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Huimei Liu and Roger L. Berger

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BY HUIMEI LIU AND ROGER L. BERGER

North Carolina State University

Let X have a multivariate, p-dimensional normal distribution ($p \geq 2$) with unknown mean μ and known, nonsingular covariance Σ. Consider testing H_0: $b_i'\mu \leq 0$, for some $i = 1,\ldots,k$, versus H_1: $b_i'\mu > 0$, for all $i = 1,\ldots,k$, where $b_1,\ldots,b_k$ ($k \geq 2$) are known vectors that define the hypotheses. We construct a test that has the same size as the likelihood ratio test (LRT) and is uniformly more powerful than the LRT. The proposed test is an intersection-union test. Other authors have presented uniformly more powerful tests under restrictions on the covariance matrix and on the hypothesis being tested. Our new test is uniformly more powerful than the LRT for all known nonsingular covariance matrices and all hypotheses. So our results show that, in a very general class of problems, the LRT can be uniformly dominated.

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1. Introduction. Let $X = (X_1,\dots,X_p)'$ $(p \geq 2)$ be a p-variate normal random variable with mean vector $\mu = (\mu_1,\dots,\mu_p)'$ and known, nonsingular covariance matrix Σ. We consider the problem of testing

$$H_0: b_i'\mu \leq 0 \text{ for some } i = 1, \dots, k$$

(1.1) versus

$$H_1: b_i'\mu > 0 \text{ for all } i = 1, \dots, k.$$

Here $b_1,\dots,b_k$ $(k \geq 2)$ are specified p-dimensional vectors that define the hypotheses. Berger (1989) gives several examples of hypotheses that can be expressed in this way. We assume H_1 is nonempty so the testing problem is meaningful. (We use the symbol H_1 to denote the set of μ vectors specified by the hypothesis, as well as the statement of the hypothesis.) We also assume that the set $\{b_1,\dots,b_k\}$ has no redundant vectors in it. That is, there is no b_j such that $\{\mu: b_i\mu > 0, i = 1,\dots,k\} = \{\mu: b_i\mu > 0, i = 1,\dots,k, \ i \neq j\}$. Sasabuchi (1980) discusses conditions that are equivalent to our two assumptions.

In this paper, for any testing problem of the form (1.1), we propose a new test that has the same size as the size-α likelihood ratio test (LRT) and is uniformly more powerful than the LRT. First we consider hypotheses that have only two linear restrictions ($k = 2$). Two new tests, ϕ_o and ϕ_a, are proposed for the cases $b_1'\Sigma b_2 \geq 0$ and $b_1'\Sigma b_2 < 0$, respectively. In both cases, the rejection region of the new test is like Berger's (1989) in that it contains the rejection region of the LRT and an additional set, but the size of the new test is still α. So the new test is uniformly more powerful than the LRT. Berger (1989) proposed a more powerful test for the $b_1'\Sigma b_2 \leq 0$ case. The test ϕ_a we propose is different than Berger's test, and, in some cases, appears to be more powerful. Then, recognizing that for $k > 2$ H_1 can be written as the intersection of sets each defined by two inequalities, we use the intersection-union method to combine tests of the form ϕ_o and ϕ_a to obtain a test, ϕ_g, that is uniformly more powerful for the general problem (1.1).

The initial work on testing problems where both null and alternative hypotheses are determined by k linear inequalities was by Sasabuchi (1980). Sasabuchi (1980) treats the problem where the null hypothesis corresponds to the boundary of a convex polyhedral cone determined by linear inequalities and the alternative corresponds to its interior. His problem is to test

$$H_{0S}: b_i'\mu \geq 0 \text{ for all } i = 1, \dots, k \text{ where equality holds for at least one value of } i$$

(1.2) versus

$$H_{1S}: b_i'\mu > 0 \text{ for all } i = 1, 2, \dots, k.$$

Sasabuchi (1980) showed that the size-α likelihood ratio test (LRT) of problem (1.2) is the test that rejects H_{0S} if

$$Z_i = \frac{b_i'X}{(b_1'\Sigma b_2)^{1/2}} \geq z_\alpha, \text{ for all } i = 1, \dots, k,$$

where z_α is the upper 100α percentile of the standard normal distribution. Berger (1989) shows that, although $H_{0S} \subset H_0$ and H_0 is a much bigger set than H_{0S}, the size-α LRT in problem (1.1) is the same as Sasabuchi's (1980). The LRT has some optimal properties. Lehmann (1952), Cohen et al. (1983) and Sierra-Cavazos (1992) prove under various conditions that the LRT is uniformly most powerful among all monotone, size-α tests. Cohen et al. (1983) also show that, in a bivariate problem, the LRT is admissible. But the LRT is a biased test. Berger (1989) points out that the power will be approximately α^p when μ is close to $\mathbf{0}$ in the sign testing problem. Lehmann (1952) showed that in some problems of this type, no unbiased, nonrandomized test exists. Iwasa (1991) also points out the LRT is d-admissible but not α-admissible in a bivariate problem. The α-admissibility would guarantee the nonexistence of a uniformly more powerful test of size α, but the d-admissibility does not. So it is possible that we can find a nonmonotone test which is uniformly more powerful than the LRT, and several researchers have worked on finding such tests.

Gutmann (1987) constructs two tests, when $X_1,\ldots,X_k$ are independent, that are uniformly more powerful than the uniformly most powerful monotone test in the sign testing problem. Nomakuchi and Sakata (1987) also give a uniformly more powerful test in the bivariate normal case, which is a special case of Sasabuchi's (1980) problem. Berger (1989) gives a class of tests which are more powerful than the LRT if $b_1'\Sigma b_2 \le 0$. If X is a normal random vector, then Gutmann's (1987) problem is a special case of Berger's (1989), and Berger's test is more powerful than Gutmann's test. Iwasa (1991) generalized the Nomakuchi – Sakata test to an exponential family. In the same paper, he also generalized Berger's test to an exponential family in the sign testing problem when $k = 2$. Shirley (1992) proposes a test that is more powerful than Gutmann's when $k = 3$.

To simplify computation, we consider the transformed version of the original problem that is similar to the one used by Sasabuchi (1980) and Berger

(1989). Let Γ be a $p \times p$ nonsigular matrix such that $\Gamma\Sigma\Gamma' = \mathrm{I}_p$, the $p \times p$ identity matrix. So $\Gamma^{-1}(\Gamma^{-1})' = \Sigma$. Make the transformation $\boldsymbol{Y} = \Gamma \boldsymbol{X}$. Then $\boldsymbol{Y} \sim \mathrm{N}_p(\boldsymbol{\theta},\mathrm{I}_p)$, where $\boldsymbol{\theta} = \Gamma\boldsymbol{\mu}$. Let $\|\boldsymbol{a}\| = (\boldsymbol{a}'\boldsymbol{a})^{1/2}$ denote the norm of a vector. Define $\boldsymbol{h}_i = \boldsymbol{b}_i'\Gamma^{-1}/\|\boldsymbol{b}_i'\Gamma^{-1}\|$. Then $\boldsymbol{b}_i'\boldsymbol{\mu} = \boldsymbol{h}_i'\boldsymbol{\theta}\|\boldsymbol{b}_i'\Gamma^{-1}\|$. Therefore, problem (1.1) is equivalent to observing $\boldsymbol{Y}$ and testing

$$H_0: \boldsymbol{h}_i'\boldsymbol{\theta} \leq 0 \quad \text{for some } i=1,\ldots,k$$

(1.3) versus

$$H_1: \boldsymbol{h}_i'\boldsymbol{\theta} > 0 \quad \text{for all } i=1,\ldots,k.$$

We will use $\boldsymbol{Y}$, $\boldsymbol{h}_i$ and $\boldsymbol{\theta}$ through the rest of the paper. In terms of these variables, the size-α LRT of (1.1) or (1.3) is the test that rejects H_0 if $\boldsymbol{h}_i'\boldsymbol{Y} \geq z_\alpha$, for all $i=1,\ldots,k$.

In Section 2 we propose a new test, ϕ_o, for the case $k=2$ and $\boldsymbol{b}_1'\Sigma\boldsymbol{b}_2 \geq 0$. To our knowledge, this is the first more powerful test described for these problems except that Berger considered the $\boldsymbol{b}_1'\Sigma\boldsymbol{b}_2 = 0$ case. We compare the power of ϕ_o and the LRT in an example. We also discuss a restriction on size-α tests that shows why some types of construction will not give uniformly more powerful, size-α tests in this case. In Section 3 we consider the $k=2$ and $\boldsymbol{b}_1'\Sigma\boldsymbol{b}_2 < 0$ case, which was also considered by Berger (1989). We imitate the strategy used in Section 2 to propose another test, ϕ_a, which is more powerful than the LRT. We compare the power functions of ϕ_a, Berger's test and the LRT in an example. In Section 4, we construct a uniformly more powerful, intersection-union test based on ϕ_o and ϕ_a, for the general, $k \geq 2$, problem (1.1). Section 5 contains some general comments on intersection-union tests.

2. Uniformly more powerful test when the cone is obtuse. In this section, we will consider the testing problem (1.3) when $k=2$ and $b_1'\Sigma b_2 = h_1'h_2 \| b_1'\Gamma^{-1} \| \| b_2'\Gamma^{-1} \| \geq 0$, i.e., $h_1'h_2 \geq 0$. Let τ be the angle between the vectors h_1 and h_2. Since $\cos(\tau) = h_1'h_2 \geq 0$, τ is acute. But the angle in the cone $\mathfrak{J} = \left\{\theta: h_1'\theta \geq 0, h_2'\theta \geq 0\right\}$ is $\xi = \pi - \tau$, which is obtuse. So we say H_1 is an obtuse cone when $h_1'h_2 \geq 0$. Figure 1 illustrates this. Berger (1989) describes a test in the opposite case, $h_1'h_2 \leq 0$. His method of test construction does not yield a size-α test when $h_1'h_2 > 0$. We discuss this in Section 2.2.

2.1 *A test that is uniformly more powerful than the LRT.* In this section we will describe a new test that is uniformly more powerful than the LRT when the alternative hypothesis is an obtuse cone. We start by defining the test, ϕ_o. Then we show that ϕ_o is a size-α test and is uniformly more powerful than the LRT.

Before describing the test, ϕ_o, we will define the functions and set which will be used to construct the rejection region for the test ϕ_o.

DEFINITION 2.1. *For any* s, $-\infty < \mathrm{s} < \infty$, *let* $\mathrm{L_s}$ *be the two dimensional set defined by*

$$\mathrm{L_s} = \left\{(u,v): \frac{u + \mathrm{s}v}{\sqrt{1+\mathrm{s}^2}} \geq z_\alpha, v \geq z_\alpha\right\}.$$

Let $\mathrm{c_s} = (\sqrt{1+\mathrm{s}^2} - \mathrm{s})z_\alpha$.

L_s is an obtuse cone if $s \geq 0$, and L_s is acute cone if $s < 0$. Examples of each are shown in Figure 2 and Figure 6. The vertex of the cone is (c_s, z_α). We will eventually express the LRT in terms of L_s. Throughout the rest of the paper, $\varphi(v)$ and $\Phi(v)$ denote the standard normal pdf and cdf, respectively.

Definition 2.2. *For any u, $-\infty < u < \infty$, define*

$$P_s(u) = \alpha - \int_{L_s(u)} \varphi(v)dv$$

where $L_s(u) = \{v: (u,v) \in L_s\}$. *Specifically, for* $s > 0$

$$P_s(u) = \begin{cases} \alpha - \left\{1 - \Phi\left(\frac{\sqrt{1+s^2}z_\alpha - u}{s}\right)\right\}, & u < c_s, \\ 0, & u \geq c_s. \end{cases}$$

For $s = 0$,

$$P_s(u) = \begin{cases} \alpha, & u < c_s = z_\alpha, \\ 0, & u \geq c_s = z_\alpha. \end{cases}$$

For $s < 0$,

$$P_s(u) = \begin{cases} \alpha, & u < c_s, \\ 1 - \Phi\left(\frac{\sqrt{1+s^2}z_\alpha - u}{s}\right), & u \geq c_s. \end{cases}$$

The specific formulas for $P_s(u)$, are easily verified by using the definition of L_s. $0 \leq P_s(u) \leq \alpha$ for all u. $P_0(u)$ is the limit of $P_s(u)$ as $s \to 0$. And, if $(U,V) \sim N_2((\mu,0), I_2)$, $P((U,V) \in L_s) = \int (\alpha - P_s(u))\varphi(u-\mu)du$. The line between the origin and (c_s, z_α), the vertex of L_s, has the equation $v = z_\alpha u / c_s = (\sqrt{1+s^2} + s)u$. We now define a set that contains this line, for $s \geq 0$.

Definition 2.3. *For* $s \geq 0$ *and* $0 < d < 1$, *let* B_s *be the set defined by*

$$B_s = \left\{(u,v): -c_s \le u \le c_s,\ l_2^o(u) \le v \le l_1^o(u)\right\}$$

where

$$l_1^o(u) = Min\left\{\Phi^{-1}\left(\ \Phi((\sqrt{1+s^2}+s)u) + d \times P_s(u)\right),\ su - sc_s + z_\alpha\right\},$$
$$l_2^o(u) = Max\left\{\Phi^{-1}\left(\ \Phi(l_1^o(u)) - P_s(u)\right),\ 0\right\}.$$

B_s is a set that touches L_s at the vertex of L_s and extends down toward the origin. An example of B_s is shown in Figure 2. Ignoring the Max and Min, the constant d is the proportion of the probability $P_s(u)$ that is placed above the line $v = (\sqrt{1+s^2}+s)u$. Increasing d moves the lines $l_1^o(u)$ and $l_2^o(u)$ upward. The following lemma is the key fact that will ensure that the size of ϕ_o is α.

Lemmma 2.1. *Let* $(U,V) \sim N_2\left((\mu,\nu), I_2\right)$ *where* $\nu \le 0$. *Let* $s \ge 0$ *and* $A_s = L_s \cup B_s$. *Then* $P_{(\mu,\nu)}\left((U,V) \in A_s\right) \le \alpha$.

Proof. For every $(u,v) \in A_s$, $v \ge 0$. Since $\nu \le 0$, by Theorem 2.2 of Berger (1989),

$$P_{(\mu,\nu)}\left((U,V) \in A_s\right) \le P_{(\mu,0)}\left((U,V) \in A_s\right)$$

$$= \int_{-\infty}^{+\infty}\left(\int_{L_s(u)} \varphi(v)dv + \int_{B_s(u)} \varphi(v)dv\right)\varphi(u-\mu)du$$

$$= \int_{-\infty}^{+\infty}\left(\alpha - P_s(u) + \int_{B_s(u)} \varphi(v)dv\right)\varphi(u-\mu)du, \tag{2.1}$$

where $L_s(u)$ is defined in Definition 2.2 and

$$B_s(u) = \left\{v: (u,v) \in B_s\right\} = \begin{cases} \emptyset, & u < -c_s \text{ or } u > c_s, \\ \left\{v: l_2^o(u) \le v \le l_1^o(u)\right\}, & -c_s \le u \le c_s. \end{cases}$$

The expression in parentheses in (2.1) is clearly bounded above by α if $u < -c_s$ or $u > c_s$. For $-c_s \le u \le c_s$, $B_s(u) = \emptyset$ and the integral over $B_s(u)$ is zero, if $l_1^o(u) < l_2^o(u)$. Otherwise,

$$\int_{B_s(u)} \varphi(v)dv = \Phi\left(l_1^o(u)\right) - \Phi\left(l_2^o(u)\right)$$

$$\le \Phi\left(l_1^o(u)\right) - \Phi\left(\Phi^{-1}\left(\Phi(l_1^o(u)) - P_s(u)\right)\right) = P_s(u).$$

So again the expression in parentheses is bounded above by α, and, hence, $P_{(\mu,\nu)}\left((U,V) \in A_s\right) \le \alpha$. □

Our new tests will be defined in terms of variables U_1, V_1, U_2 and V_2, that we now define.

DEFINITION 2.4. *Let h_1 and h_2 be noncolinear vectors ($|h_1'h_2| < \|h_1\| \, \|h_2\| = 1 \times 1 = 1$). Let $g_1 = h_2 - (h_1'h_2)h_1$ and $g_2 = h_1 - (h_1'h_2)h_2$. (g_1 and g_2 are vectors spanned by h_1 and h_2 that are orthogonal to h_1 and h_2; $g_1'h_1 = 0, g_2'h_2 = 0$.) Define $h_i'y = v_i$ and $g_i'y / \|g_i\| = u_i$, $i = 1,2$. Also define the corresponding random vectors $h_i'Y = V_i$ and $g_i'Y / \|g_i\| = U_i$, $i = 1,2$.*

Note: $\|g_1\| = \|g_2\| = \sqrt{1-(h_1'h_2)^2}$. Since $g_i'h_i = 0$, we know that U_i and V_i are independent.

Now we define the test ϕ_o. In fact, we define a whole family of tests, indexed by the constant d, $0 < d < 1$, that appears in Definition 2.3.

Definition 2.5. *Consider the testing problem* (1.3) *for vectors* h_1 *and* h_2 *that satisfy* $h_1'h_2 \geq 0$. *Fix* d, $0 < d < 1$. *Let* $\mathrm{s} = h_1'h_2(1-(h_1'h_2)^2)^{-1/2}$. *For any* α *that satisfies* $0 < \alpha < 1/2$, *define* ϕ_o *as the test that rejects* H_0 *if* $Y \in \mathrm{S}_1^* \cap \mathrm{S}_2^*$ *where* $\mathrm{S}_1^* = \{y\colon (u_1, v_1) \in \mathrm{A_s}\}$, $\mathrm{S}_2^* = \{y\colon (u_2, v_2) \in \mathrm{A_s}\}$ *and* $\mathrm{A_s} = \mathrm{L_s} \cup \mathrm{B_s}$.

The following lemma will show that the rejection region for the LRT is a subset of that for ϕ_o.

Lemma 2.2. *Consider the testing problem* (1.3) *when* $k = 2$. *The rejection region for the size-*α *LRT is* $\mathrm{R}_L = \{y\colon h_1'y \geq z_\alpha$ *and* $h_2'y \geq z_\alpha\}$. *Let* $\mathrm{L}_\mathrm{s}^i = \{y\colon (u_i, v_i) \in \mathrm{L_s}\} \subset \mathrm{S}_i^*$, $i = 1,2$, *where* $\mathrm{s} = h_1'h_2(1-(h_1'h_2)^2)^{-1/2}$. Then $\mathrm{L}_\mathrm{s}^i = \mathrm{R}_L$ *for* $i = 1,2$. *Hence, the rejection region for* ϕ_o, *namely* $\mathrm{S}_1^* \cap \mathrm{S}_2^*$, *contains* R_L.

Proof. For $i = 1$, $u_1 = \dfrac{g_1'y}{\| g_1 \|} = \dfrac{h_2'y - (h_1'h_2)h_1'y}{\sqrt{1-(h_1'h_2)^2}}$ and $v_1 = h_1'y$. Since $\dfrac{\mathrm{s}}{\sqrt{1+\mathrm{s}^2}} = h_1'h_2$, then $\| g_1 \| = \sqrt{1-(h_1'h_2)^2} = \dfrac{1}{\sqrt{1+\mathrm{s}^2}}$. Hence,

$$\frac{u_1+\mathrm{s}v_1}{\sqrt{1+\mathrm{s}^2}} = a_1'y,$$

where

$$a_1' = \left\{\frac{h_2' - (h_1'h_2)h_1'}{\| g_1 \|} \| g_1 \| + (h_1'h_2)h_1'\right\} = h_2'.$$

Therefore, $\mathrm{L}_\mathrm{s}^1 = \mathrm{R}_L$. For $i = 2$, similar algebra yields $\frac{u_2+\mathrm{s}v_2}{\sqrt{1+\mathrm{s}^2}} = h_1'y$ and $v_2 = h_2'y$. So

$L_s^2 = R_L$, also. □

Note, Lemma 2.2 is true for any h_1 and h_2, not just obtuse cones with $h_1'h_2 \geq 0$. Another way to state Lemma 2.2 is to say that the three events, $\{Y \in R_L\}$, $\{(U_1, V_1) \in L_s\}$ and $\{(U_2, V_2) \in L_s\}$, are all the same event. The following theorem shows ϕ_o is a size-α test and uniformly more powerful than the LRT.

Theorem 2.1. *For the testing problem* (1.1) *or* (1.3) *when* $k = 2$, *suppose that* $b_1'\Sigma b_2 \geq 0$ (*i.e.* $h_1'h_2 \geq 0$). *If* $0 < \alpha < 1/2$, *then* ϕ_o *has size exactly* α, *and* ϕ_o *is uniformly more powerful than the size-*α *LRT.*

Proof. From Lemma 2.2 we know the rejection region of the size-α LRT, R_L, is a subset of the rejection region of ϕ_o. Hence, ϕ_o is uniformly more powerful than the size-α LRT. Also,

(2.2) $\quad$ the size of $\phi_o \geq$ size of LRT $= \alpha$.

For any $\theta \in H_0$, $h_i'\theta \leq 0$ for either $i = 1$ or $i = 2$. For this i,

(2.3) $$P_\theta\left(Y \in S_1^* \cap S_2^*\right) \leq P_\theta\left(Y \in S_i^*\right) = P_\theta\left((U_i, V_i) \in A_s\right) \leq \alpha$$

The last inequality is from Lemma 2.1 since U_i and V_i are independent normal random variables, each with variance one, and $EV_i = h_i'\theta \leq 0$. Since, (2.3) is true for any $\theta \in H_0$, the size of $\phi_o \leq \alpha$. With (2.2) this implies ϕ_o has size exactly α. □

Figure 3 shows two examples (different s and d, $\alpha = 0.1$) of the rejection region of ϕ_o. Consider $p=2$, $h_1'=[0,1]$ and $h_2'=[1/\sqrt{1+s^2}, s/\sqrt{1+s^2}]$ so that $(y_1, y_2)=(u_1, v_1)$. In Figure 3a and 3b, the solid line above the line from (0,0) to the vertex of R_L is $l_1^o(u_1)$ and that below the line is $l_2^o(u_1)$. The lower dotted line is $l_1^o(u_2)$ and the upper dotted line is $l_2^o(u_2)$. These are the same functions, l_1^o and l_2^o, but these are graphed in the (u_2, v_2) axes. The intersection of the region between the solid lines and the region between the dotted lines is the additional set which is added to the rejection region of the LRT. Specifically, this is $C=\{y:(u_1,v_1)\in B_s\}\cap\{y:(u_2,v_2)\in B_s\}$. $R_L\cup C$ is the rejection region of the ϕ_o. When s increases as in Figure 3b, the added area decreases. The constant d that will produce the biggest intersection, and hence the highest power, depends on s.

Example 2.1. Suppose Y_1 and Y_2 are independent and $Y_i\sim N_1(\theta_i,1)$. Consider $h_1'=[0,1]$, $h_2'=[1/\sqrt{1+s^2}, s/\sqrt{1+s^2}]$, $s>0$, so that we are testing H_0: $\theta_2\le 0$ or $\theta_1+s\theta_2\le 0$ against H_1: $\theta_2>0$ and $\theta_1+s\theta_2>0$. Here we selected $\alpha=0.1$, $s=0.1$ and $d=1/2$ (as in Figure 3a) to compute the power of ϕ_o and LRT. Let $\beta_L(\boldsymbol{\theta})$ and $\beta_{\phi_o}(\boldsymbol{\theta})$ be the power functions of the LRT and ϕ_o, respectively. Values of these two functions for certain $\boldsymbol{\theta}$ values are in Table 1. These values are calculated by two steps. First, we calculate the cross-sectional probability $\int_{A(u)}\varphi(v-\theta_2)dv=f(u,\theta_2)$ which is a function of u and θ_2. Second, we calculate $\int_{-\infty}^{+\infty}f(u,\theta_2)\varphi(u-\theta_1)du$ using the trapezoidal rule with 300 points.The first part of the tables are for values of $\boldsymbol{\theta}'=(\theta, 0)$, $\theta\ge 0$. These values are on the boundary of H_0, so the powers are less than $\alpha=0.1$. If a test is unbiased, then the power is equal to α for the values of $\boldsymbol{\theta}$ which are on the boundary of H_0. Here we can see that the LRT and ϕ_o are biased, but the difference between α and the power of ϕ_o is considerably smaller than that between α and the power of the

LRT. The second part of the table is for values of $\theta' = ((\sqrt{1+s^2}-s)\theta, \theta)$ which are on the line from the origin to the vertex (c_s, z_α). For example, $\beta_{\phi_o}(0.4525,0.5)/\beta_L(0.4525,0.5) \approx 1.80$ and $\beta_{\phi_o}(0.4525,0.5) > \alpha > \beta_L(0.4525,0.5)$. $\beta_{\phi_o}(\theta)$ is clearly bigger than $\beta_L(\theta)$ for $\theta \leq 1.5$. The largest difference is 0.047. The bottom of the table is for values of $\theta' = (0.5(\sqrt{1+s^2}-s)\theta, \theta)$. $\beta_{\phi_o}(\theta)$ is clearly larger than $\beta_L(\theta)$ for $\theta \leq 2.5$. As s increases, there is less space to add to the rejection region of the LRT. Figure 3b shows this fact. So we can not improve the power as much when s is large.

Table 1 Power of LRT and ϕ_o for $s = 0.1$, $d = 1/2$ and $\alpha = 0.1$

	θ								
	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$\beta_L(\theta,0)$	0.013	0.027	0.046	0.065	0.081	0.092	0.097	0.099	0.100
$\beta_{\phi_o}(\theta,0)$	0.048	0.066	0.081	0.091	0.096	0.099	0.100	0.100	0.100
$\beta_L(0.905\theta,\theta)$	0.013	0.059	0.181	0.390	0.633	0.827	0.936	0.981	0.996
$\beta_{\phi_o}(0.905\theta,\theta)$	0.048	0.106	0.224	0.416	0.643	0.829	0.936	0.982	0.996
$\beta_L(0.4505\theta,\theta)$	0.013	0.043	0.108	0.219	0.367	0.526	0.668	0.781	0.863
$\beta_{\phi_o}(0.4505\theta,\theta)$	0.048	0.087	0.153	0.255	0.389	0.536	0.672	0.782	0.863

2.2 *Restriction on the construction of a size-α test.* The set that is added to the rejection region of the LRT to construct ϕ_o touches the LRT rejection region, R_L, at only a single point (see Figure 3). One might ask, if we can add a set along the boundary of R_L to obtain a more powerful test. Gutmann (1987) constructed such a set for a nonnormal problem. The following theory will show that a test with a rejection region like this will have size greater than α in a normal problem. It will also show that a construction like Berger (1989) used for acute cone problems, will not work for obtuse cone problems. Liu (1992) also showed this.

Lemma 2.3. *Let $Z \sim N(\mu,1)$. Let α, ϵ, c and d be real numbers such that $\epsilon > 0$ and $c < d$. Define the function $\psi(z)$ by*

$$\psi(z) = \begin{cases} 0, & z < c, \\ \alpha+\epsilon, & c \le z \le d, \\ \alpha, & d < z. \end{cases} \tag{2.4}$$

Then $E_\mu\ \psi(Z) > \alpha$ for all large values of μ.

Proof. Since

$$\begin{aligned} E_\mu\psi(Z) &= (\alpha+\epsilon)P(\ c \le Z \le d) + \alpha P(d < Z) \\ &= (\alpha+\epsilon)[\Phi(d-\mu) - \Phi(c-\mu)] + \alpha[1-\Phi(d-\mu)] \\ &= \epsilon\Phi(d-\mu) - (\alpha+\epsilon)\Phi(c-\mu) + \alpha, \\ \frac{d}{d\mu}E_\mu\psi(Z) &= -\epsilon\varphi(d-\mu) + (\alpha+\epsilon)\varphi(c-\mu). \end{aligned}$$

So $\frac{d}{d\mu}E_\mu\psi(Z)$ will be less than 0, if and only if

(2.5) $$\frac{\alpha+\epsilon}{\epsilon} < \frac{\varphi(d-\mu)}{\varphi(c-\mu)}.$$

Since φ is symmetric about 0, $\frac{\varphi(d-\mu)}{\varphi(c-\mu)} = \frac{\varphi(\mu-d)}{\varphi(\mu-c)}$. Since $d > c$, the right-hand side of (2.5) is increasing in μ (monotone likelihood ratio), and in fact, for the normal density,

$$\lim_{\mu\to\infty} \frac{\varphi(d-\mu)}{\varphi(c-\mu)} = \infty.$$

So, for all large μ, (2.5) is satisfied and

$$\frac{d}{d\mu}\mathrm{E}_\mu\psi(Z) < 0.$$

Since $\lim_{\mu\to\infty}\mathrm{E}_\mu\psi(Z) = \alpha$, it must be that $\mathrm{E}_\mu\psi(Z) > \alpha$ for all large μ. □

THEOREM 2.2. *Let $Z \sim \mathrm{N}(\mu, 1)$. Let c be real number and α be positive number. Suppose $\psi(z)$ is a function such that $\psi(z) \geq 0$ for all z, $\psi(z) \geq \alpha$ for all $z \geq c$, and $\psi(z) > \alpha$ on a set of positive Lebesgue measure contained in $[c,\infty)$. Then $E_\mu\psi(Z) > \alpha$ for all large values of μ.*

PROOF. Let $m(\cdot)$ denote Lebesgue measure. Let $\mathrm{A} = \{z: \psi(z) > \alpha,\ z \geq c\}$ and $\mathrm{B}_\epsilon = \{z: \psi(z) \geq \alpha+\epsilon,\ z \geq c\}$. Since $m(\mathrm{A}) > 0$ there exists $\epsilon_0 > 0$ and f such that $c < f < \infty$ and $m(\mathrm{B}^*_{\epsilon_0}) > 0$ where $\mathrm{B}^*_{\epsilon_0} = \mathrm{B}_{\epsilon_0} \cap [c, f]$. Define the function $\psi^*(z)$ by

$$\psi^*(z) = \begin{cases} 0, & z < c, \\ \alpha+\epsilon_0, & c \leq z \leq c+m(\mathrm{B}^*_{\epsilon_0}), \\ \alpha, & z > c+m(\mathrm{B}^*_{\epsilon_0}). \end{cases}$$

Then,

(2.6) $\quad E_\mu \psi^*(Z) > \alpha$ for all large values of μ

(by Lemma 2.3). We can bound $E_\mu \psi(Z)$ by

$$E_\mu\psi(Z) \geq \alpha P_\mu\{Z \in [c,\infty)\} + \epsilon_0 P_\mu\{Z \in B^*_{\epsilon_0}\}$$
$$= \alpha P_\mu\{c \leq Z \leq c+m(B^*_{\epsilon_0})\} + \epsilon_0 P_\mu\{Z \in B^*_{\epsilon_0}\} + \alpha P_\mu\{Z > c+m(B^*_{\epsilon_0})\}.$$

Now, $P_\mu\{Z \in B^*_{\epsilon_0}\} \geq P_\mu\{c \leq Z \leq c+m(B^*_{\epsilon_0})\}$, when $\mu > f$, because $B^*_{\epsilon_0} \subset [c,f]$ and the normal distribution is unimodal. Thus, if $\mu \geq f$,

$$E_\mu\psi(Z) \geq (\alpha+\epsilon_0)P_\mu(c \leq Z \leq c+m(B^*_{\epsilon_0})) + \alpha P_\mu(Z > c+m(B^*_{\epsilon_0})) = E_\mu\psi^*(Z).$$

From (2.6), $E_\mu\psi(Z) > \alpha$ for all large value of μ. □

Theorem 2.3. *Let* $(U,V) \sim N_2((\theta,0), I_2)$. *For a constant* c, *let* $R = \{(u,v): u \geq c, v \geq z_\alpha\}$ *and* $Q_1 = \{(u,v): u \geq c, v < z_\alpha\}$. *If* $Q \subset Q_1$ *and* $P_\theta((U,V) \in Q) > 0$, *then* $P_\theta((U,V) \in R \cup Q) > \alpha$ *for all large* θ.

Proof. Define $Q(u) = \{v: (u,v) \in Q\}$. Then

$$P_\theta((U,V) \in R \cup Q) = \int_c^{+\infty}\left\{\int_{z_\alpha}^{+\infty}\varphi(v)dv + \int_{Q(u)}\varphi(v)dv\right\}\varphi(u-\theta)du$$
$$= \int_{-\infty}^{+\infty}\psi(u)\varphi(u-\theta)du = E_\theta\psi(U),$$

where

$$\psi(u) = \begin{cases} 0 & u < c, \\ \alpha + \int_{Q(u)}\varphi(v)dv & u \geq c. \end{cases}$$

Since $0 < P_\theta((U,V) \in Q)$, $m\left(\left\{u: u \geq c, \int_{Q(u)}\varphi(v)dv > 0\right\}\right) > 0$. Thus $\psi(u)$

satisfies the conditions of Theorem 2.2, and

$$P_{\theta}\Big((U,V) \in R \cup Q\Big) = \mathrm{E}_{\theta}\psi(U) > \alpha \qquad \text{for all large } \theta. \qquad \square$$

Now consider a test ϕ^* whose rejection region, R^*, contains R_L. In Section 2.1, we saw that $\mathrm{R}_L = \Big\{(u_1,v_1):(u_1,v_1) \in \mathrm{L}_s\Big\}$ and this set contains $\Big\{(u_1,v_1): u_1 \geq \mathrm{c}_s,\ v_1 \geq z_\alpha\Big\}$. So, by Theorem 2.3, if ϕ^* has size α then R^* can not contain any part of $Q_1 = \Big\{(u_1,v_1): u_1 \geq \mathrm{c}_s,\ \ v_1 < z_\alpha\Big\}$ (except for a set with probability zero). Similarly, we can also write $\mathrm{R}_L \supset \Big\{(u_2,v_2): u_2 \geq \mathrm{c}_s,\ v_2 \geq z_\alpha\Big\}$ and R^* can not contain any part of $Q_2 = \Big\{(u_2,v_2): u_2 \geq \mathrm{c}_s,\ v_2 < z_\alpha\Big\}$. These sets are shown in Figure 4. The set added to R_L to form a more powerful, size-α test can not be in the shaded region $Q_1 \cup Q_2$. It must lie in the triangular region labeled Q_3, which is where the set that defines ϕ_o lies.

Berger (1989) constructed tests that were uniformly more powerful than the LRT for problems in which $h_1'h_2 \leq 0$. Theorem 2.3 can be used to show that Berger's method of construction will not give a size-α test if $h_1'h_2 > 0$. Figure 5 illustrates this. R_L is the rejection region of the LRT. The diamond shapes $\mathrm{R}_L \cup \mathrm{R}_2 \cup \cdots \cup \mathrm{R}_5$ would be the rejection region of Berger's test. This rejection region contains some area in the shaded region which causes the size of the test to be greater than α. Berger's method can not be applied to obtuse cone alternative hypotheses when p=2.

3. Uniformly more powerful test when the cone is acute. In this section, we describe a size-α test that is uniformly more powerful than the size-α LRT for problems in which $h_1'h_2 < 0$, that is, $b_1'\Sigma b_2 < 0$. In these problems, the cone defined by the alternative hypothesis is acute. Berger (1989) described a size-α test, ϕ_b, that is more powerful than the LRT for these problems. Here we describe a new test, ϕ_a, that has smoother boundaries and sometimes appears more powerful than ϕ_b. The method we use to construct ϕ_a is very similar to the method we used to construct ϕ_o. So we will omit the formal proofs that ϕ_a has the described properties. One difference in this acute case is that, unlike in Section 2.2, the rejection region for ϕ_a completely surrounds and enlarges upon the rejection region of the LRT.

Our description of ϕ_a will be similar to our description of ϕ_o in Section 2.1. The set $\mathrm{L_s}$, constant $\mathrm{c_s}$, function $\mathrm{P_s}(u)$ and variables (u_1, v_1) and (u_2, v_2) are as defined in Section 2.1. Lemma 2.2 remains valid, and the LRT's rejection region can be expressed in any of the following ways,

$$\mathrm{R}_L = \left\{y: h_1'y \geq z_\alpha \text{ and } h_2'y \geq z_\alpha\right\} = \left\{y: (u_1, v_1) \in \mathrm{L_s}\right\} = \left\{y: (u_2, v_2) \in \mathrm{L_s}\right\},$$

where $\mathrm{s} = h_1'h_2(1-(h_1'h_2)^2)^{-1/2}$. ϕ_a is defined in terms of a set $\mathrm{A_s}$ that we now define.

Definition 3.1. *For* $\mathrm{s} < 0$ *and* $0 < d < 1$, *let* $\mathrm{A_s}$ be the set defined by

$$\mathrm{A_s} = \left\{(u, v): u \geq 0,\ l_2^a(u) \leq v \leq l_1^a(u)\right\},$$

where

$$l_1^a(u) = \begin{cases} \Phi^{-1}\left\{\Phi\left((\sqrt{1+\mathrm{s}^2}+\mathrm{s})u\right) + d \times \mathrm{P_s}(u)\right\}, & 0 \leq u < \mathrm{c_s}, \\ \Phi^{-1}\left\{1-(1-d) \times \mathrm{P_s}(u)\right\} & u \geq \mathrm{c_s} \end{cases}$$

and

$$l_2^\alpha(u) = Max\left\{\Phi^{-1}\left(\Phi(l_1^\alpha(u))-\alpha\right),0\right\}, \qquad u \geq 0.$$

Examples of the sets L_s and A_s and the lines $l_1^\alpha(u)$ and $l_2^\alpha(u)$ are shown in Figure 6. In this figure, $s = -2$, $d = 1/2$ and $\alpha = 0.1$. The solid lines are $l_1^\alpha(u)$ and $l_2^\alpha(u)$. For $u \geq c_s$ the line $l_1^\alpha(u)$ lies above the upper boundary of L_s, which is given by the line $v = (\sqrt{1+s^2}z_\alpha - u)/s$, This is true since $1-(1-d)P_s(u) = \Phi\left((\sqrt{1+s^2}z_\alpha - u)/s\right) + dP_s(u) > \Phi\left((\sqrt{1+s^2}z_\alpha - u)/s\right)$. And, for $u \geq c_s$, $l_2^\alpha(u)$ is below the lower boundary of L_s because the lower boundary is $z_\alpha = \Phi^{-1}(1-\alpha) > l_2^\alpha(u)$. Therefore, $L_s \subset A_s$ and for $i = 1$or 2, we have

$$\text{(3.1)} \qquad R_L = \left\{y: (u_i, v_i) \in L_s\right\} \subset \left\{y: (u_i, v_i) \in A_s\right\}.$$

If $(U,V) \sim N_2\left((\mu,\nu), I_2\right)$, with $\nu \leq 0$, then $P_{(\mu,\nu)}\left((U,V) \in A_s\right) < \alpha$. This follows as in the proof of Lemma 2.1. In this case we have

$$P_{(\mu,\nu)}\left((U,V) \in A_s\right) \leq P_{(\mu,0)}\left((U,V) \in A_s\right)$$

$$= \int_0^\infty \left\{ \int_{l_2^\alpha(u)}^{l_1^\alpha(u)} \varphi(v)dv \right\} \varphi(u-\mu)du$$

$$\text{(3.2)} \qquad = \int_0^\infty \left\{\Phi\left(l_1^\alpha(u)\right) - \Phi\left(l_2^\alpha(u)\right)\right\} \varphi(u-\mu)du,$$

$$\leq \int_0^\infty \left\{\Phi\left(l_1^\alpha(u)\right) - \Phi\left(\Phi^{-1}\left(\Phi(l_1^\alpha(u))-\alpha\right)\right)\right\} \varphi(u-\mu)du$$

$$= \int_0^\infty \alpha\varphi(u-\mu)du < \alpha$$

Thus, we can define a size-α test, just as we did in Section 2.1.

Definition 3.2. *Consider the testing problem* (1.3) *for vectors* h_1 *and* h_2 *that* *satisfy* $h_1'h_2 < 0$. *Fixed* d, $0 < d < 1$. *Let* $s = h_1'h_2(1-(h_1'h_2)^2)^{-1/2}$. *For any* α *that satisfies* $0 < \alpha < 1/2$, *define* ϕ_a *as the test that rejects* H_0 *if* $Y \in S_1^* \cap S_2^*$ *where* $S_i^* = \{y: (u_i, v_i) \in A_s\}$ (A_s *is defined in Definition* 3.1).

Because (3.1) and (3.2) are true, as in Theorem 2.1, we can show that ϕ_a is a size-α test that is uniformly more powerful than size-α LRT.

Consider the testing problem with $h_1' = (0,1)$ and $h_2' = (1/\sqrt{5}, -2/\sqrt{5})$, so that $(y_1, y_2) = (u_1, v_1)$ and $s = -2$. Let $d = 1/2$ and $\alpha = 0.1$. Then in Figure 6, the solid lines are $l_1^a(u_1)$ and $l_2^a(u_1)$ and the region between them is S_1^*. The dotted lines are $l_1^a(u_2)$ (lower line) and $l_2^a(u_2)$ (upper line) and the region between them is S_2^*. The rejection region is $S_1^* \cap S_2^*$, and it contains $L_s = R_L$, the LRT's rejection region. In Figure 7, the rejection region for Berger's (1989) test, ϕ_b, for the problem is shown. The union of the diamond shaped regions, $R_L \cup R_2 \cup \cdots \cup R_5$, is the rejection region for ϕ_b. Note that the rejection region for ϕ_b is almost completely contained in the rejection region for ϕ_a. In fact, ϕ_a may be uniformly more powerful than ϕ_b. In general, as s decreases, the containment of ϕ_b in ϕ_a comes closer and closer to reality.

For this same problem, the power functions of the LRT, ϕ_b and ϕ_a are compared in Table 2. Denote these power functions by $\beta_L(\boldsymbol{\theta})$, $\beta_b(\boldsymbol{\theta})$ and $\beta_{\phi_a}(\boldsymbol{\theta})$, respectively. These values are calculated in the same way as in Example 2.2. The first part of the table is for values of $\boldsymbol{\theta}' = (\theta, 0)$, $\theta \geq 0$. These values are on the boundary of H_0, so the powers are less than $\alpha = 0.1$. Again here we can see that the LRT, ϕ_b and ϕ_a are biased, but the bias of ϕ_a is considerably smaller than the bias of the LRT. The power of the LRT is much smaller than those of ϕ_b and ϕ_a when θ is closed to 0. $\beta_{\phi_a}(0,0) > \beta_b(0,0)$. Both tests improve greatly on the LRT. $\beta_b(2,0) = 0.052$ and $\beta_{\phi_a}(2,0) = 0.091$, but $\beta_L(2,0) \approx 0.0$. The largest difference between $\beta_{\phi_a}(\theta,0)$ and $\beta_L(\theta,0)$ is 0.095. The largest difference between $\beta_{\phi_a}(\theta,0)$ and $\beta_b(\theta,0)$ is 0.042. The second part of the table is for values of $\boldsymbol{\theta}' = ((\sqrt{1+s^2}-s)\theta, \theta)$, values on the line from (0,0) to (c_s, z_α). $\beta_{\phi_a}(4.236,1)/\beta_L(4.236,1) \approx 15.3$. The largest difference is 0.143. $\beta_{\phi_a}(\boldsymbol{\theta})$ is significantly bigger than $\beta_L(\boldsymbol{\theta})$ for $\theta \leq 3$. $\beta_{\phi_a}(4.236,1)/\beta_b(4.236,1) \approx 1.75$. The largest difference is 0.066. $\beta_{\phi_a}(\boldsymbol{\theta})$ is clearly bigger than $\beta_b(\boldsymbol{\theta})$ for $\theta \leq 2$. The bottom of the table is for $\boldsymbol{\theta}' = (0.5(\sqrt{1+s^2}-s)\theta, \theta)$. Again ϕ_a improves on ϕ_b for these parameter values. $\beta_{\phi_a}(2.118,1)/\beta_L(2.118,1) > 100$. The largest difference is 0.119. $\beta_{\phi_a}(\boldsymbol{\theta})$ is significantly bigger than $\beta_L(\boldsymbol{\theta})$ and $\beta_b(\boldsymbol{\theta})$ for $\theta \leq 2$. $\beta_{\phi_a}(2.118,1)/\beta_b(2.118,1) \approx 1.8$. The largest difference is 0.071. The power of ϕ_a may be greater than that of ϕ_b for all $\boldsymbol{\theta}$, at $s = -2$.

Table 2 Power of LRT, ϕ_b and ϕ_a for s = −2.0, $d = 1/2$ and $\alpha = 0.1$

	θ								
	0	1	2	3	4	5	6	7	8
$\beta_L(\theta,0)$	0.000	0.000	0.000	0.000	0.002	0.014	0.040	0.071	0.089
$\beta_b(\theta,0)$	0.026	0.045	0.052	0.053	0.053	0.050	0.054	0.073	0.090
$\beta_{\phi_a}(\theta,0)$	0.029	0.069	0.091	0.095	0.092	0.090	0.090	0.092	0.096
$\beta_L(4.236\theta,\theta)$	0.000	0.010	0.528	0.914	0.993	1.000	1.000	1.000	1.000
$\beta_b(4.236\theta,\theta)$	0.026	0.087	0.528	0.914	0.993	1.000	1.000	1.000	1.000
$\beta_{\phi_a}(4.236\theta,\theta)$	0.029	0.153	0.536	0.914	0.993	1.000	1.000	1.000	1.000
$\beta_L(2.118\theta,\theta)$	0.000	0.000	0.135	0.481	0.691	0.830	0.920	0.968	0.989
$\beta_b(2.118\theta,\theta)$	0.026	0.067	0.151	0.481	0.691	0.830	0.920	0.968	0.989
$\beta_{\phi_a}(2.118\theta,\theta)$	0.029	0.119	0.222	0.484	0.691	0.830	0.920	0.968	0.989

4. A more powerful test in the general problem. We will now describe a size-α test that is uniformly more powerful than the size-α LRT for the general problem (1.3) when $k \geq 2$ and $0 < \alpha < 1/2$. We will denote this test by ϕ_g. The intersection-union method will be used to construct ϕ_g. A summary of this method may be found in Sections 8.2.4 and 8.3.5 of Casella and Berger (1990) or in Berger (1982).

To use the intersection-union method, $H_0{:}\, \boldsymbol{h}_i'\boldsymbol{\theta} \leq 0$ for some $i = 1,\ldots,k$, must be written as a union. Let D denote any division of the indices $\{1,\ldots,k\}$ into the minimal number of subsets of size two such that each value $1,\ldots,k$ appears at least once. Elements of D are just pairs of indices, (i,j). If k is even, D has $k^* = k/2$ elements and each index appears once. If k is odd, D has $k^* = (k+1)/2$ elements. All indices appears once except one appears twice. To construct a more powerful test, any such division of $\{1,\ldots,k\}$ will work, but different divisions will lead to different tests.

For each $(i,j) \in \mathrm{D}$, consider testing $H_{0ij}{:}\, \boldsymbol{h}_i'\boldsymbol{\theta} \leq 0$ or $\boldsymbol{h}_j'\boldsymbol{\theta} \leq 0$ versus H_{1ij}: $\boldsymbol{h}_i'\boldsymbol{\theta} > 0$ and $\boldsymbol{h}_j'\boldsymbol{\theta} > 0$. If $\boldsymbol{h}_i'\boldsymbol{h}_j \geq 0$, let C_{ij} denote the size-α rejection region of ϕ_o (for some d) from Section 2. If $\boldsymbol{h}_i'\boldsymbol{h}_j < 0$, let C_{ij} denote the size-α rejection region of ϕ_a (for some d) from Section 3. Since $H_0 = \bigcup_{(i,j)\in \mathrm{D}} H_{0ij}$, we can define an intersection-union test based on the C_{ij}.

Definition 4.1. *For the testing problem* (1.3) *with* $k \geq 2$ *and* $0 < \alpha < 1/2$, *let* ϕ_g *be the test that rejects* H_0 *if* $Y \in \bigcap_{(i,j)\in \mathrm{D}} \mathrm{C}_{ij}$.

THEOREM 4.1. *For $0 < \alpha < 1/2$, the test ϕ_g is a size-α test of H_0 versus H_1, and ϕ_g is uniformly more powerful than the size-α LRT.*

PROOF. Since each of C_{ij} is a size-α rejection region for testing H_{0ij}, by Theorem 1 in Berger (1982), ϕ_g has size $\leq \alpha$. But, the size-α LRT's rejection region is

$$\begin{aligned} R_L &= \left\{y: h_i'y \geq z_\alpha, i = 1,\ldots,k\right\} \\ &\subset \left\{y: h_i'y \geq z_\alpha \text{ and } h_j'y \geq z_\alpha\right\} \subset C_{ij}, \end{aligned}$$

for every $(i,j) \in D$. Hence R_L is contained in the rejection region of ϕ_g, the size of $\phi_g \geq$ size of the LRT $= \alpha$, and ϕ_g is uniformly more powerful than the LRT. □

ϕ_g is, in fact, strictly more powerful than the LRT because ϕ_g's rejection region contains an open set that is not in R_L. Let y denote a point satisfying $h_i'y = z_\alpha$, $i = 1\ldots,k$. (If $k \geq p$, there is only one such y. If $k < p$, there are many such ys.) Every C_{ij} contains an open set that contains the line from y to the origin. So the intersection of the C_{ij}, ϕ_g's rejection region, contains an open set containing this line, and this open set is not in R_L.

As mentioned earlier, different choices of D will lead to different tests. More work needs to be done to determine which Ds yield generally more powerful tests. But one principle seems reasonable. In Table 1 and Table 2, we see that the improvement in power over the LRT is much greater for small s (acute cones) than for large s (obtuse cones). So it seems that we will get more improvement from ϕ_g if D is chosen so that the values $s_{ij} = h_i'h_j(1-(h_i'h_j)^2)^{-1/2}$, $(i,j) \in D$, are small rather than large.

EXAMPLE 4.1. Consider the hypothesis testing problem defined by the three vectors $\boldsymbol{h}_1' = [1/\sqrt{2},\ 1/\sqrt{2},\ 0]$, $\boldsymbol{h}_2' = [1/\sqrt{3},\ \sqrt{2}/\sqrt{3},\ 0]$ and $\boldsymbol{h}_3' = [1/\sqrt{3},\ -\sqrt{2}/\sqrt{3},\ 0]$. Then $s_{12} \approx 5.83$, $s_{13} \approx -0.17$ and $s_{23} \approx -0.35$. So we conjecture that $D = \{(1,3), (2,3)\}$ will give a generally more powerful ϕ_g than will $D = \{(1,2), (1,3)\}$. But we would not expect the first test to be uniformly more powerful than the second.

5. Further comments on intersection-union tests. In Section 4, ϕ_g was explicitly constructed as an intersection-union test (IUT). In fact, most of the tests considered in this paper are naturally thought of as IUTs.

For $i = 1,\ldots,k$, $R_{L_i} = \{y: \boldsymbol{h}_i'y \geq z_\alpha\}$ is the size-α LRT of $H_{0i} : \boldsymbol{h}_i'\boldsymbol{\theta} \leq 0$ versus $H_{1i} : \boldsymbol{h}_i'\boldsymbol{\theta} > 0$. Since $H_0 = \bigcup_{i=1}^{k} H_{0i}$, the test with rejection region $R_L = \bigcap_{i=1}^{k} R_{L_i}$ is a level-α IUT of H_0 versus H_1. This test is just the size-α LRT. Theorem 1 of Berger (1982) shows that this test is level-α. A more specific analysis, such as in Berger (1989), is required to show the test is size-α.

The tests ϕ_o and ϕ_a are also constructed as IUTs for their $k = 2$ problems. For example, consider an obtuse cone problem. By Lemma 2.1 and Definition 2.5, for $i = 1$ or 2 the test with rejection region $S_i^* = \{y: (u_i, v_i) \in A_s\}$ is a size-α test of $H_{0i}: \boldsymbol{h}_i'\boldsymbol{\theta} \leq 0$ versus $H_{1i}: \boldsymbol{h}_i'\boldsymbol{\theta} > 0$. So the test with rejection region $S_1^* \cap S_2^*$, that is ϕ_o, is a level-α IUT of H_0 versus H_1. Since $R_L \subset S_1^* \cap S_2^*$, and we know R_L is size-α, ϕ_o must in fact have size equal to α.

For the $k = 2$ case, both the LRT and ϕ_o (or ϕ_a) are IUTs constructed starting from the some individual hypotheses, H_{01} and H_{02}. This illustrates that some foresight in choosing the rejection region for the individual hypotheses, foresight concerning how the rejection regions will intersect, might result in increased power in the resulting IUT. Starting with the more complicated regions S_1^* and S_2^*, rather than the simpler R_{L_1} and R_{L_2}, yields a more powerful test. Also, although the R_{L_i}s have certain optimality properties, e.g., R_{L_i} is an unbiased test of H_{0i}, whereas S_i^* is not, this optimality does not carry over to the IUTs.

The test ϕ_b could also be described as an IUT in terms of the variables (u_i, v_i), $i = 1, \ldots, k$. But it was not described in this way in Berger (1989).

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Department of Statistics
National Chung Hsing University
Taipei, Taiwan 105
R. O. C.

Department of Statistics
North Carolina State University
Raleigh, North Carolina 27695-8203

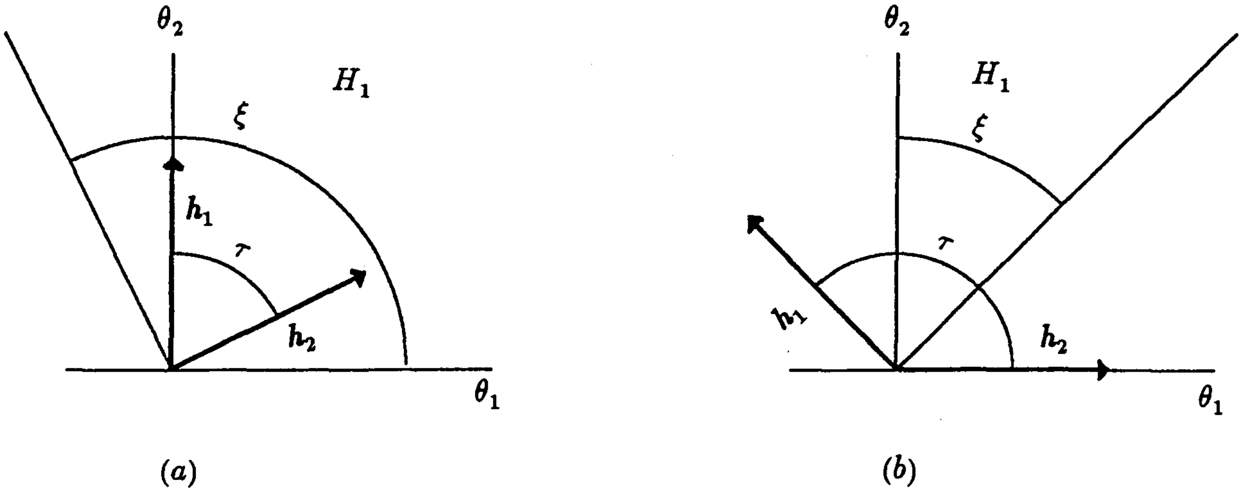

Figure 1: (a) obtuse cone, $h_1'h_2 \geq 0$

(b) acute cone, $h_1'h_2 < 0$

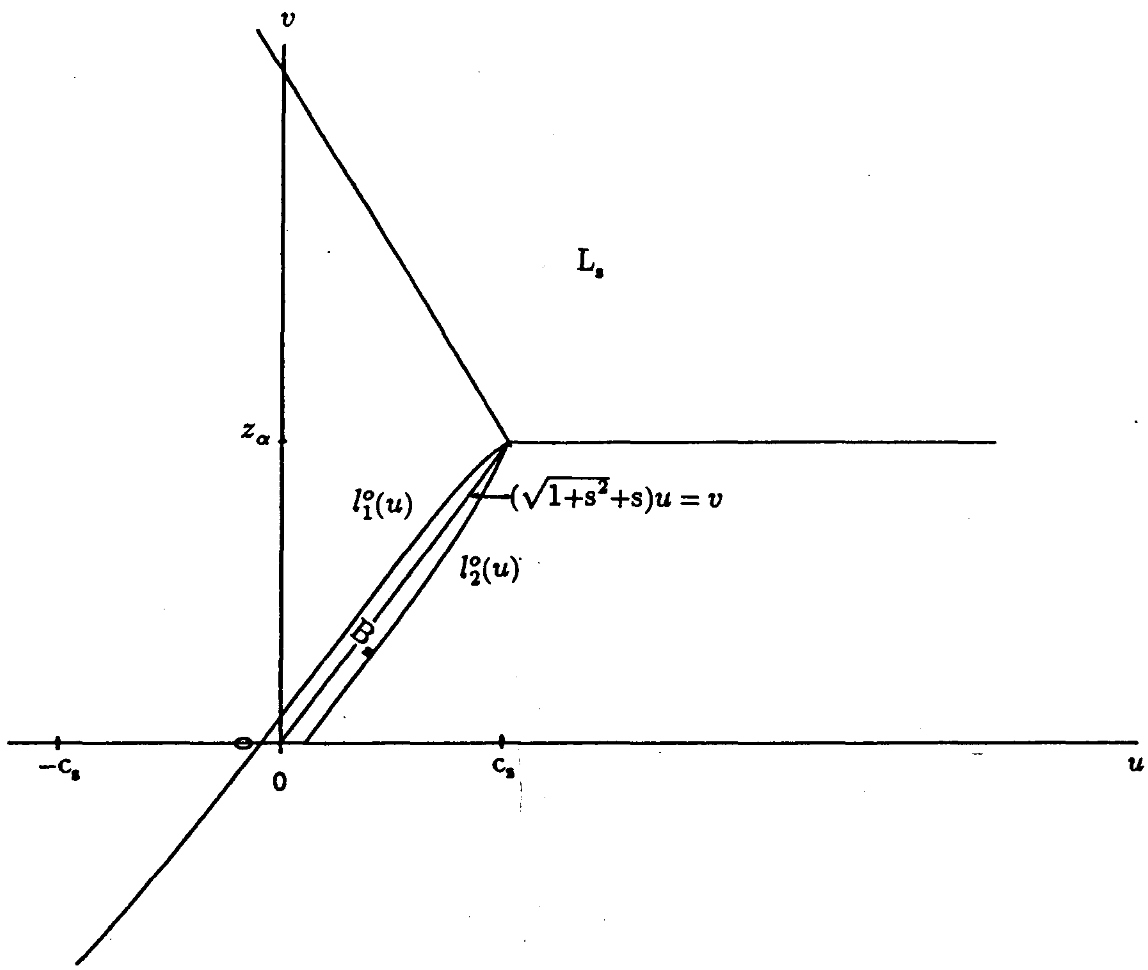

Figure 2: The set $A_s = L_s \cup B_s$ and functions $l_1^o(u)$ and $l_2^o(u)$.

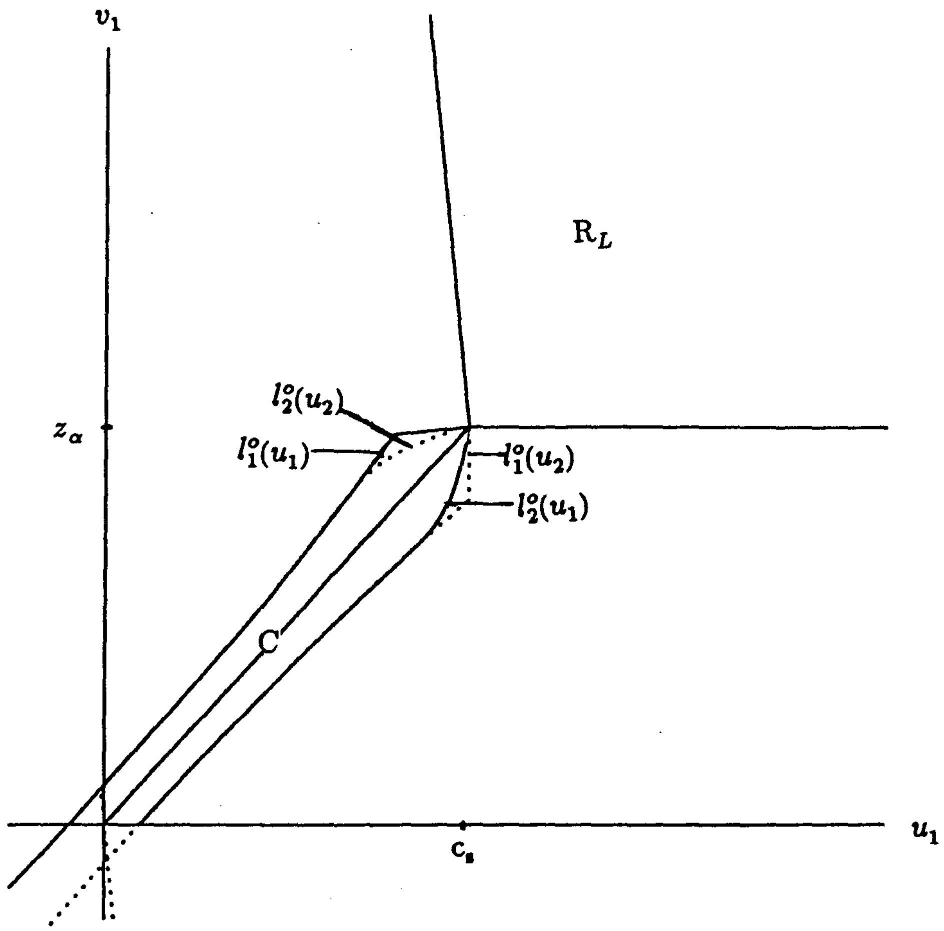

Figure 3(a): Rejection region of ϕ_o when $s = 0.1$, $d = 1/2$ and $\alpha = 0.1$.

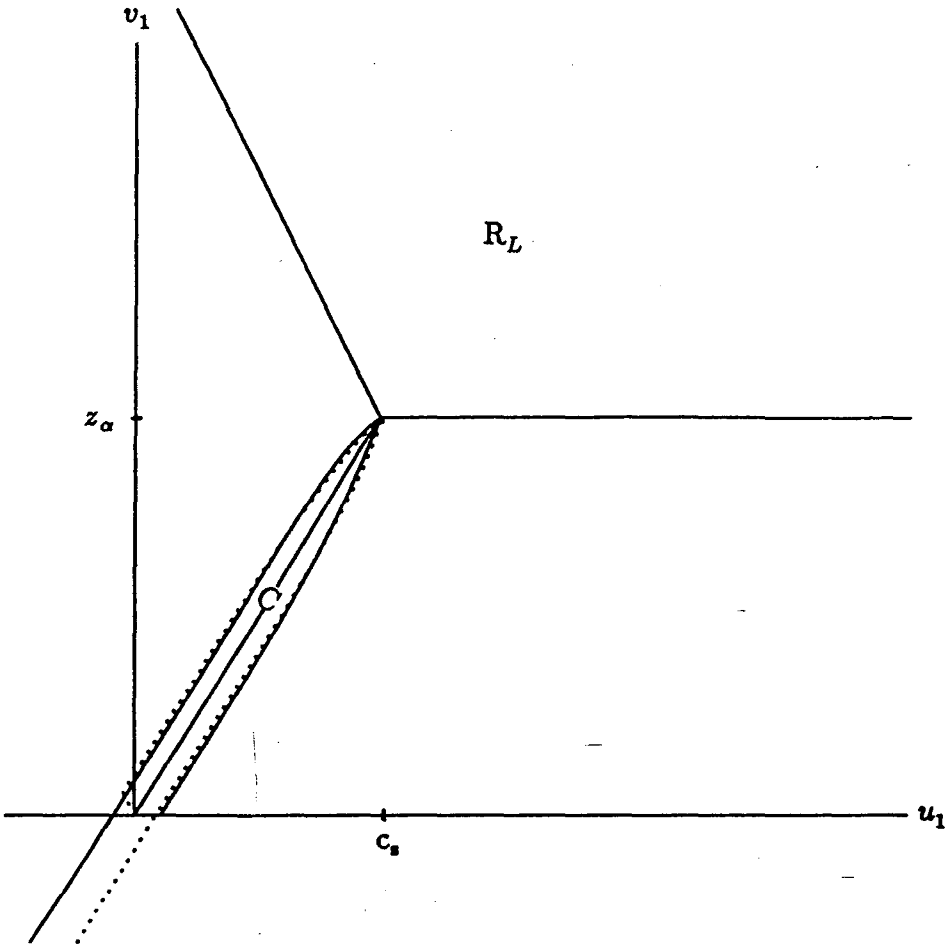

Figure 3(b): Rejection region of ϕ_o when $s = 0.5$, $d = 4/9$ and $\alpha = 0.1$.

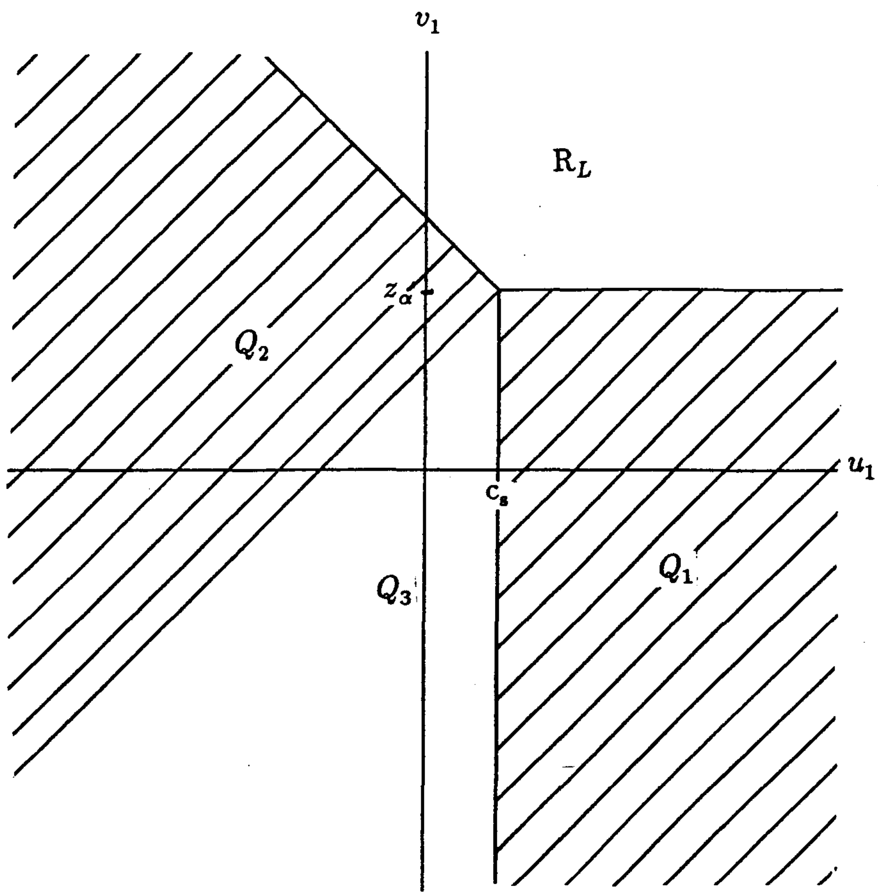

Figure 4: Constraints on a size-α rejection region.

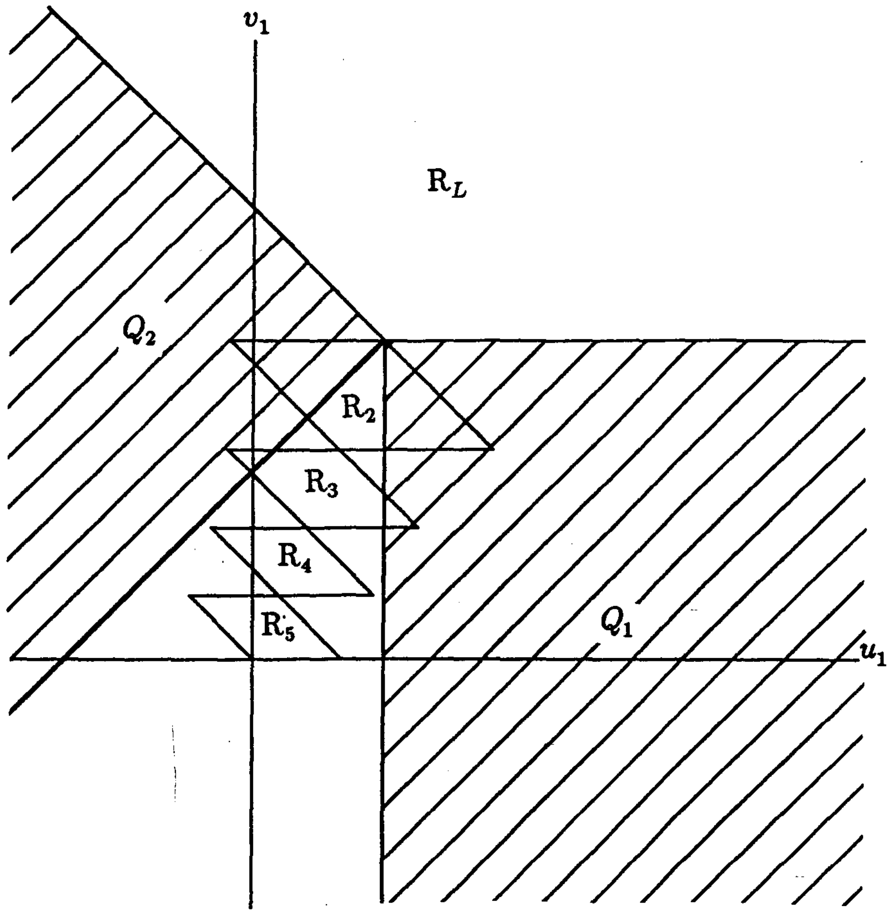

Figure 5: Rejection region of ϕ_b for an obtuse cone hypothesis.

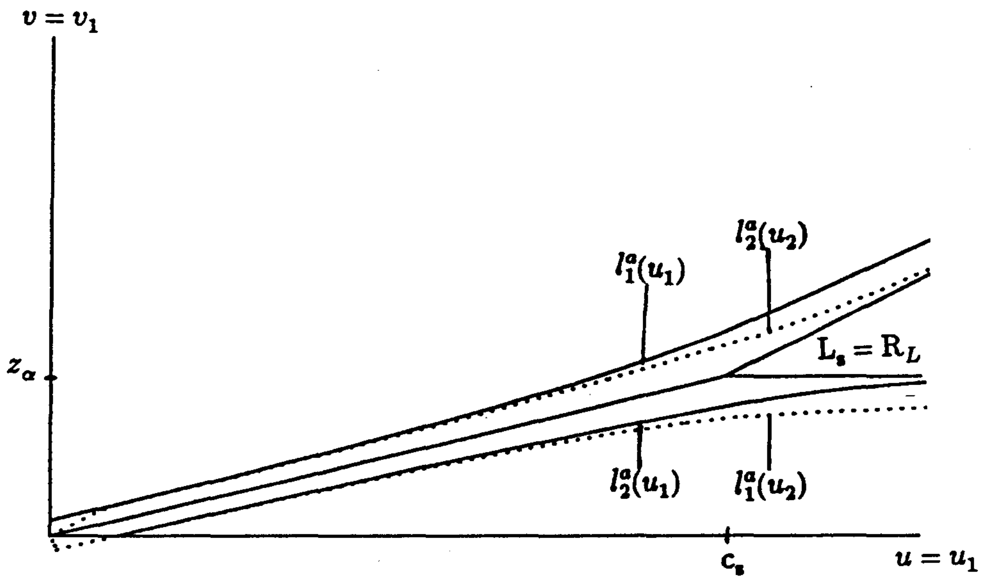

Figure 6: Rejection region of ϕ_a.

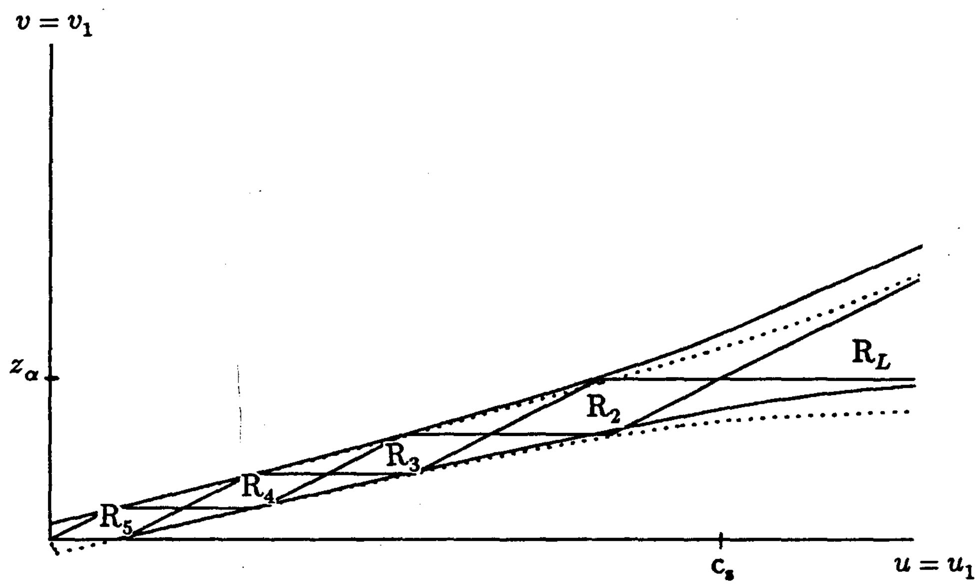

Figure 7: Rejection regions of ϕ_a and ϕ_b.