Estimation of Ratios of Variance Components in the Unbalanced One Way Random Effects Model

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Abstract

The point estimation of the ratio of variance components and the intraclass correlation coefficient for the unbalanced one way random effects model are considered. ANOVA, REML, ML, MIVQUE, MINQE type, and MU (median unbiased) estimators are compared with respect to their absolute biases and mean squared errors through a simulation study. Explicit, computable expressions with no matrix inversion necessary are given for these estimators. Our results indicate that the two types of MINQE estimators are excellent for estimating the ratio of variance components for all around performance and good for the intraclass correlation coefficient when the value of true ratio of variance components is thought to be low.

Key Words: Ratios of variance component; Mean squared error.

*This work was accomplished when he was a visiting professor of dept. of statistics, North Carolina State University. He is an associate professor at DanKook University in Seoul, Korea.

1 Introduction

The model for the one-way classification for the jth observation in the ith group is

$$y_{ij} = \mu + a_i + e_{ij}, \tag{1.1}$$

where μ is an unknown parameter, a_i and e_{ij} are mutually independent normal random variables with zero means and variance σ_a^2 and σ_e^2, respectively, $i = 1, \ldots, k$ with $k \geq 2$, $j = 1, \ldots, n_i$ and $N = \sum_i^k n_i$. When this model is analyzed, the ratio of the variance components $\theta = \sigma_a^2/\sigma_e^2$ and the intraclass correlation coefficient $\rho = \sigma_a^2/(\sigma_a^2 + \sigma_e^2)$ are often parameters of interest in applied work such as genetics, breeding, and certain industrial work (Graybill, 1966).

The problem of estimating θ in the balanced one way random effects model was considered by Loh (1986). Chaloner (1987) provided a simulation study for the estimation of θ in the unbalanced one way random effects model with a Bayesian approach. Lee and Lee (1990) gave a more extended study for the problem of estimating θ in the balanced one way random effects model. They recommended that the MSE of a MINQE type and an improved estimator have smaller than the those of other estimators that have been developed during the past 20 years. Also, an unified solution to the problem of estimating θ in the balanced one way random effects model can be found in Das (1992).

Point estimation of the intraclass correlation coefficient under the one way random effects model was investigated by Donner (1980). He recommended that maximum likelihood estimator be used if no prior knowledge concerning the value of ρ exists, or if the value of ρ is thought to be high. Palmer and Broemeling (1990) presented a simulation study of estimation of the intraclass correlation coefficient ρ under the one way random effects models. Their results showed that the Bayes median estimator would be preferred over the maximum likelihood estimator unless ρ was small or when one or more classes consisted of only one observation. Confidence intervals for θ and ρ in the unbalanced one-way random effects model are well examined by Donner and Wells (1986) using approximate methods, and Burdick et al. (1986) using exact methods.

In this paper, we consider eight popular point estimators of variance components. These are ANOVA estimator, two of MIVQUE's, ML estimator, REML estimator, two of MINQE's, and MU (median unbiased) estimator. For a review on this subject one may be refer to the recent book by Rao and Kleffe (1988). In Section 2, explicit, computable expressions with no matrix inversion necessary for these estimators are investigated. Section 3 describes the simulation study performed in this article to compare eight estimators. Also, the absolute biases and mean squared errors of the estimates of θ and ρ for eight estimators are provided. Section 4 gives some comparisons of other procedures. Finally, Section 5 presents some remarks and conclusions.

2 Model and Estimators

The model (1.1) may be written in matrix notation as

$$\mathbf{y} = 1_N\mu + Z_1\mathbf{a} + Z_2\mathbf{e},$$

where $\mathbf{a}' = (a_1, ..., a_k)$, $\mathbf{e}' = (e_{11}, e_{12}, ..., e_{kn_k})$, 1_N an N-vector with all elements unity, the $N \times k$ matrix $Z_1 = \Sigma^+ 1_{n_i}$, $Z_2 = I_N$, and Σ^+ denotes a direct sum of matrices.

Thus $\mathbf{y}$ is a vector of multivariate normal random variables with mean $1_N\mu$ and variance matrix

$$V = \sigma_e^2 I_N + \sigma_a^2 Z_1 Z_1'. \tag{2.1}$$

When σ_a^2 and σ_e^2 are replaced in (2.1) by "estimates" $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$, we have

$$\tilde{V} = \hat{\sigma}_e^2 I_N + \hat{\sigma}_a^2 Z_1 Z_1'. \tag{2.2}$$

Let

$$P = \tilde{V}^{-1}(I_N - 1_N(1_N'\tilde{V}^{-1}1_N)^{-1}1_N'\tilde{V}^{-1}), \tag{2.3}$$

$$S = \{s_{ij}\} = \mathrm{tr}(\tilde{P}Z_iZ_i'\tilde{P}Z_jZ_j'), \quad i,\ j = 1, 2, \tag{2.4}$$

$$Q = \{q_{ij}\} = \mathrm{tr}(\tilde{V}^{-1}Z_iZ_i'\tilde{V}^{-1}Z_jZ_j'), \quad i,\ j = 1, 2, \tag{2.5}$$

and

$$\mathbf{u} = \{u_i\} = \mathbf{y}'\tilde{P}Z_iZ_i'\tilde{P}\mathbf{y}, \quad i = 1, 2. \tag{2.6}$$

Also, we use the following notations throughout this section.

$$m_i = n_i/(1 + n_i r), \quad m = 1/\sum m_i,$$

$$\bar{y}_{i.} = \sum_j y_{ij}/n_i \quad \text{and} \quad \tilde{y}_{..} = (\sum_{i=1}^{k} m_i \bar{y}_{i.})/(\sum_{i=1}^{k} m_i), \tag{2.7}$$

where r is a prior chosen values of θ.
The main variance component function to be estimated in this paper is $\theta = \sigma_a^2/\sigma_e^2$. Note that the intraclass correlation coefficient $\rho = \theta/(1+\theta)$ can be regarded as a nonlinear function of θ. In order to provide desirable point estimators of θ, we consider natural estimators, the ratio of the readily available estimators of σ_a^2 and σ_e^2. Six methods of estimating the variance components are considered as follows.

2.1 Analysis of Variance (ANOVA) Estimator

The ANOVA estimators are obtained by equating the among and within groups mean squares from an analysis of variance to their expectation, and solving the resulting equations for $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$. This gives

$$\hat{\sigma}_e^2 = \frac{\sum_i \sum_j (y_{ij} - \bar{y}_{i.})^2}{(N-k)} \quad \text{and} \quad \hat{\sigma}_a^2 = \frac{[\sum n_i(\bar{y}_{i.} - \bar{y}_{..})^2 - (k-1)\hat{\sigma}_e^2]}{(N - \sum_i n_i^2/N)}, \tag{2.8}$$

where $\bar{y}_{i.}$ and $\bar{y}_{..}$ are the ith group mean and grand mean, respectively. They are uniformly minimum variance unbiased estimates for balanced data. However, this property is lost in unbalanced cases. Since ANOVA estimators of σ_e^2 and σ_a^2 are given in (2.8), respectively, it is natural to define the ANOVA estimator of θ as $\hat{\theta} = \hat{\sigma}_a^2/\hat{\sigma}_e^2$.

2.2 Restricted Maximum Likelihood (REML) Estimator

The REML estimators $\hat{\sigma}^2(\mathbf{r}) = (\hat{\sigma}_a^2, \hat{\sigma}_e^2)'$ obtained as the solution of REML equations solved iteratively are given by

$$\hat{\sigma}^2(\mathbf{r}) = S^{-1}\mathbf{u}. \tag{2.9}$$

Simplifying the elements of matrix expressions (2.9), we use the following expressions which are a modified version of a result proved in Swallow and Searle (1978).

$s_{11}\sigma_e^4 = \sum m_i^2 - 2m \sum m_i^3 + m^2(\sum m_i^2)^2$,

$s_{12}\sigma_e^4 = \sum (m_i^2/n_i) - 2m \sum (m_i^3/n_i) + m^2 \sum m_i^2 \sum (m_i^2/n_i),$
$s_{22}\sigma_e^4 = N - k + \sum (m_i^2/n_i^2) - 2m \sum (m_i^3/n_i^2) + m^2(\sum (m_i^2/n_i))^2,$
$u_1\sigma_e^4 = \sum m_i^2[\bar{y}_{i.} - m \sum m_i\bar{y}_{i.}]^2,$
and
$u_2\sigma_e^4 = [\sum\sum y_{ij}^2 - \sum n_i\bar{y}_{i.}^2] + \sum (m_i^2/n_i)[\bar{y}_{i.} - m \sum m_i\bar{y}_{i.}]^2.$
For the simplicity of notation, define $s_{ij}^* = s_{ij}\sigma_e^4$ and $u_i^* = u_i\sigma_e^4$ for i, j = 1, 2. Then, the first step REML of θ is

$$\hat{\theta} = (s_{22}^*u_1^* - s_{12}^*u_2^*)/(s_{11}^*u_2^* - s_{12}^*u_1^*). \tag{2.10}$$

Thereafter, r for each iteration is $\hat{\theta}$ from the previous iteration, and we continue iteration procedure until the convergence criterion is satisfied.

2.3 Maximum Likelihood (ML) Estimator

The ML estimators $\hat{\sigma}^2(\mathbf{m}) = (\hat{\sigma}_a^2, \hat{\sigma}_e^2)'$ can be obtained by iterative solutions of ML equations given by

$$\hat{\sigma}^2(\mathbf{m}) = Q^{-1}\mathbf{u}. \tag{2.11}$$

The expressions needed in (2.11) for computing ML estimators are:
$q_{11}^* = \sum m_i^2, \qquad q_{12}^* = \sum(m_i^2/n_i), \qquad q_{22}^* = N - 2r\sum m_i + r^2 \sum m_i{}^2,$
$u_1^* = \sum m_i^2(\bar{y}_{i.} - m \sum m_i\bar{y}_i)^2,$
and
$u_2^* = (\sum\sum y_{ij}^2 - \sum n_i\bar{y}_i^2) + \sum(m_i^2/n_i)(\bar{y}_{i.} - m \sum m_i\bar{y}_{i.})^2.$
Then, the first step MLE of θ is

$$\hat{\theta} = (q_{22}^*u_1^* - q_{12}^*u_2^*)/(q_{11}^*u_2^* - q_{12}^*u_1^*). \tag{2.12}$$

The relationships among q_{ij}^*, q_{ij}, u_i^*, and u_i are $q_{ij}^* = q_{ij}\sigma_e^4$ and $u_i^* = u_i\sigma_e^4$ for i, j = 1, 2, respectively.
The procedure then iterates until the convergence criterion is satisfied. Note that the ML estimator of θ is a direct function of the ML estimators of σ_a^2 and σ_e^2.

2.4 MIVQUE

Minimum variance quadratic unbiased estimators (MIVQUE) are given by Rao (1971). Also Swallow and Searle (1978) consider these estimators precisely. In applications of MIVQUE, two MIVQUE's are considered generally.

The one is the MIVQUE(A) that uses equation (2.8) without iteration, inserting the ANOVA estimates of $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$ in a prior value r for θ. The MIVQUE(A)'s are the first iterates of the REML estimators of section 2.2. The other is MIVQUE(0) by virtue of their inclusion as the default estimators in SAS's procedure VARCOMP. MIVQUE(0) estimators are defined as those obtained from equation (2.10) without iteration using $r = 0$. However, MIVQUE(0) is poor for estimating σ_a^2 and very poor for σ_e^2, even for just mildly unbalanced data (Swallow and Monahan, 1984).

2.5 MINQE

The MINQE are the minimum norm quadratic estimators without the condition of unbiasedness. The explicit forms of MINQE for σ_a^2 and σ_e^2 were given Ahrens et al. (1981) as

$$\hat{\sigma}_a^2 = r^2 \sum_{i=1}^{k} m_i{}^2 (\bar{y}_{i.} - \tilde{y}_{..})^2 / k \tag{2.13}$$

and

$$\hat{\sigma}_e^2 = (\sum_{i}^{k} \sum_{j}^{n_i} (y_{ij} - \bar{y}_{i.})^2 + \sum_{i=1}^{k} (m_i^2 / n_i)(\bar{y}_{i.} - \tilde{y}_{..})^2) / , \tag{2.14}$$

where r is a prior guess of θ. Since MINQE of σ_a^2 and σ_e^2 are given in (2.13) and (2.14), respectively, we define MINQE of θ as $\hat{\theta} = \hat{\sigma}_a^2 / \hat{\sigma}_e^2$. Another modification of MINQE for estimating θ is using the equation (2.7) instead of (2.13) as estimator of $\hat{\sigma}_e^2$, which we shall denote by MINQE(A). We note that ANOVA estimator of σ_e^2 are one of the best estimators.

From the point of view of a smaller MSE, Conerly and Webster (1987) pointed out that the MSE of MINQE is smaller than that of the MLE when $\theta > 1$. Moreover, the MINQE is a better estimator of σ_a^2 than any unbiased quadratic estimator $\theta > 0.5$. Lee and Lee (1990) also recommended that the MINQE type estimator could provide useful estimator of ratio θ for the balanced one-way random model.

2.6 MU Estimator

Since no unbiased estimator of θ is known, one possibility is to use the median unbiased estimator (MUE) of θ. Burdick et al. (1986) pointed out the possibility of using this for the point estimator of θ. Median unbiased estimates of θ is the solution of Wald's (1940) equation

$$Z(\theta) = \sum_{i=1}^{k} m_i(\bar{y}_{i.} - \tilde{y}_{..})^2 / [(k-1)MSE] = F(0.5; k-1, N-k), \quad (2.15)$$

where MSE is the usual within groups mean squares and $F(\delta; v_1, v_2)$ represents the upper δ probability point of an F-distribution with v_1 and v_2 degrees of freedom. But MU estimator has the disadvantages, in comparison to other methods, of requiring an iterative solution of non - linear equations.

3 Simulation Study

A simulation study involving several unbalanced designs was conducted to compare the MSE and the bias of several estimators described in section 2. The procedures of a simulation were conducted as being similar to those used in Burdick et al. (1986). For any particular estimator, the MSE and bias of estimator depends on the value of θ. The value of θ selected in the study were 0.1, 0.2, 0.5, 1.0, 2.0 and 5.0. The comparison of estimators of the parameter θ for unbalanced data required a variety of patterns of imbalance, numbers of classes, and a full range of the values of θ. To overcome this problem, the concept of N-patterns used in previous researach, such as Swallow and Monahan (1978), Burdick and Graybill (1984) and Chaloner (1987), is introduced. These N-patterns are listed in the first column of Table 1.

For all data sets, each value was calculated using 10000 replications of each of the 13 model N-pattern and six values of θ. Therefore, 780,000 distinct data sets were created. Without loss of generality, the value $\sigma_a^2 + \sigma_e^2 = 1$ was assigned and μ was set at 0. Also, we generate simulated data sets using RANNOR and RANGAM function of SAS.

ML and REML procedures iterates until their log-likelihood objective function converge. Both ML and REML were allowed up to 50 iterations to converge. Convergence was said to have occurred when the estimates at the kth and $(k+1)$th iterations satisfied

$$|\hat{\theta}_{k+1} - \hat{\theta}_k| < 10^{-8}.$$

We use the ANOVA estimates as the starting point when ML and REML estimates can be calculated. Also, prior guess value $r = 1$ was used on

MINQE's. As we know, this is the common choice of prior weigt in MINQUE theory.

3.1 Estimation of θ

The MSE and absolute Bias of eight estimators of θ are shown in Table 1 and Table 2. These are ratios of MSE and absolute biases to the MIVQUE(A) estimators. The efficiency of MIVQUE(0) is not reported because they give the worst efficiency among eight estimators. The results of our investigation for estimating θ are as follows:

1. Two types of MINQE estimators consistently have the smallest MSE and absolute Bias. The MINQE(A) has considerably smaller MSE and absolute Bias than other estimators, an the MINQE comes next when θ is small. When θ is large, the MINQE is best and the MINQE(A) comes next. The absolute bias behavior of two MINQE's repoted is surprising. We know that the MINQE have nontrivial bias generally (Conerly and Webster 1987).

2. The MSE advantage of ANOVA over MIVQUE(A) estimators depends on the value of θ, the number of classes, and unbalance of design. The MSE superiority of ANOVA esitmator increases when θ is very small, the number of classes is small and the data are severly unbalanced. The size of absolute bias for ANOVA is similar to that of MIVQUE(A). It decreases as θ increases generally.

3. MIVQUE(0) performs well when θ is nearly 0. MIVQUE(0) has no advantage and should be used only when θ is sufficiently small. When $\theta \geq 0.2$, MIVQUE(0) performs very poorly, even in mildly unbalanced designs and it may have a tremendous MSE and absolute bias as θ becomes large. This is similar to the results of Swallow and Monahan (1984) but the degree of poor efficiency for estimating ratio of variance components are higher than those of estimating individual variance components.

4. There is no apparent advantage of REML over MIVQUE(A), the first iterate of REML. For all around performance, MIVQUE(A) should be preferred to REML estimators of the parameter θ except $\theta = 5.0$ under

Table 1. Ratios of MSE of Estimators of θ to the MIVQUE(A).

Name	Estimators	θ 0.1	0.2	0.5	1.0	2.0	5.0
P1 = (3,5,7)	ANOVA	0.947	0.943	1.026	1.005	1.024	1.032
	REML	1.018	1.015	1.010	1.006	1.004	1.001
	ML	0.357	0.389	0.430	0.443	0.451	0.456
	MINQE	0.364	0.290	0.242	0.219*	0.215*	0.206*
	MINQE(A)	0.244*	0.209*	0.213*	0.230	0.256	0.285
	MU	7.516	5.511	3.543	2.792	2.377	2.150
P2 = (1,5,9)	ANOVA	0.646	0.651	0.969	1.027	1.150	1.242
	REML	1.264	1.221	1.132	1.102	1.066	0.997
	ML	0.365	0.405	0.442	0.464	0.471	0.468
	MINQE	0.202	0.145	0.145	0.150*	0.179*	0.216*
	MINQE(A)	0.156*	0.123*	0.143*	0.170	0.214	0.263
	MU	8.534	6.349	4.068	3.144	2.580	2.222
P3 = (1,7,7)	ANOVA	0.635	0.644	0.978	1.041	1.167	1.262
	REML	1.298	1.250	1.143	1.110	1.071	0.998
	ML	0.372	0.414	0.446	0.466	0.473	0.465
	MINQE	0.283	0.284	0.354	0.383	0.427	0.474
	MINQE(A)	0.215*	0.220*	0.272*	0.302*	0.340*	0.383*
	MU	8.919	6.589	4.113	3.168	2.591	2.226
P4 = (3,3,5,5,7,7)	ANOVA	0.950	0.977	1.001	1.037	1.077	1.059
	REML	1.022	1.023	1.018	1.011	1.004	1.001
	ML	0.595	0.647	0.688	0.697	0.700	0.698
	MINQE	0.894	0.608	0.430	0.394	0.390*	0.386*
	MINQE(A)	0.539*	0.375*	0.329*	0.377*	0.441	0.495
	MU	7.056	4.219	2.055	1.487	1.336	1.309
P5 = (1,1,5,5,9,9)	ANOVA	0.810	0.865	0.979	1.116	1.242	1.254
	REML	1.260	1.214	1.162	1.113	1.061	1.025
	ML	0.630	0.682	0.733	0.747	0.736	0.715
	MINQE	0.721	0.434	0.295	0.312*	0.372*	0.447*
	MINQE(A)	0.480*	0.297*	0.257*	0.330	0.426	0.510
	MU	10.34	6.152	2.923	1.856	1.459	1.347
P6 = (1,1,7,7,7,7)	ANOVA	0.907	0.886	1.021	1.090	1.172	1.171
	REML	1.198	1.201	1.143	1.113	1.067	1.030
	ML	0.583	0.668	0.731	0.751	0.740	0.725
	MINQE	0.523	0.499	0.555	0.606	0.671	0.725
	MINQE(A)	0.385*	0.378*	0.441*	0.498*	0.561*	0.616*
	MU	11.50	6.505	2.904	1.856	1.482	1.349
P7 = (1,1,1,1,13,13)	ANOVA	0.586	0.741	1.013	1.392	1.708	1.914
	REML	1.566	1.489	1.369	1.222	1.108	1.028
	ML	0.695	0.736	0.794	0.798	0.769	0.722
	MINQE	0.427	0.249	0.167*	0.233*	0.357*	0.540*
	MINQE(A)	0.312*	0.189*	0.170	0.266	0.401	0.552
	MU	7.701	5.399	3.044	2.003	1.540	1.363

* means the best choice.

Table 1. *(Continued)*

Name	Estimators	θ 0.1	0.2	0.5	1.0	2.0	5.0
P8 = (3,3,3,5,5,5,7,7,7)	ANOVA	0.979	0.995	1.023	1.034	1.052	1.070
	REML	1.015	1.023	1.017	1.010	1.004	1.001
	ML	0.702*	0.755	0.796	0.795	0.798	0.793
	MINQE	1.376	0.849	0.510	0.457	0.472*	0.514*
	MINQE(A)	0.785	0.476*	0.374*	0.445*	0.556	0.642
	MU	8.186	4.307	1.768	1.257	1.179	1.180
P9 = (1,1,1,5,5,5,9,9,9)	ANOVA	0.929	0.963	1.028	1.100	1.175	1.262
	REML	1.080	1.104	1.097	1.087	1.049	1.018
	ML	0.673*	0.753	0.825	0.841	0.831	0.807
	MINQE	1.260	0.680	0.358	0.373*	0.475*	0.636*
	MINQE(A)	0.782	0.417*	0.305*	0.409	0.565	0.710
	MU	14.44	8.356	3.031	1.679	1.295	1.206
P10 = (1,1,1,7,7,7,7,7,7)	ANOVA	0.952	0.965	1.013	1.064	1.125	1.198
	REML	1.067	1.087	1.094	1.084	1.050	1.019
	ML	0.669	0.743	0.821	0.838	0.831	0.807
	MINQE	0.626	0.593	0.618	0.676	0.757	0.831
	MINQE(A)	0.458*	0.449*	0.512*	0.586*	0.671*	0.723*
	MU	15.25	8.586	3.107	1.686	1.299	1.207
P11 = (1,1,1,1,1,1,1,19,19)	ANOVA	0.623	0.815	1.185	1.672	1.949	2.673
	REML	1.632	1.577	1.374	1.213	1.080	1.010
	ML	0.837	0.903	0.952	0.933	0.865	0.806*
	MINQE	0.672	0.333	0.168*	0.285*	0.522*	0.915
	MINQE(A)	0.470*	0.234*	0.182	0.340	0.581	0.890
	MU	7.295	5.107	2.662	1.723	1.317	1.208
P12 = (2,10,18)	ANOVA	0.810	0.889	1.156	1.197	1.243	1.340
	REML	1.250	1.207	1.113	1.066	1.036	1.011
	ML	0.383	0.448	0.508	0.518	0.512	0.503
	MINQE	0.405	0.314	0.302	0.320	0.334*	0.337*
	MINQE(A)	0.331*	0.267*	0.282*	0.315*	0.345	0.376
	MU	13.23	8.261	4.353	3.160	2.621	2.313
P13 = (3,15,27)	ANOVA	0.913	0.997	1.250	1.236	1.256	1.395
	REML	1.232	1.169	1.083	1.044	1.021	1.006
	ML	0.429	0.491	0.530	0.536	0.525	0.520
	MINQE	0.472	0.372	0.365	0.384	0.394*	0.405*
	MINQE(A)	0.409*	0.333*	0.346*	0.377*	0.397	0.427
	MU	16.17	8.735	4.237	3.092	2.593	2.334

* means the best choice.

Table 2. Ratios of Absolute Bias of Estimators of θ to the MIVQUE(A).

Name	Estimators	θ 0.1	0.2	0.5	1.0	2.0	5.0
P1 = (3,5,7)	ANOVA	0.991	0.991	1.007	0.998	1.000	1.007
	REML	1.011	1.013	1.008	1.004	1.001	1.000
	ML	0.660	0.721	0.773	0.789	0.792	0.796
	MINQE	0.805	0.519*	0.315*	0.193*	0.119*	0.055*
	MINQE(A)	0.551*	0.521	0.574	0.630	0.677	0.721
	MU	6.347	6.374	6.726	8.973	12.67	25.15
P2 = (1,5,9)	ANOVA	0.919	0.931	1.011	1.001	1.023	1.049
	REML	1.086	1.101	1.084	1.058	1.033	1.002
	ML	0.613	0.705	0.799	0.821	0.816	0.804
	MINQE	0.438*	0.282*	0.219*	0.154*	0.107*	0.056*
	MINQE(A)	0.503	0.460	0.522	0.588	0.654	0.718
	MU	11.68	11.49	9.724	11.49	14.78	26.62
P3 = (1,7,7)	ANOVA	0.917	0.930	1.014	1.003	1.028	1.054
	REML	1.097	1.112	1.090	1.062	1.035	1.003
	ML	0.620	0.711	0.802	0.822	0.817	0.804
	MINQE	0.469*	0.361*	0.301*	0.205*	0.136*	0.067*
	MINQE(A)	0.549	0.581	0.662	0.703	0.741	0.774
	MU	11.77	11.68	9.688	11.46	14.75	26.58
P4 = (3,3,5,5,7,7)	ANOVA	0.997	1.003	1.001	1.007	1.019	1.023
	REML	1.012	1.016	1.011	1.005	1.001	1.000
	ML	0.824	0.877	0.894	0.898	0.897	0.894
	MINQE	2.775	1.602	0.830	0.503*	0.293*	0.129*
	MINQE(A)	0.769*	0.616*	0.631*	0.712	0.793	0.850
	MU	2.604	2.043	1.685	2.061	3.331	7.673
P5 = (1,1,5,5,9,9)	ANOVA	0.972	0.988	1.004	1.028	1.065	1.090
	REML	1.036	1.062	1.068	1.051	1.023	1.006
	ML	0.796	0.877	0.925	0.933	0.919	0.900
	MINQE	2.207	1.221	0.655	0.438*	0.283*	0.140*
	MINQE(A)	0.775*	0.583*	0.589*	0.692	0.800	0.889
	MU	4.667	3.566	2.685	2.760	3.800	8.070
P6 = (1,1,7,7,7,7)	ANOVA	0.991	0.990	1.013	1.030	1.049	1.065
	REML	1.023	1.055	1.064	1.048	1.026	1.006
	ML	0.788	0.872	0.932	0.938	0.920	0.910
	MINQE	2.004	1.398	0.898	0.565*	0.347*	0.155*
	MINQE(A)	0.691*	0.703*	0.771*	0.828	0.862	0.908
	MU	4.501	3.600	2.632	2.815	3.803	8.099
P7 = (1,1,1,1,13,13)	ANOVA	0.891	0.949	1.005	1.066	1.149	1.236
	REML	1.155	1.208	1.183	1.096	1.031	1.001
	ML	0.781	0.908	0.992	0.972	0.934	0.900
	MINQE	1.239	0.721	0.435*	0.355*	0.268*	0.155*
	MINQE(A)	0.711*	0.511*	0.517	0.654	0.799	0.951
	MU	5.481	4.483	3.478	3.383	4.299	8.422

* means the best choice.

Table 2. *(Continued)*

Name	Estimators	θ = 0.1	0.2	0.5	1.0	2.0	5.0
P8 = (3,3,3,5,5,5,7,7,7)	ANOVA	1.000	1.003	1.005	1.010	1.020	1.022
	REML	1.010	1.015	1.009	1.004	1.001	1.000
	ML	0.884	0.921	0.938	0.934	0.935	0.931
	MINQE	5.101	2.705	1.229	0.723*	0.426*	0.196*
	MINQE(A)	0.925*	0.668*	0.652*	0.754	0.865	0.938
	MU	2.048	1.433	1.026	1.262	2.144	5.022
P9 = (1,1,1,5,5,5,9,9,9)	ANOVA	0.995	1.007	1.012	1.032	1.059	1.088
	REML	1.011	1.039	1.044	1.035	1.018	1.003
	ML	0.859*	0.921	0.961	0.962	0.950	0.935
	MINQE	4.607	2.281	0.986	0.641*	0.422*	0.220*
	MINQE(A)	0.953	0.646*	0.613*	0.746	0.894	1.008
	MU	3.980	3.064	1.923	1.798	2.463	5.252
P10 = (1,1,1,7,7,7,7,7,7)	ANOVA	1.001	1.006	1.010	1.027	1.045	1.069
	REML	1.004	1.027	1.041	1.036	1.018	1.004
	ML	0.855	0.911	0.957	0.962	0.951	0.936
	MINQE	3.123	2.265	1.342	0.835*	0.492*	0.222*
	MINQE(A)	0.733*	0.737*	0.809*	0.873	0.928	0.952
	MU	4.081	3.081	1.949	1.796	2.460	5.246
P11 = (1,1,1,1,1,1,1,19,19)	ANOVA	0.904	0.968	1.029	1.128	1.231	1.389
	REML	1.187	1.260	1.176	1.074	1.014	0.994
	ML	0.866*	1.013	1.056	1.008	0.956	0.928
	MINQE	2.242	1.142	0.570	0.500*	0.417*	0.265*
	MINQE(A)	0.883	0.540*	0.508*	0.702	0.939	1.176
	MU	3.628	2.973	2.191	2.169	2.796	5.532
P12 = (2,10,18)	ANOVA	0.969	0.989	1.041	1.036	1.051	1.085
	REML	1.097	1.095	1.056	1.026	1.009	1.000
	ML	0.732	0.816	0.854	0.849	0.836	0.828
	MINQE	1.386	0.936	0.577*	0.349*	0.196*	0.084*
	MINQE(A)	0.642*	0.601*	0.653	0.701	0.742	0.787
	MU	6.734	5.604	4.974	6.298	9.728	20.95
P13 = (3,15,27)	ANOVA	0.996	1.009	1.057	1.058	1.061	1.091
	REML	1.098	1.080	1.037	1.014	1.003	0.999
	ML	0.799	0.852	0.860	0.853	0.843	0.838
	MINQE	2.146	1.326	0.719	0.427*	0.228*	0.097*
	MINQE(A)	0.688*	0.656*	0.700*	0.744	0.775	0.806
	MU	5.489	4.462	4.158	5.373	8.712	18.92

* means the best choice.

Table 3. Estimated MSE of Estimators of ρ

		θ					
Name	Estimators	0.1	0.2	0.5	1.0	2.0	5.0
P1=(3,5,7)	ANOVA	0.0309	0.0414	0.0655	0.0905	0.1046	0.0908
	MIVQUE(A)	0.0312	0.0416	0.0651	0.0902	0.1044	0.0904
	MIVQUE(0)	0.0313	0.0425	0.0705	0.0985	0.1178	0.1090
	REML	0.0316	0.0423	0.0659	0.0909	0.1046	0.0903
	ML	0.0173	0.0305	0.0671	0.1094	0.1386	0.1273
	MINQE	0.0188	0.0214	0.0380*	0.0690	0.1001	0.1069
	MINQE(A)	0.0138*	0.0178*	0.0399	0.0780	0.1154	0.1234
	MU	0.1563	0.1307	0.0781	0.0493*	0.0379*	0.0327*
P2 = (1,5,9)	ANOVA	0.0353	0.0452	0.0713	0.1010	0.1220	0.1095
	MIVQUE(A)	0.0377	0.0468	0.0699	0.1001	0.1217	0.1079
	MIVQUE(0)	0.0331	0.0457	0.0820	0.1228	0.1629	0.1758
	REML	0.0411	0.0516	0.0761	0.1064	0.1258	0.1087
	ML	0.0200	0.0346	0.0778	0.1322	0.1729	0.1616
	MINQE	0.0205	0.0209	0.0378*	0.0743*	0.1156	0.1311
	MINQE(A)	0.0160*	0.0183*	0.0407	0.0840	0.1309	0.1464
	MU	0.2892	0.2757	0.1955	0.1549	0.1056*	0.0720*
P3 = (1,7,7)	ANOVA	0.0337	0.0439	0.0705	0.1001	0.1212	0.1091
	MIVQUE(A)	0.0362	0.0455	0.0688	0.0993*	0.1209	0.1074
	MIVQUE(0)	0.0319	0.0448	0.0815	0.1225	0.1634	0.1780
	REML	0.0400	0.0507	0.0753	0.1055	0.1248	0.1079
	ML	0.0193	0.0338	0.0763	0.1300	0.1699	0.1591
	MINQE	0.0199	0.0294	0.0608	0.1046	0.1420	0.1390
	MINQE(A)	0.0167*	0.0264*	0.0604*	0.1091	0.1521	0.1523
	MU	0.2977	0.2777	0.2085	0.1453	0.1043*	0.0701*
P4 = (3,3,5,5,7,7)	ANOVA	0.0172	0.0247	0.0394	0.0459	0.0436	0.0258
	MIVQUE(A)	0.0173	0.0245	0.0392	0.0458	0.0432	0.0252
	MIVQUE(0)	0.0178	0.0266	0.0447	0.0565	0.0608	0.0475
	REML	0.0176	0.0251	0.0399	0.0461	0.0431	0.0250
	ML	0.0126	0.0215	0.0417	0.0538	0.0542	0.0333
	MINQE	0.0174	0.0155	0.0200*	0.0303*	0.0399	0.0340
	MINQE(A)	0.0115*	0.0111*	0.0209	0.0376	0.0518	0.0433
	MU	0.1090	0.0874	0.0536	0.0378	0.0295*	0.0181*
P5 = (1,1,5,5,9,9)	ANOVA	0.0184	0.0263	0.0430	0.0527	0.0535	0.0337
	MIVQUE(A)	0.0191	0.0265	0.0422	0.0522	0.0530	0.0319
	MIVQUE(0)	0.0182	0.0286	0.0521	0.0723	0.0859	0.0781
	REML	0.0201	0.0286	0.0457	0.0552	0.0538	0.0309*
	ML	0.0134	0.0235	0.0480	0.0659	0.0697	0.0427
	MINQE	0.0184	0.0147	0.0195*	0.0344*	0.0512*	0.0484
	MINQE(A)	0.0129*	0.0111*	0.0216	0.0429	0.0637	0.0572
	MU	107.70	32.286	26.687	10.332	5.5942	0.1540
P6 = (1,1,7,7,7,7)	ANOVA	0.0176	0.0259	0.0412	0.0531	0.0527	0.0331
	MIVQUE(A)	0.0179	0.0261	0.0403	0.0523*	0.0525*	0.0317
	MIVQUE(0)	0.0169	0.0267	0.0464	0.0639	0.0725	0.0619
	REML	0.0186	0.0280	0.0436	0.0552	0.0536	0.0310*
	ML	0.0123	0.0230	0.0460	0.0658	0.0692	0.0426
	MINQE	0.0122	0.0190	0.0365*	0.0561	0.0637	0.0437
	MINQE(A)	0.0100*	0.0169*	0.0372	0.0616	0.0732	0.0527
	MU	40.027	151.61	79.728	255.58	4.7196	0.5201
P7 = (1,1,1,1,13,13)	ANOVA	0.0223	0.0307	0.0515	0.0678	0.0753	0.0501
	MIVQUE(A)	0.0258	0.0323	0.0504	0.0672	0.0731	0.0438
	MIVQUE(0)	0.0212	0.0360	0.0741	0.1164	0.1599	0.1816
	REML	0.0312	0.0406	0.0606	0.0727	0.0719	0.0400*
	ML	0.0183	0.0305	0.0643	0.0929	0.1022	0.0602
	MINQE	0.0208	0.0145	0.0193*	0.0415*	0.0703*	0.0747
	MINQE(A)	0.0155*	0.0116*	0.0226	0.0510	0.0832	0.0817
	MU	1.4652	2.3543	2.9181	1.2688	0.9073	0.1671

* means the best choice.

Table 3. *(Continued)*

Name	Estimators	θ 0.1	0.2	0.5	1.0	2.0	5.0
P8 = (3,3,3,5,5,5,7,7,7)	ANOVA	0.0121	0.0181	0.0275	0.0307	0.0255	0.0131
	MIVQUE(A)	0.0122	0.0180	0.0273	0.0304	0.0251	0.0126
	MIVQUE(0)	0.0127	0.0195	0.0326	0.0402	0.0393	0.0309
	REML	0.0124	0.0184	0.0278	0.0305	0.0249	0.0125
	ML	0.0098*	0.0167	0.0293	0.0347	0.0301	0.0158
	MINQE	0.0164	0.0135	0.0133*	0.0194*	0.0243	0.0193
	MINQE(A)	0.0102	0.0087*	0.0137	0.0256	0.0340	0.0256
	MU	0.0991	0.0824	0.0512	0.0327	0.0232*	0.0108*
P9 = (1,1,1,5,5,5,9,9,9)	ANOVA	0.0128	0.0191	0.0306	0.0357	0.0314	0.0172
	MIVQUE(A)	0.0130	0.0189	0.0300	0.0351	0.0307*	0.0155
	MIVQUE(0)	0.0128	0.0209	0.0384	0.0517	0.0559	0.0503
	REML	0.0133	0.0201	0.0319	0.0366	0.0308	0.0147*
	ML	0.0101*	0.0178	0.0339	0.0424	0.0382	0.0190
	MINQE	0.0171	0.0126	0.0128*	0.0229*	0.0332	0.0298
	MINQE(A)	0.0113	0.0087*	0.0144	0.0304	0.0439	0.0357
	MU	23.645	30.991	23.616	11.452	1.1366	0.0630
P10 = (1,1,1,7,7,7,7,7,7)	ANOVA	0.0126	0.0188	0.0300	0.0347	0.0304	0.0165
	MIVQUE(A)	0.0127	0.0186	0.0295	0.0343*	0.0301*	0.0152*
	MIVQUE(0)	0.0120	0.0189	0.0333	0.0421	0.0417	0.0318
	REML	0.0128	0.0195	0.0313	0.0358	0.0303	0.0145
	ML	0.0097	0.0172	0.0330	0.0413	0.0375	0.0188
	MINQE	0.0094	0.0141	0.0266*	0.0371	0.0381	0.0220
	MINQE(A)	0.0076*	0.0125*	0.0278	0.0425	0.0468	0.0284
	MU	28.266	27.576	12.691	12.155	1.0990	0.0610
P11 = (1,1,1,1,1,1,1,19,19)	ANOVA	0.0171	0.0252	0.0430	0.0572	0.0568	0.0361
	MIVQUE(A)	0.0200	0.0264	0.0421	0.0554	0.0523	0.0253
	MIVQUE(0)	0.0174	0.0326	0.0700	0.1145	0.1533	0.1885
	REML	0.0258	0.0352	0.0511	0.0581	0.0493*	0.0222*
	ML	0.0164	0.0283	0.0561	0.0753	0.0686	0.0304
	MINQE	0.0210	0.0125	0.0127*	0.0331*	0.0590	0.0652
	MINQE(A)	0.0152*	0.0094*	0.0160	0.0421	0.0704	0.0680
	MU	0.3379	0.4910	0.6157	0.3561	0.1554	0.0352
P12 = (2,10,18)	ANOVA	0.0184	0.0302	0.0579	0.0868	0.1066	0.0972
	MIVQUE(A)	0.0193	0.0304	0.0565	0.0859	0.1045	0.0926
	MIVQUE(0)	0.0193	0.0334	0.0689	0.1083	0.1453	0.1572
	REML	0.0221	0.0339	0.0595	0.0871	0.1028*	0.0898
	ML	0.0116	0.0254	0.0643	0.1105	0.1416	0.1302
	MINQE	0.0116	0.0156	0.0370*	0.0728*	0.1077	0.1145
	MINQE(A)	0.0099*	0.0145*	0.0383	0.0774	0.1152	0.1226
	MU	0.1482	0.1244	0.0813	0.0576	0.0422	0.0340*
P13 = (3,15,27)	ANOVA	0.0140	0.0256	0.0533	0.0809	0.1002	0.0918
	MIVQUE(A)	0.0142	0.0252	0.0516	0.0789	0.0963	0.0853
	MIVQUE(0)	0.0156	0.0294	0.0646	0.1030	0.1385	0.1499
	REML	0.0163	0.0279	0.0533	0.0784	0.0936	0.0823
	ML	0.0094	0.0221	0.0580	0.0987	0.1276	0.1182
	MINQE	0.0085	0.0139	0.0370*	0.0716	0.1040	0.1080
	MINQE(A)	0.0077*	0.0133*	0.0378	0.0744	0.1089	0.1134
	MU	0.1245	0.1051	0.0649	0.0451*	0.0348*	0.0292*

* means the best choice.

Table 4. Estimated Absolute Bias Estimators of ρ

		θ					
Name	Estimators	0.1	0.2	0.5	1.0	2.0	5.0
P1=(3,5,7)	ANOVA	0.1314	0.1696	0.2246	0.2535	0.2509	0.2019
	MIVQUE(A)	0.1314	0.1693	0.2240	0.2534	0.2507	0.2008
	MIVQUE(0)	0.1303	0.1700	0.2315	0.2667	0.2758	0.2475
	REML	0.1326	0.1714	0.2257	0.2540	0.2505	0.2003
	ML	0.1058	0.1544	0.2317	0.2825	0.2955	0.2517
	MINQE	0.0976	0.1147	0.1673*	0.2211	0.2535	0.2427
	MINQE(A)	0.0837*	0.1071*	0.1734	0.2388	0.2796	0.2681
	MU	0.3736	0.3317	0.2290	0.1806*	0.1597*	0.1272*
P2 = (1,5,9)	ANOVA	0.1394	0.1779	0.2366	0.2712	0.2750	0.2267
	MIVQUE(A)	0.1416	0.1777	0.2330	0.2703	0.2741	0.2224
	MIVQUE(0)	0.1321	0.1770	0.2565	0.3088	0.3379	0.3217
	REML	0.1457	0.1880	0.2467	0.2793	0.2762	0.2187
	ML	0.1099	0.1645	0.2558	0.3183	0.3354	0.2844
	MINQE	0.1026	0.1141	0.1672*	0.2314*	0.2797	0.2846
	MINQE(A)	0.0895*	0.1083*	0.1757	0.2506	0.3045	0.3031
	MU	0.4613	0.4225	0.3091	0.2451	0.1975*	0.1459*
P3 = (1,7,7)	ANOVA	0.1365	0.1751	0.2347	0.2699	0.2745	0.2270
	MIVQUE(A)	0.1388	0.1750	0.2310	0.2691	0.2734	0.2222
	MIVQUE(0)	0.1301	0.1752	0.2553	0.3082	0.3387	0.3240
	REML	0.1435	0.1860	0.2450	0.2779	0.2749	0.2180
	ML	0.1079	0.1619	0.2526	0.3153	0.3325	0.2824
	MINQE	0.1071	0.1449	0.2181*	0.2784	0.3059	0.2721
	MINQE(A)	0.0993*	0.1390*	0.2187	0.2866	0.3216	0.2922
	MU	0.4648	0.4236	0.3144	0.2410*	0.1969*	0.1453*
P4 = (3,3,5,5,7,7)	ANOVA	0.1032	0.1314	0.1664	0.1720	0.1566	0.1070
	MIVQUE(A)	0.1028	0.1303	0.1659	0.1716	0.1554	0.1051
	MIVQUE(0)	0.1032	0.1340	0.1766	0.1939	0.1957	0.1681
	REML	0.1041	0.1325	0.1675	0.1719	0.1549	0.1047
	ML	0.0924	0.1274	0.1719	0.1852	0.1734	0.1217
	MINQE	0.1001	0.0954	0.1166*	0.1404*	0.1538	0.1329
	MINQE(A)	0.0788*	0.0816*	0.1202	0.1587	0.1806	0.1532
	MU	0.3161	0.2709	0.1713	0.1417	0.1285*	0.0924*
P5 = (1,1,5,5,9,9)	ANOVA	0.1062	0.1357	0.1751	0.1856	0.1747	0.1225
	MIVQUE(A)	0.1065	0.1344	0.1729	0.1844	0.1716	0.1164
	MIVQUE(0)	0.1032	0.1384	0.1928	0.2225	0.2355	0.2154
	REML	0.1080	0.1404	0.1814	0.1896	0.1711*	0.1131*
	ML	0.0934	0.1331	0.1872	0.2072	0.1956	0.1338
	MINQE	0.1036	0.0934	0.1151*	0.1506*	0.1796	0.1686
	MINQE(A)	0.0837*	0.0819*	0.1226	0.1717	0.2056	0.1831
	MU	1.7565	1.3908	0.9003	0.5050	0.2857	0.1141
P6 = (1,1,7,7,7,7)	ANOVA	0.1046	0.1346	0.1709	0.1863	0.1736	0.1215
	MIVQUE(A)	0.1040	0.1333	0.1685	0.1841*	0.1716*	0.1163
	MIVQUE(0)	0.1013	0.1361	0.1827	0.2087	0.2099	0.1770
	REML	0.1047	0.1381	0.1766	0.1891	0.1716*	0.1131*
	ML	0.0905	0.1307	0.1836	0.2075	0.1955	0.1345
	MINQE	0.0883	0.1151	0.1608*	0.1930	0.1937	0.1436
	MINQE(A)	0.0806*	0.1099*	0.1639	0.2049	0.2116	0.1627
	MU	1.6783	1.7167	1.0851	0.7304	0.2891	0.1193
P7 = (1,1,1,1,13,13)	ANOVA	0.1155	0.1468	0.1942	0.2135	0.2120	0.1542
	MIVQUE(A)	0.1201	0.1467	0.1914	0.2131	0.2034	0.1341
	MIVQUE(0)	0.1091	0.1567	0.2407	0.2999	0.3416	0.3436
	REML	0.1291	0.1683	0.2147	0.2191	0.1954*	0.1242*
	ML	0.1040	0.1531	0.2253	0.2511	0.2363	0.1535
	MINQE	0.1122	0.0931	0.1149*	0.1692*	0.2221	0.2282
	MINQE(A)	0.0929*	0.0834*	0.1265	0.1922	0.2447	0.2325
	MU	0.6465	0.6546	0.5317	0.3655	0.2499	0.1271

* means the best choice.

Table 4. *(Continued)*

Name	Estimators	θ 0.1	0.2	0.5	1.0	2.0	5.0
P8 = (3,3,3,5,5,5,7,7,7)	ANOVA	0.0883	0.1118	0.1358	0.1389	0.1209	0.0789
	MIVQUE(A)	0.0879	0.1110	0.1354	0.1383	0.1194	0.0771
	MIVQUE(0)	0.0891	0.1144	0.1480	0.1618	0.1575	0.1381
	REML	0.0889	0.1128	0.1366	0.1384	0.1190	0.0769
	ML	0.0822	0.1100	0.1406	0.1465	0.1295	0.0858
	MINQE	0.1029	0.0903	0.0938*	0.1117*	0.1196	0.1024
	MINQE(A)	0.0777*	0.0718*	0.0965	0.1303	0.1462	0.1197
	MU	0.3015	0.2582	0.1529	0.1243	0.1108*	0.0725*
P9 = (1,1,1,5,5,5,9,9,9)	ANOVA	0.0906	0.1152	0.1440	0.1507	0.1341	0.0900
	MIVQUE(A)	0.0901	0.1130	0.1424	0.1488	0.1309	0.0837
	MIVQUE(0)	0.0890	0.1180	0.1623	0.1862	0.1896	0.1762
	REML	0.0909	0.1170	0.1475	0.1516	0.1301*	0.0814*
	ML	0.0829	0.1134	0.1528	0.1626	0.1437	0.0920
	MINQE	0.1059	0.0876	0.0921*	0.1221*	0.1443	0.1352
	MINQE(A)	0.0824*	0.0717*	0.0994	0.1440	0.1708	0.1471
	MU	1.9331	1.9227	1.1026	0.5394	0.1983	0.0818
P10 = (1,1,1,7,7,7,7,7,7)	ANOVA	0.0899	0.1144	0.1426	0.1487	0.1319	0.0878
	MIVQUE(A)	0.0892	0.1124	0.1411	0.1471	0.1297	0.0833
	MIVQUE(0)	0.0878	0.1135	0.1512	0.1661	0.1578	0.1274
	REML	0.0895	0.1152	0.1458	0.1502*	0.1294*	0.0811*
	ML	0.0816	0.1113	0.1504	0.1607	0.1427	0.0916
	MINQE	0.0780	0.0983	0.1343*	0.1547	0.1486	0.1021
	MINQE(A)	0.0707*	0.0935*	0.1390	0.1679	0.1680	0.1193
	MU	2.0320	1.8656	1.0563	0.5683	0.2066	0.0814
P11 = (1,1,1,1,1,1,1,19,19)	ANOVA	0.1032	0.1327	0.1747	0.1959	0.1848	0.1352
	MIVQUE(A)	0.1071	0.1313	0.1740	0.1912	0.1689	0.1018
	MIVQUE(0)	0.1002	0.1493	0.2320	0.2987	0.3343	0.3561
	REML	0.1191	0.1576	0.1941	0.1912	0.1580*	0.0933
	ML	0.0994	0.1467	0.2067	0.2192	0.1853	0.1081
	MINQE	0.1201	0.0877	0.0927*	0.1526*	0.2104	0.2250
	MINQE(A)	0.0977*	0.0744*	0.1061	0.1765	0.2311	0.2217
	MU	0.4333	0.4529	0.3822	0.2702	0.1732	0.0891*
P12 = (2,10,18)	ANOVA	0.1056	0.1469	0.2097	0.2485	0.2564	0.2152
	MIVQUE(A)	0.1059	0.1458	0.2071	0.2471	0.2511	0.2044
	MIVQUE(0)	0.1067	0.1540	0.2328	0.2865	0.3173	0.3042
	REML	0.1129	0.1556	0.2133	0.2472	0.2463	0.1989
	ML	0.0915	0.1446	0.2262	0.2833	0.2984	0.2541
	MINQE	0.0783	0.1021	0.1657*	0.2284	0.2665	0.2550
	MINQE(A)	0.0732*	0.0999*	0.1697	0.2374	0.2790	0.2671
	MU	0.3646	0.3253	0.2315	0.1856*	0.1588*	0.1257*
P13 = (3,15,27)	ANOVA	0.0936	0.1346	0.2000	0.2394	0.2487	0.2101
	MIVQUE(A)	0.0930	0.1335	0.1969	0.2346	0.2398	0.1963
	MIVQUE(0)	0.0970	0.1445	0.2245	0.2796	0.3090	0.2971
	REML	0.0999	0.1412	0.1998	0.2321	0.2344	0.1914
	ML	0.0849	0.1344	0.2122	0.2642	0.2820	0.2435
	MINQE	0.0695	0.0981	0.1656*	0.2251	0.2602	0.2443
	MINQE(A)	0.0668*	0.0970*	0.1681	0.2310	0.2685	0.2529
	MU	0.3378	0.3033	0.2097	0.1701*	0.1499*	0.1211*

* means the best choice.

Table 5. MSE of Four Estimators of θ

Name	Estimators	θ 0.1	0.5	1.0	5.0
PB = (5,5,5,5,5,5,5,5,5)	ML	0.02	0.14	0.42	7.95
	MINQE	0.02	0.13	0.45	9.46
	MINQE(A)	0.01*	0.07*	0.24*	6.35*
	PM	0.01*	0.10	0.30	6.38
P8 = (3,3,3,5,5,5,7,7,7)	ML	0.02	0.14	0.45	7.95
	MINQE	0.02	0.13	0.45	9.34
	MINQE(A)	0.01*	0.10*	0.35	7.33
	PM	0.01*	0.10*	0.30*	6.13*
P11 = (1,1,1,1,1,1,1,19,19)	ML	0.06	0.35	0.88	9.60
	MINQE	0.02	0.14	0.48	9.18
	MINQE(A)	0.01*	0.13*	0.44*	9.03
	PM	0.02	0.15	0.48	8.49*

* means the best choice.

Table 6. Bias of Four Estimators of θ

Name	Estimators	θ 0.1	0.5	1.0	5.0
PB = (5,5,5,5,5,5,5,5,5)	ML	-0.01*	-0.04	-0.07	-0.30
	MINQE	0.03	0.00*	-0.02*	-0.01*
	MINQE(A)	0.09	-0.04	-0.21	-1.60
	PM	-0.02	-0.14	-0.39	-1.70
P8 = (3,3,3,5,5,5,7,7,7)	ML	0.00*	-0.04	-0.07	-0.31
	MINQE	0.03	-0.01*	-0.05*	-0.08*
	MINQE(A)	0.01	-0.09	-0.21	-0.89
	PM	-0.02	-0.14	-0.30	-1.72
P11 = (1,1,1,1,1,1,1,19,19)	ML	0.00*	-0.04*	-0.05*	-0.30*
	MINQE	0.00*	-0.16	-0.36	-1.67
	MINQE(A)	0.00*	-0.20	-0.44	-2.00
	PM	-0.03	-0.24	-0.46	-2.00

* means the best choice for absolute value.

Table 7. MSE of Four Estimators of ρ

Name	Estimators	θ 0.1	0.2	0.5	1.0	2.0	5.0
P1=(3,5,7)	ML	0.0173	0.0305	0.0671	0.1094	0.1386	0.1273
	MINQE	0.0188	0.0214	0.0380*	0.0690	0.1001	0.1069
	MINQE(A)	0.0138*	0.0178*	0.0399	0.0780	0.1154	0.1234
	MEDIAN	0.0263	0.0271	0.0416	0.0655*	0.0970*	0.0914*
P2 = (1,5,9)	ML	0.0200	0.0346	0.0778	0.1322	0.1729	0.1616
	MINQE	0.0205	0.0209	0.0378*	0.0743	0.1156	0.1311
	MINQE(A)	0.0160*	0.0183*	0.0407	0.0840	0.1309	0.1464
	MEDIAN	0.0665	0.0571	0.0495	0.0622*	0.0778*	0.0830*
P3 = (1,7,7)	ML	0.0193	0.0338	0.0763	0.1300	0.1699	0.1591
	MINQE	0.0199	0.0294	0.0608	0.1046	0.1420	0.1390
	MINQE(A)	0.0167*	0.0264*	0.0604	0.1091	0.1521	0.1523
	MEDIAN	0.0611	0.0584	0.0530*	0.0597*	0.0831*	0.0853*
P4 = (3,3,5,5,7,7)	ML	0.0126	0.0215	0.0417	0.0538	0.0542	0.0333
	MINQE	0.0174	0.0155	0.0200*	0.0303	0.0399	0.0340
	MINQE(A)	0.0115*	0.0111*	0.0209	0.0376	0.0518	0.0433
	MEDIAN	0.0298	0.0258	0.0234	0.0284*	0.0300*	0.0223*
P5 = (1,1,5,5,9,9)	ML	0.0134	0.0235	0.0480	0.0659	0.0697	0.0427
	MINQE	0.0184	0.0147	0.0195*	0.0344	0.0512	0.0484
	MINQE(A)	0.0129*	0.0111*	0.0216	0.0429	0.0637	0.0572
	MEDIAN	0.0722	0.0576	0.0375	0.0296*	0.0273*	0.0180*
P6 = (1,1,7,7,7,7)	ML	0.0123	0.0230	0.0460	0.0658	0.0692	0.0426
	MINQE	0.0122	0.0190	0.0365	0.0561	0.0637	0.0437
	MINQE(A)	0.0100*	0.0169*	0.0372	0.0616	0.0732	0.0527
	MEDIAN	0.0720	0.0608	0.0341*	0.0303*	0.0259*	0.0218*
P7 = (1,1,1,1,13,13)	ML	0.0183	0.0305	0.0643	0.0929	0.1022	0.0602
	MINQE	0.0208	0.0145	0.0193*	0.0415	0.0703	0.0747
	MINQE(A)	0.0155*	0.0116*	0.0226	0.0510	0.0832	0.0817
	MEDIAN	0.1282	0.0995	0.0558	0.0366*	0.0240*	0.0156*
P8 = (3,3,3,5,5,5,7,7,7)	ML	0.0098*	0.0167	0.0293	0.0347	0.0301	0.0158
	MINQE	0.0164	0.0135	0.0133*	0.0194	0.0243	0.0193
	MINQE(A)	0.0102	0.0087*	0.0137	0.0256	0.0340	0.0256
	MEDIAN	0.0272	0.0269	0.0191	0.0184*	0.0172*	0.0096*
P9 = (1,1,1,5,5,5,9,9,9)	ML	0.0101*	0.0178	0.0339	0.0424	0.0382	0.0190
	MINQE	0.0171	0.0126	0.0128*	0.0229	0.0332	0.0298
	MINQE(A)	0.0113	0.0087*	0.0144	0.0304	0.0439	0.0357
	MEDIAN	0.0774	0.0616	0.0351	0.0220*	0.0168*	0.0086*
P10 = (1,1,1,7,7,7,7,7,7)	ML	0.0097	0.0172	0.0330	0.0413	0.0375	0.0188
	MINQE	0.0094	0.0141	0.0266*	0.0371	0.0381	0.0220
	MINQE(A)	0.0076*	0.0125*	0.0278	0.0425	0.0468	0.0284
	MEDIAN	0.0723	0.0569	0.0329	0.0215*	0.0159*	0.0083*
P11 = (1,1,1,1,1,1,1,19,19)	ML	0.0164	0.0283	0.0561	0.0753	0.0686	0.0304
	MINQE	0.0210	0.0125	0.0127*	0.0331*	0.0590	0.0652
	MINQE(A)	0.0152*	0.0094*	0.0160	0.0421	0.0704	0.0680
	MEDIAN	0.1593	0.1214	0.0674	0.0369	0.0172*	0.0061*
P12 = (2,10,18)	ML	0.0116	0.0254	0.0643	0.1105	0.1416	0.1302
	MINQE	0.0116	0.0156	0.0370*	0.0728	0.1077	0.1145
	MINQE(A)	0.0099*	0.0145*	0.0383	0.0774	0.1152	0.1226
	MEDIAN	0.0253	0.0274	0.0401	0.0634*	0.0829*	0.0843*
P13 = (3,15,27)	ML	0.0094	0.0221	0.0580	0.0987	0.1276	0.1182
	MINQE	0.0085	0.0139	0.0370	0.0716	0.1040	0.1080
	MINQE(A)	0.0077*	0.0133*	0.0378	0.0744	0.1089	0.1134
	MEDIAN	0.0162	0.0192	0.0366*	0.0655*	0.0874*	0.0991*

* means the best choice.

$P2$ and $\theta = 5.0$ under $P3$ in the sense of MSE criterion. Similarly, MIVQUE(A) has smaller absolute bias than REML estimators except $\theta = 5.0$ under $P11$ and $\theta = 5.0$ under $P13$.

5. The MU estimator has poor efficiency for any values of k and θ. MU has highest absolute bias and smallest MSE efficiency except MIVQUE(0).

3.2 Estimation of ρ

It is difficult to obtain good estimator of intraclass correlation coefficient ρ under MSE and absolute bias criterion. However, one possibility is to find the approximation for MSE of $g(X)$, where $g(X)$ means that the estimator of intraclass correlation coefficient $g(\theta) = \theta/(1+\theta)$. If we expand g in a Taylor series, keeping only to one term, then $E(g(X) - g(\theta))^2 \doteq [g'(\theta)]^2 E(X - \theta)^2$. This means that good estimator of θ is also desirable estimator of $g(\theta)$, approximately. But it is not easy to improve this approximation by using higher-order moments of X.

To draw general conclusions for the estimation of intraclass correlation coefficient ρ, a simulation study was performed using the procedure described in section 3. The results of the simulations are shown in Table 3 and Table 4. The following conclusions are drawn from all the computed values.

1. For all around performance except P8 and P9 designs, the MINQE(A) estimator of the ρ may have smaller MSE than any other estimator if $\theta \leq 0.2$. In the case of absolute bias, MINQE(A) is the best estimator when $\theta \leq 0.2$. MINQE performs very well if $\theta = 0.5$ and $\theta = 1.0$. However, the best winner in the sense of MSE and absolute bias is not clear. Because the difference of efficiency to their performances is very small.

2. The MU estimator of ρ performs well if the value of k is small and θ is thought to be large. However when $\theta \leq 0.5$, the MU estimators is a terrible estimator of ρ.

3. The ANOVA, MIVQUE(A) and REML estimators of ρ have similar efficiency. Further iteration to the REML estimates is unnecessary for the estimation of ρ.

4. The efficiency of ML estimator increases as ρ becomes small.

5. The MIVQUE(0) is a poor estimator for ρ when θ is large. The MIVQUE(0) is a desirable estimator only when one is confident that $\theta \simeq 0$.

4 Comparisons of Other Procedures

4.1 Attention on Point estimation of θ.

Several other estimators of θ have also been cited in the literature. Chaloner (1987) used the bayesian posterior mode estimator of θ in the consideration of estimation of θ. In his paper, the posterior mode estimator (PME) is superior to MLE. We reproduced a table regarding the efficiencies of MLE, PME, and two types of MINQE. Table 5 illustrates MSE and average bias of four estimators of θ. The values of MSE and bias for ML and PM estimators were duplicated in Chaloner (1987) except the case of design PB. MSE and average bias of ML and MINQE(A), respectively, can be calculated exactly when the data is balanced. By this reason, we used the exact values of MSE and bias at Table 5 and Table 6. Table 5 illustrates the following points:

1. The MSE of the MINQE(A) is less than the MSE of any other estimator for $\theta \leq 0.5$.

2. Irrespective of small MSE of MINQE(A) and PME, they may have a nontrivial bias. Moreover, the absolute value of bias for PM is large than that of MINQE(A) except $\theta = 0.1$ undere PB.

3. Every estimators are generally biased downward for $\theta \leq 0.5$.

4. PME is not generally much more efficient than the MINQE(A). Of course, the expense of computation for PME is very high, although the MINQE(A) can be easily computed by hand.

4.2 Attention on Point estimation of ρ.

Palmer and Broemeling (1990) pointed out the possibility of using the Bayes estimator used the median of a conditional posterior density as its estimator of ρ. We reproduced a table regarding to the efficiencies of MLE, PME, and two types of MINQE. Table 6 illustrates MSE and average bias of four estimators of θ. Again the values for the Bayes estimators are taken from the Monte Carlo study of Palmer and Broemeling (1990). Table 7 gives the MSE and bias for four estimators. The MSE of the Bayes estimator is less than the MSE of any other estimators for $\rho \geq 1.0$. If $\rho \leq 0.2$ then MSE of MINQE(A) is the smallest except $P11$. MINQE is the best around $\rho = 0.5$ approximately.

5 Summary and Conclusions

To sum up, we recommend the followings:

1. For estimating of θ, two types of MINQE estimators seem to be the best or nearly best estimators unless the value of θ is thought to be high. For low values of θ it is recommended that MINQE(A) be used.

2. For estimating of ρ, the MINQE(A) estimators seems to be the overall best choice in the case of $\rho \leq 0.2$. When the value of $\rho \geq 1.0$, the Bayes median estimates of ρ is recommended.

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