P(A_n \mid \pi(1) = r) = \frac{m_n(r)}{(n-1)!}.$$

Further,

$$M_n = \sum_{r=1}^{n} m_n(r),$$

so that

(3.1) $$P_n := P(A_n) = \frac{M_n}{n!} = \frac{1}{n} \sum_{r=1}^{n} p_{nr}\,.$$

Finally, it is easily seen that the $m_n(r)$'s are linked together through the recursion

$$m_2(1) = 0,\ m_2(2) = 1;\ m_n(r) = \sum_{k=1}^{r-1} m_{n-1}(n-k),\ 1 \le r \le n,\ n \ge 3,$$

so that

(3.2) $$p_{n+1,r} = \frac{1}{n} \sum_{k=1}^{r-1} p_{n,n+1-k} = P_n - \frac{1}{n} \sum_{k=r}^{n} p_{n,n+1-k},\ 1 \le r \le n+1.$$

Here, and below, summations over empty index sets (when $r = 1$) are defined as zero.

Our initial objective is to establish the (asymptotic) sine law described in (1.12) (under a slightly weaker assumption):

THEOREM 4.

(3.3) $$\frac{p_{nr}}{2\,(\frac{2}{\pi})^n} = \sin\{\tfrac{\pi}{2} u_{nr}\} + O(\tfrac{1}{n}) \quad \textit{as } n \to \infty,\ \textit{uniformly in } r\ (r = 1,\dots,n),$$

whenever

$$\max\{|u_{nr} - \tfrac{2r-1}{2n}| : r = 1,\dots,n\} = O(\tfrac{1}{n}) \quad \textit{as } n \to \infty.$$

PROOF. Since

$$|\sin\{\tfrac{\pi}{2} u_{nr}\} - \sin\{\tfrac{\pi}{2} \tfrac{2r-1}{2n}\}| \le \tfrac{\pi}{2}\, |u_{nr} - \tfrac{2r-1}{2n}| = O(\tfrac{1}{n}),$$

it is sufficient to prove (3.3) for the special case $u_{nr} = \frac{2r-1}{2n}$ $(r = 1,\dots,n)$. To this end, let

$$H_n = \max\{|h_{nr}| : r = 1,\dots,n\},$$

where

(3.4) $$h_{nr} := \frac{p_{nr}}{2\,(\frac{2}{\pi})^n} - \sin\{\tfrac{\pi}{2} \tfrac{2r-1}{2n}\}.$$

Lemma 2 below clearly implies that H_n is of order $1/n$, completing this proof. □

Lemma 2. *There exists a constant* $C > 0$, *such that for all* $n \ge 2$,

(3.5) $$H_{n+1} \le \tfrac{\pi}{4} H_n + \frac{C}{n}\,.$$

Proof. Separate arguments will show $|h_{n+1,r}| \le \frac{\pi}{4} H_n + \frac{C}{n}$ for $r-1 \le \frac{n}{2}$, and for $r-1 > \frac{n}{2}$.

For $r-1 \leq \frac{n}{2}$, (3.4) and the first equality in (3.2) yield

$$(\tfrac{2}{\pi})(h_{n+1,r} + \sin\{\tfrac{\pi}{2}\tfrac{2r-1}{2(n+1)}\}) = \frac{p_{n+1,r}}{2\,(\frac{2}{\pi})^n} = \frac{1}{n}\sum_{k=1}^{r-1} \frac{p_{n,n+1-k}}{2(\frac{2}{\pi})^n}$$
$$= \frac{1}{n}\sum_{k=1}^{r-1} h_{n,n+1-k} + \frac{\cos\{\frac{\pi}{2}\frac{n+1-r}{n}\}}{2n\,\sin\frac{\pi}{4n}}.$$

Hence,

$$|h_{n+1,r}| \leq \frac{\pi(r-1)}{2n} H_n + \left|\frac{\sin\{\frac{\pi}{2}\frac{r-1}{n}\}}{\frac{4n}{\pi}\,\sin\frac{\pi}{4n}} - \sin\{\tfrac{\pi}{2}\tfrac{2r-1}{2(n+1)}\}\right| \leq \frac{\pi}{4} H_n + \frac{C}{n},$$

for $r-1 \leq \frac{n}{2}$ and some $C > 0$ (independent of n and r).

For $r-1 > \frac{n}{2}$, (3.4) and the second equality in (3.2) yield

$$(\tfrac{2}{\pi})(h_{n+1,r} + \sin\{\tfrac{\pi}{2}\tfrac{2r-1}{2(n+1)}\}) = \frac{p_{n+1,r}}{2\,(\frac{2}{\pi})^n} = \frac{P_n}{2\,(\frac{2}{\pi})^n} - \frac{1}{n}\sum_{k=r}^{n} \frac{p_{n,n+1-k}}{2(\frac{2}{\pi})^n}$$
$$= \frac{P_n}{2\,(\frac{2}{\pi})^n} - \frac{1}{n}\sum_{k=r}^{n} h_{n,n+1-k} - \frac{1-\cos\{\frac{\pi}{2}\frac{n+1-r}{n}\}}{2n\,\sin\frac{\pi}{4n}}.$$

Hence,

$$|h_{n+1,r}| \leq \frac{\pi(n+1-r)}{2n} H_n + \left|\frac{\sin\{\frac{\pi}{2}\frac{r-1}{n}\}}{\frac{4n}{\pi}\,\sin\frac{\pi}{4n}} - \sin\{\tfrac{\pi}{2}\tfrac{2r-1}{2(n+1)}\}\right| + \left|\frac{P_n}{2\,(\frac{2}{\pi})^{n+1}} - 1\right| + \left|1 - \frac{\frac{\pi}{4n}}{\sin\frac{\pi}{4n}}\right|$$
$$\leq \frac{\pi}{4} H_n + \frac{C}{n}$$

for $r-1 > \frac{n}{2}$ and some $C > 0$. Note that the latter two absolute values are of orders 3^{-n} (see (1.4)) and $1/n^2$, respectively, and thus of smaller orders than $1/n$. □

It is possible to improve upon the accuracy shown in (3.3). An examination of the proof of Lemma 2 reveals that

$$\text{(3.6)} \qquad \frac{p_{nr}}{2\,(\frac{2}{\pi})^n} = s_{nr}^{(t)} + O(\tfrac{1}{n^t}) \quad \text{as } n \to \infty, \text{ uniformly in r } (r = 1,\dots,n),$$

for some integer $t \geq 1$ and a set of approximates $\{s_{nr}^{(t)} : r = 1,2,\dots,n;\ n \geq 2\}$, if

$$\text{(3.7)} \qquad s_{n+1,r}^{(t)} = \frac{\pi}{2n}\sum_{k=1}^{r-1} s_{n,n+1-k}^{(t)} + O(\tfrac{1}{n^t}) \quad \text{as } n \to \infty, \text{ uniformly in r, } r-1 \leq \tfrac{n}{2},$$

and

$$(3.8)\qquad s^{(t)}_{n+1,r} = 1 - \frac{\pi}{2n}\sum_{k=r}^{n} s^{(t)}_{n,n+1-k} + O(\frac{1}{n^t}) \quad \text{as } n \to \infty, \text{ uniformly in r, } r-1 > \frac{n}{2}.$$

Conversely, (3.6) (together with (1.4), (3.1), (3.2)) readily implies (3.7), (3.8), and also

$$(3.9)\qquad \frac{\pi}{2n}\sum_{k=1}^{n} s^{(t)}_{n,k} = 1 + O(\frac{1}{n^t}) \text{ as } n \to \infty.$$

Further, (3.7), (3.8) and (3.9) jointly imply

$$(3.10)\qquad s^{(t)}_{n+1,r} = \frac{\pi}{2n}\sum_{k=1}^{r-1} s^{(t)}_{n,n+1-k} + O(\frac{1}{n^t}) \quad \text{as } n \to \infty, \text{ uniformly in r } (r = 1,\dots,n).$$

To summarize:

Lemma 3. *For each fixed integer* $t \geq 1$, *the following are equivalent:*

(a) (3.6),

(b) (3.7) *and* (3.8),

(c) (3.9) *and* (3.10).

Now suppose (3.6) holds for a set of approximates $\{s^{(t)}_{nr}: r = 1,2,..,n;\ n \geq 2\}$ and some fixed t, so that (3.9) and (3.10) are valid statements. The objective is to find improved (easily computed) approximates $\{s^{(t+1)}_{nr}: r = 1,2,..,n;\ n \geq 2\}$ which satisfy

$$(3.11)\qquad \frac{p_{nr}}{2\ (\frac{2}{\pi})^n} = s^{(t+1)}_{nr} + O(\frac{1}{n^{t+1}}) \quad \text{as } n \to \infty, \text{ uniformly in r } (r = 1,\dots,n).$$

To this end, let

$$(3.12)\qquad \alpha^{(t)}_{nr} := n^t \left\{ \frac{p_{nr}}{2\ (\frac{2}{\pi})^n} - s^{(t)}_{nr} \right\} \quad (r = 1,2,\dots,n;\ n \geq 2),$$

$$(3.13)\qquad \beta^{(t)}_{nr} := n^t \left\{ \frac{\pi}{2n}\sum_{k=1}^{r-1} s^{(t)}_{n,n+1-k} - s^{(t)}_{n+1,r} \right\} \quad (r = 1,2,\dots,n+1;\ n \geq 2),$$

and

$$(3.14)\qquad \gamma^{(t)}_{n} := n^t \left\{ 1 - \frac{\pi}{2n}\sum_{k=1}^{n} s^{(t)}_{n,k} \right\} \quad (n \geq 2).$$

Then (3.9) and (3.10) (because of (1.4) and (3.2)) are equivalent, respectively, to

$$\frac{\pi}{2n}\sum_{k=1}^{n} \alpha_{n,k}^{(t)} = \gamma_n^{(t)} + O(n^t\, 3^{-n}) \text{ as } n \to \infty \tag{3.15}$$

and

$$\left\{\frac{n}{n+1}\right\}^t \alpha_{n+1,r}^{(t)} - \frac{\pi}{2n}\sum_{k=1}^{r-1} \alpha_{n,n+1-k}^{(t)} = \beta_{nr}^{(t)}. \tag{3.16}$$

Now suppose there exist a C^∞ function $B^{(t)}$ on [0,1] and a constant $C^{(t)}$ such that, as $n \to \infty$,

$$\beta_{nr}^{(t)} = B^{(t)}\left(\tfrac{r}{n+1}\right) + O\left(\tfrac{1}{n}\right) \quad \text{uniformly in } r\ (1 \le r \le n+1), \tag{3.17}$$

and

$$\gamma_n^{(t)} = C^{(t)} + O\left(\tfrac{1}{n}\right). \tag{3.18}$$

Lemma 4 below shows that there exists a C^∞ function $A^{(t)}$ on [0,1] satisfying

$$A^{(t)}(x) - \frac{\pi}{2}\int_{1-x}^{1} A^{(t)}(w)\, dw = B^{(t)}(x),\ 0 \le x \le 1, \tag{3.19}$$

and

$$\frac{\pi}{2}\int_0^1 A^{(t)}(w)\, dw = C^{(t)}. \tag{3.20}$$

By (3.15)–(3.20), it follows that, as $n \to \infty$,

$$\frac{\pi}{2n}\sum_{k=1}^{n} \left[\alpha_{n,k}^{(t)} - A^{(t)}\left(\tfrac{k}{n}\right)\right] = O\left(\tfrac{1}{n}\right) \tag{3.21}$$

and

$$\left[\alpha_{n+1,r}^{(t)} - A^{(t)}\left(\tfrac{r}{n+1}\right)\right] - \frac{\pi}{2n}\sum_{k=1}^{r-1} \left[\alpha_{n,n+1-k}^{(t)} - A^{(t)}\left(\tfrac{n+1-k}{n}\right)\right] = O\left(\tfrac{1}{n}\right), \tag{3.22}$$

Let

$$\bar{\alpha}_{n,r} := \alpha_{n,r}^{(t)} - A^{(t)}\left(\tfrac{r}{n}\right).$$

From (3.21) and (3.22), it readily follows that

$$|\bar{\alpha}_{n+1,r}| \leq \frac{\pi}{4}\max\{|\bar{\alpha}_{n,k}|: 1 \leq k \leq n\} + \frac{C'}{n}$$

for some constant C'. Hence,

$$\max\{|\bar{\alpha}_{n,k}|: 1 \leq k \leq n\} = O(\tfrac{1}{n}),$$

and

$$s_{nr}^{(t+1)} := s_{nr}^{(t)} + \frac{A^{(t)}(\frac{r}{n})}{n^t}$$

meets the requirements of (3.11).

Lemma 4. *Assume* (3.17) *and* (3.18). *Then there exists a unique* C^∞ *function* $A^{(t)}$ *satisfying* (3.19) *and* (3.20).

Proof. For ease of notation, we shall drop the superscript (t). Equation (3.19) is equivalent to

$$A(x) - \frac{\pi}{2}\int_0^1 K(x,y)\, A(y)\, dy = B(x)$$

where K(x,y) is the self adjoint kernel defined in the proof of Theorem 1 of Section 2. If

$$\int_0^1 B(x)\sin\{\tfrac{\pi}{2}x\}\, dx = 0, \tag{3.23}$$

then as an element of $L^2[0,1]$,

$$A(x) = \sum_{k=-\infty}^{\infty} C_k\sqrt{2}\sin\{\tfrac{\pi}{2}(4k+1)\,x\},$$

where

$$C_k = (1 - \tfrac{1}{4k+1})^{-1}\int_0^1 B(x)\sqrt{2}\sin\{\tfrac{\pi}{2}(4k+1)\,x\}\, dx, \;\; k \neq 0,$$

and

$$C_0 = \frac{C}{\sqrt{2}} - \sum_{k\neq 0}\frac{C_k}{4k+1}$$

by (3.20). Clearly, as an element of $L^2[0,1]$,

$$\bar{A}(x) := B(x) - \frac{\pi}{2}\int_{1-x}^1 A(w)\, dw$$

also satisfies (3.19) and (3.20), since $\bar{A} = A$ in $L^2[0,1]$. The fact that $\bar{A}$ is continuous implies $\bar{A}$ satisfies (3.19) *pointwise*, which, in turn, implies (via induction) that $\bar{A}$ is C^∞. The uniqueness of a C^∞ solution follows from the uniqueness of an L^2 solution to (3.19) and (3.20).

It remains to verify (3.23). Let

$$\mu := \int_0^1 B(x) \sin\{\tfrac{\pi}{2} x\}\, dx.$$

Then by (3.17),

$$b_n := \frac{1}{n} \sum_{k=1}^{n} \beta_{n,k} \sin\{\tfrac{\pi}{2}\tfrac{k}{n}\} \longrightarrow \mu \text{ as } n \to \infty. \tag{3.24}$$

Since $\alpha_{n,r} = O(1)$ (uniformly in r as $n \to \infty$), we have

$$a_n := \frac{1}{n} \sum_{k=1}^{n} \alpha_{n,k} \sin\{\tfrac{\pi}{2}\tfrac{k}{n}\} = O(1) \quad \text{as } n \to \infty. \tag{3.25}$$

By (3.16),

$$b_n = b_{1n} - b_{2n}$$

where

$$b_{1n} := \left\{\frac{n}{n+1}\right\}^t \frac{1}{n} \sum_{r=1}^{n} \alpha_{n+1,r} \sin\{\tfrac{\pi}{2}\tfrac{r}{n}\}$$

and

$$b_{2n} := \frac{\pi}{2n}\frac{1}{n} \sum_{r=1}^{n} \sum_{k=1}^{r-1} \alpha_{n,n+1-k} \sin\{\tfrac{\pi}{2}\tfrac{r}{n}\} = \frac{1}{n} \sum_{k=1}^{n} \alpha_{n,k} \sin\{\tfrac{\pi}{2}\tfrac{k}{n}\} + O(\tfrac{1}{n}).$$

Thus $b_{1n} = a_{n+1} + O(\frac{1}{n})$ and $b_{2n} = a_n + O(\frac{1}{n})$, so that

$$b_n = a_{n+1} - a_n + O(\tfrac{1}{n}) \text{ as } n \to \infty.$$

By (3.24), if $\mu \neq 0$, then a_n will be unbounded, contradicting (3.25). Thus $\mu = 0$, completing the proof. □

We are now ready to improve the first order approximation

$$\frac{p_{nr}}{2\ (\frac{2}{\pi})^n} = s_{nr}^{(1)} + O(\tfrac{1}{n}) \ \text{ as } n \to \infty,$$

for

$$s_{nr}^{(1)} := \sin\{\tfrac{\pi}{2}\tfrac{r}{n}\}$$

(corresponding to $u_{nr} = \frac{r}{n}$ in Theorem 4). By (3.13) and (3.14) with $t = 1$, we get

$$\beta_{nr}^{(1)} = B^{(1)}(\tfrac{r}{n+1}) + O(\tfrac{1}{n}) \ \text{(uniformly in } r,\ 1 \le r \le n+1) \ \text{ and } \ \gamma_n^{(1)} = C^{(1)} + O(\tfrac{1}{n}),$$

where

$$B^{(1)}(x) = \tfrac{\pi}{4}\{1 + (2x-3)\cos\tfrac{\pi}{2}x\} \ \text{ and } \ C^{(1)} = -\tfrac{\pi}{4}.$$

Solving (3.19) and (3.20) yields

$$A^{(1)}(x) = \tfrac{\pi^2}{8}(x - x^2)\sin\tfrac{\pi}{2}x - \tfrac{\pi}{2}(1-x)\cos\tfrac{\pi}{2}x\,. \tag{3.26}$$

Note that (3.19) and (3.20) can be reduced to the second order differential equation

$$A^{(t)\,''}(x) + \tfrac{\pi^2}{4}A^{(t)}(x) = B^{(t)\,''}(x) - \tfrac{\pi}{2}B^{(t)\,'}(1-x)$$

with initial conditions

$$A^{(t)}(0) = B^{(t)}(0) \ \text{ and } \ A^{(t)\,'}(0) = B^{(t)\,'}(0) + \tfrac{\pi}{2}B^{(t)}(1) + \tfrac{\pi}{2}C^{(t)}.$$

To conclude, we have shown that

$$\frac{p_{nr}}{2\ (\frac{2}{\pi})^n} = \sin\{\tfrac{\pi}{2}\tfrac{r}{n}\} + \tfrac{1}{n}A^{(1)}(\tfrac{r}{n}) + O(\tfrac{1}{n^2}) \ \text{ as } n \to \infty,$$

where $A^{(1)}$ is given in (3.26).

We can obtain the next order term by repeating the process. Letting $t = 2$ and inserting

$$s_{nr}^{(2)} := \sin\{\tfrac{\pi}{2}\tfrac{r}{n}\} + \tfrac{1}{n}A^{(1)}(\tfrac{r}{n})$$

into (3.13) and (3.14), we obtain

$$\beta_{nr}^{(2)} = B^{(2)}(\tfrac{r}{n+1}) + O(\tfrac{1}{n}) \ \text{(uniformly in } r,\ 1 \le r \le n+1) \ \text{ and } \ \gamma_n^{(2)} = C^{(2)} + O(\tfrac{1}{n}),$$

where

$$B^{(2)}(x) = (-\frac{\pi^2}{2}x^2 + \frac{7\pi^2}{8}x - \frac{13\pi^2}{48})\sin\frac{\pi}{2}x$$
$$+ (-\frac{\pi^3}{16}x^3 + \frac{5\pi^3}{32}x^2 + (\frac{\pi}{2} - \frac{3\pi^3}{32})x - \frac{\pi}{2})\cos\frac{\pi}{2}x,$$

and

$$C^{(2)} = -\frac{5\pi^2}{48}.$$

Solving (3.19) and (3.20) yields

$$A^{(2)}(x) = (\frac{\pi^4}{128}x^4 - \frac{\pi^4}{64}x^3 + (\frac{\pi^4}{128} - \frac{5\pi^2}{8})x^2 + \frac{7\pi^2}{8}x - \frac{\pi^2}{4})\sin\frac{\pi}{2}x$$
$$+ (-\frac{7\pi^3}{48}x^3 + \frac{\pi^3}{4}x^2 + (\frac{\pi}{2} - \frac{5\pi^3}{48})x - \frac{\pi}{2})\cos\frac{\pi}{2}x.$$

Thus

$$\frac{p_{nr}}{2\ (\frac{2}{\pi})^n} = \sin\{\frac{\pi}{2}\frac{r}{n}\} + \frac{1}{n}A^{(1)}(\frac{r}{n}) + \frac{1}{n^2}A^{(2)}(\frac{r}{n}) + O(\frac{1}{n^3}) \text{ as } n \to \infty .$$

Since the computational burden in finding $B^{(2)}$, $C^{(2)}$ and, in turn, $A^{(2)}$ is substantial, we resorted to the use of a symbolic processing computer package (Macsyma).

We do not know whether there is a closed-form representation of $A^{(t)}$ for all t.

Table 1 below shows the accuracy of the three approximations of $\frac{p_{nr}}{2\ (\frac{2}{\pi})^n}$ by

$$\sin\{\frac{\pi}{2}\frac{r}{n}\}, \quad \sin\{\frac{\pi}{2}\frac{r}{n}\} + \frac{1}{n}A^{(1)}(\frac{r}{n}), \quad \text{and} \quad \sin\{\frac{\pi}{2}\frac{r}{n}\} + \frac{1}{n}A^{(1)}(\frac{r}{n}) + \frac{1}{n^2}A^{(2)}(\frac{r}{n}),$$

for sample values of $n \le 100$. In this table, Δ_{ni} denotes the maximum with respect to r, $1 \le r \le n$, of the absolute error in the i–th approximation, i = 1,2,3. The quality of the second and especially the third approximations is apparent, even when n is fairly small.

It seems a bit surprising that the accuracy improves with i at *every* value of n, even when n = 2. We have no reason to believe this monotonicity continues indefinitely with increasing orders of approximation, for fixed values of n.

TABLE 1

n	Δ_{n1}	Δ_{n2}	Δ_{n3}
2	0.707	0.53847	0.4197399
3	0.500	0.24339	0.0926487
4	0.383	0.13271	0.0857672
5	0.309	0.08219	0.0311274
10	0.156	0.02108	0.0036226
15	0.105	0.00901	0.0009768
20	0.078	0.00502	0.0004002
30	0.052	0.00220	0.0001151
40	0.039	0.00123	0.0000479
50	0.031	0.00078	0.0000243
60	0.026	0.00054	0.0000140
70	0.022	0.00040	0.0000088
80	0.020	0.00030	0.0000059
90	0.017	0.00024	0.0000041
100	0.016	0.00019	0.0000030

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