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A STRONG LAW OF LARGE NUMBERS FOR TRIMMED SUMS, WITH APPLICATIONS TO GENERALIZED ST. PETERSBURG GAMES

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A strong law of large numbers for trimmed sums, with applications to generalized St. Petersburg games*

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Abstract: Extending a result of Einmahl, Haeusler and Mason (1988), a characterization of the almost sure asymptotic stability of lightly trimmed sums of upper order statistics is given when the right tail of the underlying distribution with positive support is surrounded by tails that are regularly varying with the same index. The result is motivated by applications to cumulative gains in a sequence of generalized St. Petersburg games in which a fixed number of the largest gains of the player may be withheld.

Keywords : Lightly trimmed sums of order statistics, almost sure asymptotic stability, generalized St. Petersburg games

1. Introduction and the main result

Let $X_1, X_2, \ldots$ be independent random variables, distributed as X, with distribution function $F(x) := P\{X \le x\}$, $x \in \mathbb{R}$, and quantile function $Q(s) := \inf\{x : F(x) \ge s\}$, $0 < s < 1$. For each $n \in \mathbb{N}$, let $X_{1,n} \le \ldots \le X_{n,n}$ be the order statistics of the first n variables, and for $0 < \alpha < 2$ and an integer $k_n \in \{1, \ldots, n\}$, consider the centering sequence

$$\mu_n(\alpha, k_n) := \begin{cases} 0 & \text{, if } 0 < \alpha < 1, \\ \int_{1-\frac{k_n}{n}}^{1-\frac{1}{n}} Q(s)\,ds & \text{, if } \alpha = 1, \\ \int_{1-\frac{k_n}{n}}^{1} Q(s)\,ds & \text{, if } 1 < \alpha < 2, \end{cases} \tag{1.1}$$

and norming sequence

$$a_n := b_n\, Q\Big(1 - \frac{1}{n}\Big), \quad \text{where} \quad b_1 \le b_2 \le \cdots \quad \text{and} \quad \lim_{n\to\infty} b_n = \infty. \tag{1.2}$$

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The main result of this note, motivated by direct applicability to the case when $X_1, X_2, \ldots$ are the gains in a sequence of games of the St. Petersburg type, is the following.

Theorem. *Suppose that* $F(0-) = 0$ *and that there exist two distribution functions* G *and* H *and two constants* $0 < \alpha < 2$ *and* $0 < c \leq 1$ *such that*

$$G(0-) = 0 = H(0-) \quad \text{and} \quad 1 - G(x) = \frac{g(x)}{x^\alpha}, \; 1 - H(x) = \frac{h(x)}{x^\alpha}, \quad x > 0, \tag{1.3}$$

for functions $g(\cdot)$ *and* $h(\cdot)$ *that are both slowly varying at infinity, and*

$$1 - G(x) \leq 1 - F(x) \leq 1 - H(x) \quad \text{and} \quad c \leq \frac{1 - G(x)}{1 - H(x)}, \quad x > 0. \tag{1.4}$$

Let $m \in \{0, 1, 2, \ldots\}$ *be any fixed integer and let* $\{k_n\}_{n=1}^{\infty}$ *be any sequence of integers such that* $m + 1 \leq k_n \leq n$ *and* $\lim_{n\to\infty} k_n = \infty$. *Then equivalent are:*

$$\sum_{n=1}^{\infty} n^m \left[1 - F(a_n)\right]^{m+1} < \infty, \tag{1.5}$$

$$\lim_{n\to\infty} \frac{X_{n-m,n}}{a_n} = 0 \quad \text{almost surely}, \tag{1.6}$$

and

$$\lim_{n\to\infty} \frac{1}{a_n} \left\{ \sum_{j=n+1-k_n}^{n-m} X_{j,n} - c_n \right\} = 0 \quad \text{almost surely} \tag{1.7}$$

for some sequence $\{c_n\}_{n=1}^{\infty}$ *of constants, where* a_n *is as in (1.2). If any one of these holds, then the choice* $c_n \equiv n\mu_n(\alpha, k_n)$ *is possible, given by (1.1). Furthermore, the following three statements are also equivalent:*

$$\sum_{n=1}^{\infty} n^m \left[1 - F(a_n)\right]^{m+1} = \infty, \tag{1.8}$$

$$\limsup_{n\to\infty} \frac{X_{n-m,n}}{a_n} = \infty \quad \text{almost surely}, \tag{1.9}$$

and

$$\limsup_{n\to\infty} \frac{1}{a_n} \left| \sum_{j=n+1-k_n}^{n-m} X_{j,n} - c_n \right| = \infty \quad \text{almost surely} \tag{1.10}$$

for all sequences $\{c_n\}_{n=1}^{\infty}$ *of constants.*

The investigation of the influence on strong laws of removing a few largest summands from a full sum ($k_n \equiv n$) has been initiated by Feller (1968b) in the context of the law of the iterated logarithm. Mori (1976, 1977) addresses the almost sure stability of lightly trimmed full sums without any condition on F and with some restrictions on his norming sequence. Mori's (1977) remarkable Theorem 1 is applicable to St. Petersburg-type sums $\sum_{j=1}^{n-m} X_{j,n}$ considered in Section 3 below, but not so directly as the theorem above. Mori (1976, 1977) and Maller (1984) obtain their results by classical methods; the reader is referred to Kesten and Maller (1992) and their references using the same global approach. None of these papers appears to be directly applicable to lightly trimmed extreme sums $\sum_{j=n+1-k_n}^{n-m} X_{j,n}$ in general, when $k_n \to \infty$ such that $k_n/n \to 0$ as $n \to \infty$.

On the other hand, Einmahl, Haeusler and Mason (1988), henceforth referred to as E-H-M, use the quantile-transform – empirical-process approach to the same problem and derive their Theorem 2 for the sums $\sum_{j=n+1-k_n}^{n-m} X_{j,n}$ from a corresponding result for weighted uniform empirical processes. (Concerning this approach in general, and many references to related problems for trimmed sums, see Hahn, Mason and Weiner (1991).) The latter result, Theorem 1 of E-H-M, is a far-reaching extension of a theorem of Csáki (1975). The theorem above is an extension of Theorem 2 in E-H-M, obtained by taking $c = 1$ in (1.4). When $c = 1$, we must have $G = F = H$, so that by (1.3)

$$F(0-) = 0 \quad \text{and} \quad 1 - F(x) = \frac{\ell(x)}{x^\alpha}, \quad x > 0, \tag{1.11}$$

where $\ell(\cdot)$ is slowly varying at infinity, which is too specialized for the generalized St. Petersburg distributions considered in Section 3.

Conditions (1.3) and (1.4) of the theorem can be formally relaxed by requiring only that (1.11) holds for some constant $\alpha \in (0,2)$ and some function $\ell(\cdot)$ such that for some constants $c^* \in (0,1]$ and $x_0 > 0$, $g^*(x) \le \ell(x) \le h^*(x)$ and $c^* \le g^*(x)/h^*(x)$ for all $x > x_0$, where $g^*(\cdot)$ and $h^*(\cdot)$ are some functions slowly varying at infinity. Assuming the latter condition, one can always construct distribution functions G and H satisfying (1.3) and (1.4), so this is in fact a convenient equivalent form of what we assume.

Set $S(n) := \left[\sum_{j=1}^{n} X_j - n\mu_n(\alpha, n)\right]/Q(1-1/n)$, $n \in \mathbb{N}$. The inequalities in (2.1) below, when substituted into the 'quantile theory' in Csörgő, Haeusler and Mason (1988), imply that a distribution function F satisfying (1.3) and (1.4) is in Feller's stochastically compact class. In particular, every unbounded subsequence $\{n'\} \subset \mathbb{N}$ contains a further unbounded subsequence $\{n''\}$ such that, as $n'' \to \infty$, the sequence $S(n'')$ converges in distribution to an infinitely divisible random variable with characteristic function

$$\exp\left\{i\theta t + \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dR(x)\right\}, \quad t \in \mathbb{R},$$

where i is the imaginary unit, $\theta = \theta(\alpha, c) \in \mathbb{R}$ is some constant, and the non-decreasing Lévy function $R : (0,\infty) \mapsto (-\infty, 0)$ is such that

$$-\frac{1}{c}\left(\frac{2}{2-\alpha}\right)^{\alpha/2} \frac{1}{x^\alpha} \le R(x) \le -c\left(\frac{2}{2-\alpha}\right)^{\alpha/2} \frac{1}{x^\alpha} \quad \text{for all } x > 0 .$$

The bounding functions here, as Lévy functions, produce (modulo scaling constants) all completely asymmetric stable distributions of exponent α. Then, writing $\xrightarrow{P}$ for convergence in probability, since $\{S(n)\}_{n=1}^\infty$ is stochastically bounded, it can be derived that

$$\frac{1}{a_n}\left\{ \sum_{j=n+1-k_n}^{n} X_{j,n} - n\mu_n(\alpha, k_n)\right\} \xrightarrow{P} 0, \quad \text{as } n \to \infty, \tag{1.12}$$

for any sequence of integers $1 \le k_n \le n$ and any, not necessarily monotone sequence $b_n > 0$ in (1.2) such that $k_n \to \infty$ and $b_n \to \infty$ as $n \to \infty$. This is the basic underlying weak law of large numbers, which becomes the strong law appearing in (1.7) after a possibly necessary light trimming of the largest values in the sum.

The prototypical example, when light trimming is definitely needed for a strong law, is provided by the classical St. Petersburg game, a generalized version of which is discussed in Section 3. Feller (1945, cf. also Section X.4 of 1968a) used this game to illustrate a weak law in the spirit of (1.12), thereby inaugurating a genuinely mathematical phase in the history of the St. Petersburg paradox while effectively terminating the speculative and extremely fascinating initial phase that lasted for 232 years. As noticed by Chow and Robbins (1961), Feller's law cannot be upgraded to a strong law. Hence the idea of trimming arises naturally. It was this historical example, still frequently discussed in text-books on probability and, particularly, on economic theory, that has provided the primary motivation for the present paper.

2. Proof of the theorem

The equivalences of conditions (1.5) and (1.6), and of (1.8) and (1.9), follow from Lemma 3 of Mori (1976), as in E-H-M; for these one only needs $a_n \uparrow \infty$, and this is ensured by

(1.2). (Here and in what follows, any convergence relation is meant to hold as $n \to \infty$.) Thus, it will be enough for us to show that (1.5) implies (1.7) for the particular case $c_n \equiv n\mu_n(\alpha, k_n)$, and the falsity of (1.10) implies (1.5); simple *ad absurdum* arguments yield all of the remaining implications. We begin by establishing some useful inequalities.

Let $Q_G(s)$ and $Q_H(s)$, $0 < s < 1$, denote the quantile functions pertaining to G and H, respectively. Conditions (1.3) and (1.4) imply that $Q_G(0+), Q_H(0+) \geq 0$ and

$$\frac{K(s)}{s^{1/\alpha}} = Q_G(1-s) \leq Q(1-s) \leq Q_H(1-s) = \frac{M(s)}{s^{1/\alpha}} \leq Q_G(1-cs) = \frac{1}{c^{1/\alpha}}\frac{K(cs)}{s^{1/\alpha}}, \quad (2.1)$$

or simply that $Q(1-s) = L(s)/s^{1/\alpha}$ for some functions $K(\cdot)$, $L(\cdot)$ and $M(\cdot)$ such that $K(s) \leq L(s) \leq M(s) \leq K(cs)/c^{1/\alpha}$, for all $0 < s < 1$, and hence also $c^{1/\alpha}M(t/c) \leq K(t)$, $0 < t < c$, where $K(\cdot)$ and $M(\cdot)$ are slowly varying at zero. Setting $a_n^G := b_n\, Q_G(1-n^{-1})$ and $a_n^H := b_n\, Q_H(1-n^{-1})$ for the norming sequences that belong to G and H, where b_n is the same as in (1.2), these inequalities and the slow variation of $K(\cdot)$ and $M(\cdot)$ imply that for some $n_0 \in \mathbb{N}$,

$$a_n^G \leq a_n \leq \frac{2}{c^{1/\alpha}}\, a_n^G \quad \text{and} \quad \frac{c^{1/\alpha}}{2}\, a_n^H \leq a_n \leq a_n^H, \quad n \geq n_0. \quad (2.2)$$

Using these inequalities, (1.3), (1.4) and the slow variation of $g(\cdot)$ and $h(\cdot)$, further elementary considerations also give that if $n_0 \in \mathbb{N}$ is chosen sufficiently large, then

$$\begin{aligned} \frac{c}{2^{\alpha+1}}\left[1 - G(a_n^G)\right] \leq 1 - F(a_n) \leq \frac{1}{c}\left[1 - G(a_n^G)\right] \quad &\text{and} \\ c\left[1 - H(a_n^H)\right] \leq 1 - F(a_n) \leq \frac{2^{\alpha+1}}{c}\left[1 - H(a_n^H)\right], \quad &n \geq n_0. \end{aligned} \quad (2.3)$$

The implication (1.5)⇒(1.7), with $c_n \equiv n\mu_n(\alpha, k_n)$, requires repeating the corresponding part of the proof of Theorem 2 in E-H-M, and adjusting it to the present situation when needed. This goes as a line-by-line inspection. First one corrects a trivial misprint on page 68 of E-H-M, in the third line from the bottom: their minus sign has to be a plus, as their own notation suggests. One crucial spot is their (3.2), which by (1.4) and (2.2) now becomes

$$\begin{aligned} 1 - F(a_n) \geq 1 - G(a_n) &\geq \frac{1 - G(a_n^H)}{1 - H(Q_H(1-\frac{1}{n}))}\left[1 - H\Big(Q_H\Big(1 - \frac{1}{n}\Big)\Big)\right] \\ &\geq \frac{c}{b_n^\alpha}\frac{g(b_n Q_H(1-\frac{1}{n}))}{g(Q_H(1-\frac{1}{n}))}\left[1 - H\Big(Q_H\Big(1-\frac{1}{n}\Big)\Big)\right] \geq \frac{c\,(1-\varepsilon)}{n\, b_n^{\alpha+\delta}} \end{aligned} \quad (2.4)$$

for every $\varepsilon \in (0,1)$ and $\delta > 0$ for all n large enough, using, in the last step, their Lemmas 3.3 and 3.4 as they do. Following this, all of their bounds remain effective after we express them, using (2.1), in terms of Q_H at the price of inserting factors like $2/c^{1/\alpha}$, so that certain asymptotic inequalities in their bounds will hold for upper bounds of their quantities. As a rule, most of the ingredients of their proof can be built in the modified proof as applied to Q_H; further details are unnecessary here.

To complete the proof, it suffices to show that if (1.10) does not hold for all sequences $\{c_n\}$ of constants, then we have (1.5). Suppose, therefore, that

$$P\left\{\limsup_{n\to\infty} \frac{1}{a_n}\left| \sum_{j=n+1-k_n}^{n-m} X_{j,n} - c_n\right| < \infty\right\} > 0 \tag{2.5}$$

for some sequence $\{c_n\}$ of constants.

Notice first that $a_n^{-1}\sum_{j=n+1-m}^{n} X_{j,n} \xrightarrow{P} 0$. Indeed, as E-H-M point out on their page 72, this holds in their special case, that is, when $c = 1$ in (1.4). But then a simple argument and (2.2) show that it also holds in the present generality. This fact, (1.12) and (2.5) itself then easily imply that (2.5) holds with $c_n \equiv n\mu_n(\alpha, k_n)$. Set $\bar{b}_n := \max\{b_n, \widetilde{b}_n\}$ and $\bar{a}_n := \bar{b}_n Q(1-n^{-1})$, $n \geq 3$, where

$$\widetilde{b}_n := \max_{3\leq j\leq n} \frac{Q\Big(1 - \frac{1}{j\,(\log j)^{\frac{1}{2(m+1)}}}\Big)}{Q\Big(1-\frac{1}{j}\Big)} \leq \max_{3\leq j\leq n} \frac{Q_G\Big(1 - \frac{c}{j\,(\log j)^{\frac{1}{2(m+1)}}}\Big)}{Q_G\Big(1-\frac{1}{j}\Big)}, \tag{2.6}$$

and where the inequality is by (2.1). Then, using (1.4), (2.1), the already established implication (1.5)$\Rightarrow$(1.7), with $c_n \equiv n\mu_n(\alpha, k_n)$, and the non-negativity of the underlying random variables, it is easy to adjust the E-H-M argument on their pages 72–73 to see that (2.5) implies $P\{\limsup_{n\to\infty} X_{n-m,n}/\bar{a}_n < \infty\} > 0$. (It is a small stylistic oversight in E-H-M that they assume in their (3.13) that the probability in (2.5) is 1, implying directly that the last probability is also 1, in their special case.) Since (1.5) and (1.6) are equivalent with $\bar{a}_n$ replacing a_n by Mori's (1976) Lemma 3, as are (1.8) and (1.9) also with $\bar{a}_n$ replacing a_n, it follows that $P\{\limsup_{n\to\infty} X_{n-m,n}/\bar{a}_n = 0\} = 1$, and hence also that $\sum_{n=1}^{\infty} n^m[1-F(\bar{a}_n)]^{m+1} < \infty$. This, (2.4) with $\bar{a}_n$ and $\bar{b}_n$ replacing a_n and b_n, respectively, and the inequality in (2.6), all substituted into the rest of the E-H-M argument on their page 73, now give that $\bar{a}_n = a_n$ for all n large enough, and so $\sum_{n=1}^{\infty} n^m[1-F(a_n)]^{m+1} < \infty$. Thus, indeed, (2.5) implies (1.5).

3. Application to generalized St. Petersburg games

Let $0 < p < 1$ and $0 < \alpha < 2$ be fixed, put $q := 1 - p$, consider a generalized St. Petersburg game in which the gain X of the player is such that $P\{X = q^{-k/\alpha}\} = q^{k-1}p$, $k = 1, 2, \ldots$, and let $X_1, X_2, \ldots$ be the gains of the player in a sequence of independent repetitions of the game. This is the classical St. Petersburg game if $\alpha = 1$ and $p = 1/2$. The generalized version, at least for $\alpha = 1$, was recently considered in a related context of the strong law of large numbers, very different from ours below, by Adler and Rosalsky (1989) and Adler (1990). The notation of Section 1, with the restricted meaning that belongs to the presently given concrete X, will continue to be used. Importantly, the α introduced here will play the same role as the α in Section 1.

With the notation $\lfloor y \rfloor := \max\{k \in \mathbb{Z} : k \le y\}$ and $\lceil y \rceil := \min\{k \in \mathbb{Z} : k \ge y\}$, $y \in \mathbb{R}$, where $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, elementary calculations give

$$1 - F(x) = q^{\lfloor \alpha \log_{1/q} x \rfloor} =: \frac{\ell(x)}{x^\alpha}, \quad x \ge 1, \quad \text{and} \quad Q(1-s) = q^{-\frac{1}{\alpha}\lceil \log_{1/q} \frac{1}{s} \rceil}, \ 0 < s < 1,$$

where $\log_{1/q} u$ stands for the logarithm of $u > 0$ to the base $1/q$. Even though the oscillating function $\ell(x) = x^\alpha q^{\lfloor \alpha \log_{1/q} x \rfloor}$ is *not* slowly varying as $x \to \infty$, we have $1 \le \ell(x) < 1/q$ for all $x \ge 1$. Since

$$Q\Big(1 - \frac{1}{n}\Big) = q^{-\frac{1}{\alpha}\lceil \log_{1/q} n \rceil} = \frac{n^{1/\alpha}}{\gamma_n}, \quad \text{where} \quad q^{1/\alpha} < \gamma_n := n^{1/\alpha} q^{\frac{1}{\alpha}\lceil \log_{1/q} n \rceil} \le 1, \quad (3.1)$$

noticing the inequalities

$$\frac{q}{n\,v} \le 1 - F\bigg(Q\Big(1 - \frac{1}{n}\Big) v^{1/\alpha}\bigg) < \frac{1}{q\,n\,v}, \quad v \ge \frac{1}{n},$$

we see that for all fixed $m \in \{0, 1, 2, \ldots\}$,

$$q^{m+1} \sum_{n=1}^\infty \frac{1}{n\, d_n^{m+1}} \le \sum_{n=1}^\infty n^m \bigg[1 - F\bigg(Q\Big(1 - \frac{1}{n}\Big) d_n^{1/\alpha}\bigg)\bigg]^{m+1} \le \frac{1}{q^{m+1}} \sum_{n=1}^\infty \frac{1}{n\, d_n^{m+1}} \quad (3.2)$$

for every sequence $\{d_n\}$ of numbers such that $nd_n \ge 1$ for all $n \in \mathbb{N}$.

Also, by definition $\mu_n(\alpha, k_n) = 0$ for any $k_n \in \{1, \ldots, n\}$ if $0 < \alpha < 1$, from (1.1), and straightforward calculations give

$$\mu_n(\alpha, k_n) = \begin{cases} \delta_p\big(\frac{1}{n}\big) - \delta_p\big(\frac{k_n}{n}\big) + \frac{p}{q} \log_{1/q} k_n & , \text{if } \alpha = 1, \\ \frac{p}{q^{\frac{1}{\alpha}} - q}\, q^{\frac{\alpha-1}{\alpha}\lceil \log_{1/q} \frac{n}{k_n} \rceil} - q^{\frac{\alpha-1}{\alpha}\lceil \log_{1/q} \frac{n}{k_n} \rceil} + \frac{k_n}{n}\, q^{-\frac{1}{\alpha}\lceil \log_{1/q} \frac{n}{k_n} \rceil} & , \text{if } 1 < \alpha < 2, \end{cases}$$

where, with $\langle y \rangle := y - \lfloor y \rfloor$ denoting the fractional part of $y \in \mathbb{R}$,

$$0 \le \delta_p(u) := 1 + \frac{p}{q} \langle \log_{1/q} u \rangle - q^{-\langle \log_{1/q} u \rangle} \le 1 - \frac{p}{q \log \frac{1}{q}} \left[1 - \log \frac{p}{q} + \log\log \frac{1}{q} \right] \tag{3.3}$$

for all $u > 0$, where log stands for the natural logarithm. The inequalities are obtained using the facts that $\delta_p(uq^{-j}) = \delta_p(u)$, $u > 0$, $j \in \mathbb{Z}$, and $\delta_p(1) = 0 = \delta_p(q^{-1})$, and for $u \in (1, q^{-1})$ the function $\delta_p(u) = 1 + \frac{p}{q} \log_{1/q} u - u$ is concave with the indicated maximum value taken at $u = p/\big(q \log \frac{1}{q}\big)$.

First we use the theorem for $m = 0$. If $1 < \alpha < 2$, then, choosing $b_n \equiv d_n^{1/\alpha}$ for some sequence $d_n \uparrow \infty$ of positive numbers, so that $a_n \equiv (n d_n)^{1/\alpha}/\gamma_n$ by (3.1), and using (3.2) with $m = 0$, we obtain

$$\text{if} \quad \sum_{n=1}^{\infty} \frac{1}{n\, d_n} < \infty, \quad \text{then} \quad \frac{1}{n} \sum_{j=n+1-k_n}^{n} X_{j,n} - \mu_n(\alpha, k_n) = o\left(\frac{d_n^{1/\alpha}}{n^{1-\frac{1}{\alpha}}} \right) \quad \text{a.s.}$$

for any $k_n \in \{1, \ldots, n\}$ such that $k_n \to \infty$. In particular, since $\mu_n(\alpha, n) = \int_0^1 Q(s)\, ds \equiv E(X) = p/(q^{1/\alpha} - q)$, this contains the ordinary strong law of large numbers for the present case $\alpha \in (1,2)$ when $k_n \equiv n$, with a rate, in which the typical examples for $\{d_n\}$ are $d_n \equiv \ell_\nu^{(\varepsilon)}(n)$ for all n large enough, where

$$\ell_\nu^{(\varepsilon)}(n) := (\log n)(\log\log n) \cdots (\log \cdots \log n)^{1+\varepsilon} \tag{3.4}$$

with any fixed number $\nu \in \mathbb{N}$ of factors, where $\varepsilon > 0$ is as small as we wish. This rate is sharp because, by (1.10) and (3.2),

$$\text{if} \quad \sum_{n=1}^{\infty} \frac{1}{n\, d_n} = \infty, \quad \text{then} \quad \limsup_{n \to \infty} \frac{1}{(n d_n)^{1/\alpha}} \left| \sum_{j=n+1-k_n}^{n} X_{j,n} - c_n \right| = \infty \quad \text{a.s.} \tag{3.5}$$

for any $k_n \in \{1, \ldots, n\}$ such that $k_n \to \infty$ and for any centering sequence $\{c_n\}$. The typical examples for $\{d_n\}$ are $d_n \equiv \ell_\nu(n)$ for all n large enough, where

$$\ell_\nu(n) := (\log n)(\log\log n) \cdots (\log \cdots \log n) \tag{3.6}$$

with any fixed number $\nu \in \mathbb{N}$ of factors.

If $0 < \alpha \le 1$, so that $E(X) = \infty$, then, by Theorem 2 of Chow and Robbins (1961), either $\liminf_{n \to \infty} \sum_{j=1}^{n} X_j / a_n^* = 0$ a.s. or $\limsup_{n \to \infty} \sum_{j=1}^{n} X_j / a_n^* = \infty$ a.s.

for any sequence of positive constants a_n^*. Here the theorem can be used to determine the asymptotic size of the sums $\sum_{j=n+1-k_n}^n X_{j,n}$. Letting $b_n \equiv d_n^{1/\alpha} \uparrow \infty$, it follows from (3.2) that (3.5) holds again for any $k_n \in \{1,\ldots,n\}$ such that $k_n \to \infty$ and any centering sequence $\{c_n\}$, with typical examples for $\{d_n\}$ as $d_n \equiv \ell_\nu(n)$ in (3.6) for an arbitrary $\nu \in \mathbb{N}$. On the other hand, using also (1.5),

$$\text{if} \quad \sum_{n=1}^\infty \frac{1}{n\,d_n} < \infty, \quad \text{then} \quad \lim_{n\to\infty} \frac{1}{(nd_n)^{1/\alpha}} \sum_{j=n+1-k_n}^n X_{j,n} = 0 \quad \text{a.s.}$$

for any $k_n \in \{1,\ldots,n\}$ such that $k_n \to \infty$, the typical examples for $\{d_n\}$ being $d_n \equiv \ell_\nu^{(\varepsilon)}(n)$ in (3.4), for all n large enough, with any $\nu \in \mathbb{N}$ and $\varepsilon > 0$. (Here, in the subcase $\alpha = 1$, we also used (3.3) and the fact that $0 \le (\log_{1/q} k_n)/d_n \le (\log_{1/q} n)/d_n \to 0$, resulting from the conditions that $\sum_{n=1}^\infty (n\,d_n)^{-1} < \infty$ and $d_n \uparrow \infty$.)

The most interesting case above is $\alpha = 1$, when there is a weak law of large numbers. Extending Feller (1945), cf. also in Section X.4 of Feller (1968a), who proved this for $k_n \equiv n$ in the subcase when $p = 1/2$, when $\alpha = 1$ in the generalized St. Petersburg distribution, (1.12) with $b_n \equiv \log_{1/q} n$ and any $k_n \in \{1,\ldots,n\}$, $k_n \to \infty$, reduces to

$$\frac{1}{n\log_{1/q} n} \sum_{j=n+1-k_n}^n X_{j,n} - \frac{p}{q}\frac{\log_{1/q} k_n}{\log_{1/q} n} \xrightarrow{P} 0 \quad \text{so that} \quad \frac{1}{n\log_{1/q} n}\sum_{j=1}^n X_j \xrightarrow{P} \frac{p}{q}. \tag{3.7}$$

When $k_n \equiv n$, this is also contained as a special case of Theorem 4 of Adler and Rosalsky (1989), the general result of which is a non-trivial extension in a different direction. In this case, the Chow–Robbins phenomenon can be made very precise:

$$\liminf_{n\to\infty} \frac{1}{n\log_{1/q} n}\sum_{j=1}^n X_j = \frac{p}{q} \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n\log_{1/q} n}\sum_{j=1}^n X_j = \infty \quad \text{a.s.}$$

Here the second statement was shown by Chow and Robbins (1961) for $p = 1/2$, but their easy argument is applicable for the general case $p \in (0,1)$. The first statement is a special case of Example 4 of Adler (1990), a delicate result.

Now let $m \in \mathbb{N}$ be fixed, so that the largest observations will be discarded, that is, the player renounces his largest m winnings when playing n games. We use the theorem with $b_n \equiv d_n^{1/((m+1)\alpha)}$, for some sequence $d_n \uparrow \infty$ of positive numbers, and a trivial manipulation of (3.2). For all $0 < \alpha < 2$,

$$\text{if} \quad \sum_{n=1}^\infty \frac{1}{n\,d_n} = \infty, \quad \text{then} \quad \limsup_{n\to\infty} \frac{1}{\left(n\,d_n^{\frac{1}{m+1}}\right)^{\frac{1}{\alpha}}} \left| \sum_{j=n+1-k_n}^{n-m} X_{j,n} - c_n \right| = \infty \quad \text{a.s.} \tag{3.8}$$

for any $k_n \in \{1, \ldots, n\}$ such that $k_n \to \infty$ and for any centering sequence $\{c_n\}$. Again, the natural examples for $\{d_n\}$ are $d_n \equiv \ell_\nu(n)$ in (3.6) for any $\nu \in \mathbb{N}$.

For results in the positive direction, consider first the case $1 < \alpha < 2$, when

$$\text{if} \quad \sum_{n=1}^{\infty} \frac{1}{n\, d_n} < \infty, \quad \text{then} \quad \frac{1}{n} \sum_{j=n+1-k_n}^{n-m} X_{j,n} - \mu_n(\alpha, k_n) = o\left(\frac{d_n^{\frac{1}{(m+1)\alpha}}}{n^{1-\frac{1}{\alpha}}} \right) \quad \text{a.s.}$$

for any $k_n \in \{1, \ldots, n\}$ such that $k_n \to \infty$. Again, $\mu_n(\alpha, n) \equiv E(X) = p/(q^{1/\alpha} - q)$ in the special case when $k_n \equiv n$.

Next, when $0 < \alpha < 1$, we have that

$$\text{if} \quad \sum_{n=1}^{\infty} \frac{1}{n\, d_n} < \infty, \quad \text{then} \quad \lim_{n\to\infty} \frac{1}{\left(n\, d_n^{\frac{1}{m+1}}\right)^{\frac{1}{\alpha}}} \sum_{j=n+1-k_n}^{n-m} X_{j,n} = 0 \quad \text{a.s.}$$

for any $k_n \in \{1, \ldots, n\}$ such that $k_n \to \infty$.

Finally, to accompany (3.7), for $\alpha = 1$ and any $m \in \mathbb{N}$ we obtain

$$\text{if} \quad \sum_{n=1}^{\infty} \frac{1}{n\, d_n} < \infty, \quad \text{then} \quad \frac{1}{n \log_{1/q} n} \sum_{j=n+1-k_n}^{n-m} X_{j,n} - \frac{p}{q} \frac{\log_{1/q} k_n}{\log_{1/q} n} = o\left(\frac{d_n^{\frac{1}{m+1}}}{\log n} \right) \quad \text{a.s.}$$

for all $k_n \in \{1, \ldots, n\}$ such that $k_n \to \infty$. In particular,

$$\text{if} \quad \sum_{n=1}^{\infty} \frac{1}{n\, d_n} < \infty, \quad \text{then} \quad \frac{1}{n \log_{1/q} n} \sum_{j=1}^{n-m} X_{j,n} - \frac{p}{q} = o\left(\frac{d_n^{\frac{1}{m+1}}}{\log n} \right) \quad \text{a.s.}$$

For all three results, the typical choices of $\{d_n\}$ are again as above, i.e. $d_n \equiv \ell_\nu^{(\varepsilon)}(n)$ in (3.4), for all n large enough, with any fixed number $\nu \in \mathbb{N}$ of factors, where $\varepsilon > 0$ is as small as we wish. With $\{d_n\}$ such as these, $d_n^{1/(m+1)}/\log n \to 0$ for each fixed $m \in \mathbb{N}$. The rates, for all three results, are optimal in view of (3.8). It is interesting to compare these results with those for the untrimmed cases, and to observe the improving rates of convergence as $m = 0, 1, 2, \ldots$ increases.

An algorithm to compute the distribution of $\sum_{j=1}^{n-m} X_{j,n}$ for non-negative integer-valued random variables is given in Csörgő and Simons (1994) and is illustrated on the classical St. Petersburg game.

References

Adler, A. (1990), Generalized one-sided laws of the iterated logarithm for random variables barely with or without finite mean, *J. Theoret. Probab.* **3**, 587–597.

Adler, A. and A. Rosalsky (1989), On the Chow–Robbins "fair games" problem, *Bull. Inst. Math. Acad. Sinica* **17**, 211–227.

Chow, Y.S. and H. Robbins (1961), On sums of independent random variables with infinite moments and "fair" games, *Proc. Nat. Acad. Sci. U.S.A.* **47**, 330–335.

Csáki, E. (1975), Some notes on the law of the iterated logarithm for the empirical distribution function, In *Colloquia Math. Soc. J. Bolyai* **11**, *Limit theorems of Probability Theory* (P. Révész, ed.), pp. 47–58 (North-Holland, Amsterdam).

Csörgő, S., E. Haeusler and D.M. Mason (1988), A probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables, *Adv. in Appl. Math.* **9**, 259–333.

Csörgő, S. and G. Simons (1994), Precision calculation of distributions for trimmed sums. Submitted for publication.

Einmahl, J.H.J., E. Haeusler and D.M. Mason (1988), On the relationship between the almost sure stability of weighted empirical distributions and sums of order statistics, *Probab. Theory Related Fields* **79**, 59–74.

Feller, W. (1945), Note on the law of large numbers and "fair" games, *Ann. Math. Statist.* **16**, 301–304.

Feller, W. (1968a), *An Introduction to Probability Theory and its Applications*, Vol. **I.** (Wiley, New York)

Feller, W. (1968b), An extension of the law of the iterated logarithm to variables without variance, *J. Math. Mech.* **18**, 343–355.

Hahn, M.G., D.M. Mason and D.C. Weiner, eds. (1991), *Sums, Trimmed Sums and Extremes* (Birkhäuser, Boston).

Kesten, H. and R.A. Maller (1992), Ratios of trimmed sums and order statistics, *Ann. Probab.* **20**, 1805–1842.

Maller, R.A. (1984), Relative stability of trimmed sums, *Z. Wahrsch. verw. Gebiete* **66**, 61–80.

Mori, T. (1976), The strong law of large numbers when extreme terms are excluded from sums, *Z. Wahrsch. verw. Gebiete* **36**, 189–194.

Mori, T. (1977), Stability for sums of i.i.d. random variables when extreme terms are excluded, *Z. Wahrsch. verw. Gebiete* **40**, 159–167.