Local Lyapunov Exponents: Predictability depends on where you are

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Abstract

The dominant Lyapunov exponent of a dynamical system measures the average rate at which nearby trajectories of a system diverge. Even though a positive exponent provides evidence for chaotic dynamics and upredictability, there may predictability of the time series over some finite time periods. In this paper one version of a local Lyapunov exponent is defined for a dynamic system perturbed by noise. These local Lyapunov exponents are used to detect the parts of the time series that may be more predictable than others. An examination of the fluctuations of the local Lyapunov exponents about the average exponent may provide important information in understanding the heterogeneity of a system. We will discuss the theoretical properties of these local exponents and propose a method of estimating these quantities using nonparametric regression. Also we will present an application of local exponents for interpreting surface pressure data.

1 Introduction

The characteristic feature of a chaotic system is that initially nearby trajectories diverge in time [4]. This feature may be seen by imagining the system to be started twice, but from slightly different initial conditions. For a chaotic system, this difference or error grows exponentially in time, so that the state of this system is essentially unknown after a "short" time. Initially small perturbations that grow exponentially over time indicates "sensitive dependence on initial conditions" and is responsible for the unpredictability of a chaotic system. This unpredictable time evolution of nonlinear systems occurs in a variety of fields such as physics, meteorology, and epidemiology (see e.g. ref. [8], [2], [11]).

The Lyapunov exponent measures the asymptotic average exponential divergence (or convergence) of nearby trajectories in phase space and has proven to be a useful diagnostic to detect and quantify chaos [4]. A positive Lyapunov exponent or exponential divergence implies that the system is unpredictable. The predictability of a system is an important feature of its dynamics. Predictability, not only quantified by the Lyapunov exponent, which by definition is a "global" quantity, but by the identification of regions "locally" in the phase space that have different short term sensitivity to small perturbations. This paper will discuss local Lyapunov exponents, a finite time version of the Lyapunov exponent and describe useful information that can be obtained from the analysis of these local exponents. The identification of regions in phase space that may be more predictable than others and the examinations of the fluctuations of these local exponents about the average exponent may be useful in describing the dynamics and heterogeneity of a system.

The paper is organized as follows. Section 2 reviews the mathematical ideas and basic properties of Lyapunov exponents of a system. In Section 3 a definition for local Lyapunov exponents is presented. Some technical details on the computational methods are also included. The statistical properties of local Lyapunov exponents will be discussed using Markov process theory. In Section 4, results for two examples, the Rossler system and surface pressure data are presented, showing how local Lyapunov exponents can be used to gain useful information about the system. Section 5 contains a discussion that concludes the paper. Finally, Section 6 is an outline of a proposal for research.

2 Background and Assumptions

We assume that the data $\{x_t\}$ are a time series generated by a nonlinear autoregressive model

$$x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-d}) + e_t \tag{1}$$

where $x_t \in R$ and $\{e_t\}$ is a sequence of independent random variables or perturbations with $\mathrm{E}(e_t)= 0$ and $\mathrm{Var}(e_t)= \sigma^2$. It is important to note that the error in (1) is not measurement error, but dynamic noise, an inherent part of the dynamics of the system.

It is useful to express system (1) in terms of a state vector $X_t = (x_t, \ldots, x_{t-d+1})^T$, an error vector $E_t = (e_t, 0, \ldots, 0)^T$ and a function $F : R^d \to R^d$ such that

$$X_t = F(X_{t-1}) + E_t \tag{2}$$

where $\{X_t\}$ is the systems trajectory.

The state space representation (2) is motivated by Taken's method of reconstruction in time delay coordinates for a deterministic system ($e_t = 0$ in (1)) [15]. Intuitively, the data $\{x_t\}$ are observed from a system of unknown dimension, so that the idea of state space reconstruction is to incorporate enough lags into the state vector so that the dynamics of X_t are qualitatively the same as the full system. The time lags of a single variable x are a surrogate for the unobserved variables of the real system. Casdaglia [3] derived the important result that state space reconstruction also applies to a system such as (1), where x_t follows a nonlinear autoregressive model.

To approximate the action of the map F on two initial state vectors X_0 and Y_0, use a Talor series expansion

$$\begin{aligned} X_1 - Y_1 &= F(X_0) - F(Y_0) \\ &\approx DF(X_0)(X_0 - Y_0). \end{aligned}$$

The difference at time n between these two initial state vectors is

$$\begin{aligned} X_n - Y_n &= F^n(X_0) - F^n(Y_0) \\ &\approx DF^n(X_0)(X_0 - Y_0). \end{aligned}$$

Let $J_t = DF(X_t)$, the Jacobian matrix of F evaluated at X_t. By the chain rule for differentiation

$$X_n - Y_n \approx J_{n-1} \cdot J_{n-2} \cdots J_0(X_0 - Y_0).$$

Let $u_0 = X_0 - Y_0$, then the above equation becomes

$$\begin{aligned} u_n &\approx DF^n(X_0)u_0 \\ &\approx J_0^n u_0. \end{aligned}$$

Therefore the evolution of initially small differences in initial conditions behave approximately like the solution to the linear system.

One way of defining the global Lyapunov exponent for (2) is

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \left[\sup_{u \in R^d, \|u\|=1} \|(J_{n-1} J_{n-2} \cdots J_0)u\| \right]. \tag{3}$$

By the Multiplicative Ergodic Theorem of Oseledec [12], if X_t is ergodic, stationary, and $\max(0, \ln \|J_t\|)$ has finite expectation, then λ exists, is constant, and is independent of the trajectory. The Multiplicative Ergodic Theorem also implies that for almost all $u \in R^d$ (with respect to Lebesgue measure), the *sup* over all directions in (3) can be omitted and

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \|J_{n-1} J_{n-2} \ldots J_0 u\|. \tag{4}$$

The examples in Section 4 use $u = (1, 0, \ldots, 0)^T$ and in both (3) and (4), $\| \cdot \|$ is a vector norm. The global Lyapunov exponent measures sensitive dependence on initial conditions, that is, if λ is positive then initially close trajectories will diverge exponentially over time.

The concept of an attractor or attracting set in dynamical systems theory is to ensure that trajectories starting near this set converge onto it and their motion is confined to the set. If one studies the behavior of the evolution of trajectories of system (2), the physical long term behavior settles near a subset set of R^n. It will be necessary to require bounded fluctuations of the state vector to ensure the existence of this attractor. For a system such as (2), an operational definition of chaos is bounded fluctuations in the state variable X_t with sensitive dependence on initial conditions [4].

3 Local Lyapunov Exponents

3.1 Definition

The concept of a local Lyapunov exponent was first examined in the area of chemical physics [9]. The author proposed a local criterion for the transition from quasiperiodic to chaotic motion in classical mechanical Hamiltonian systems based on geodesic flow on a manifold.

There has been renewed interest in local Lyapunov exponents [1],[14], [17], where the approach is to average the exponential divergence of nearby trajectories of the system over short time periods. Abarbanel [1] defines a local Lyapunov exponent as a finite time version of the global Lyapunov, which provides information on how a perturbation to a system's orbit will exponentially increase or decrease in finite time. The experimental data is assumed to come from a system without noise and the method of attractor reconstruction using time delays was implemented. The estimation of Jacobians from observed data involved using the time evolution of the trajectories to construct an approximate neighborhood-to-neighborhood map using local polynomials. In order to describe the variation of the local exponents over the attractor, the mean value of the local Lyapunov exponent weighted with the density of the points on the attractor was introduced. This phase space average of the local Lyapunov exponents defined the average local Lyapunov exponent and the moments about this average were defined in a similar manner.

Wolf [17] defines local Lyapunov exponents by choosing trajectories starting at a certain distance from a trajectory and averages across all such trajectories after n time steps. The goal was to quantify the divergence of initially nearby trajectories originating in a small region in phase space.The analysis was applied to data arising from one-dimensional chaotic systems and examples with observational and system noise were examined. The local Lyapunov exponent as defined by Wolf was very sensitive to the bandwidth parameter or the distance chosen to average across.

Because Wolf's definition of local Lyapunov exponent is sensitive to the choice of the bandwidth, the finite time definition based on the approach by Abarbanel will be chosen for the definition (see equation (5)). However, because of the assumption of noise in the system the method of Jacobian estimation discussed in Section 3.2 will be quite different from the method used by Abarbanel. Also, the variation of the local exponents about their mean will be described in a different manner in Section 3.3.

The local Lyapunov exponent, a finite time version of the global Lyapunov exponent (3), is defined as

$$\lambda_n(t) = \frac{1}{n} \ln \| J_{n+t-1} J_{n+t-2} \cdots J_t u \| \tag{5}$$

where $\| \cdot \|$ and u are as described in equation (4). Since $\lambda_n(t)$ is a function of time, the local Lyapunov exponent depends on the trajectory and can be thought of as an "n-step ahead" local Lyapunov exponent process.

3.2 Computational Aspects

The calculation of local Lyapunov exponents in (5) requires an estimation of the Jacobian matrix J_t. LENNS, a neural network program to estimate the dominant Lyapunov exponent of a noisy nonlinear system from a time series was used to estimate the map f of (1) [6]. Neural networks have been shown to be successful in the modeling of time series data [3]. LENNS uses nonlinear regression to generates the estimate of f and its partial derivatives that are used to compute the local Lyapunov exponent. In LENNS, the map f is approximated by a feedfoward single hidden layer network with a single output. The form of this model is

$$f(x_1, x_2, \ldots, x_d) = \beta_0 + \sum_{j=1}^{k} \beta_j \varphi(X^T \gamma_j + \mu_j) \tag{6}$$

where $\gamma_j = (\gamma_{j1}, \gamma_{j2}, \ldots, \gamma_{jd})^T$ and φ is a sigmoid function $\varphi(u) = e^u/(1+e^u)$. The model consists of d inputs, where X is the vector of inputs, and k hidden units.

The complexity of the model, i.e., the embedding dimension and the number of hidden units was chosen based on generalized cross validation (GCV). Cross validation is a standard approach for selecting smoothing parameters in nonparametric regression [16]. Prediction accuracy is determined by deleting each point from the dataset one at a time and fitting the model to the reduced dataset and predicting the ommited point. The modified GCV function is

$$V_c = \frac{\frac{1}{n} RSS}{\left(1 - p\frac{c}{n}\right)^2} \tag{7}$$

where RSS is the residual sum of squares, n is the number of data points used to fit the model, and p is the effective number of parameters. The cost parameter, c gives larger weight to the larger values of p. The standard GCV function is V_c with $c = 1$.

3.3 Statistical Properties

In this section the asymptotic theory and limiting distribution of $\lambda_n(t)$ will be investigated. Since local Lyapunov exponents involve the products of Jacobians, the analysis using products of random matrices may appear to be quite reasonable. However, convergence in distribution of products of random matrices requires conditions on nonnegativity of the entries (see e.g. ref [10], [7]). For Jacobians, this assumption is not possible. The use of Markov process theory will make for a richer analysis. Let X_k be a stationary Markov process. Consider the vector Markov process

$$W_k = \begin{bmatrix} X_k \\ \cdots \\ u_k \end{bmatrix}$$

where X_k is the state vector and

$$u_k = \frac{J_k u_{k-1}}{\|u_{k-1}\|}.$$

The information needed to compute a "n-step ahead" local Lyapunov exponent is the product of n Jacobians. To describe this process it is useful to think of u_k as a one-step action of the map F.

The motivation for the construction of the vector Markov process W_k is to write λ_n as the sum of random variables, i.e., λ_n can be written as $\frac{1}{n}\sum_{k=1}^{n} g(W_k)$. This can be seen more clearly by working backwards from the definition of λ_n. The dependency on time has been dropped to simplify notation. In this derivation, $t = 1$ and $\frac{u_0}{\|u_0\|} = u$ in equation (5).

$$\begin{aligned}
\lambda_n &= \frac{1}{n} \ln \left\| J_n J_{n-1} \cdots J_1 \frac{u_0}{\|u_0\|} \right\| \\
&= \frac{1}{n} \ln \left(\frac{\|J_n J_{n-1} \cdots J_1 \frac{u_0}{\|u_0\|}\| \cdots \|J_1 \frac{u_0}{\|u_0\|}\|}{\|J_{n-1} \cdots J_1 \frac{u_0}{\|u_0\|}\| \cdots \|J_1 \frac{u_0}{\|u_0\|}\|} \right) \\
&= \frac{1}{n} \left(\ln \left\| \frac{J_n J_{n-1} \cdots J_1 \frac{u_0}{\|u_0\|}}{\|J_{n-1} \cdots J_1 \frac{u_0}{\|u_0\|}\|} \right\| + \cdots + \ln \left\| J_1 \frac{u_0}{\|u_0\|} \right\| \right) \\
&= \frac{1}{n} \sum_{k=1}^{n} \log \|u_k\| \\
&= \frac{1}{n} \sum_{k=1}^{n} g(W_k)
\end{aligned}$$

The LLE can now be defined as

$$\lambda_n = \frac{1}{n} \sum_{k=1}^{n} \log \|u_k\| \tag{8}$$

$$= \frac{1}{n}\sum_{k=1}^{n} g(W_k). \qquad (9)$$

There are assumptions that need to be made on W_k and g in order to obtain distributional results for the local Lyapunov process. Distributional results will involve conditions on the one-step transition probabilities for the Markov chain. Let (Ω, F, m) be the space on which the Markov chain is defined. Usually, $\Omega = R^n$ and F is the σ-algebra of Borel sets. Let $P(x, A)$ be the one-step transition probability function for the Markov process.

$$\begin{aligned} P(x,A) &= P[X_{t+1} \in A \mid X_t = x] \\ &= \int_A k(x,y)dy \end{aligned}$$

where $k(x, y)$ is a transition density function. If there is a probability measure μ that is invariant with respect to $P(\cdot,\cdot)$, then

$$\mu(A) = \int_\Omega P(x,A)d\mu(x)$$

A sufficient condition for the existence of such an invariant measure is the Doeblin condition D.

Condition 1 (D) *There is a finite-valued measure ϕ, an integer m and an $\varepsilon > 0$, such that*

$$P^m(x,A) \leq 1-\varepsilon \quad if \;\; \phi(A) \leq \varepsilon$$

for all x.

Here, $P^n(x, A)$ is the n-step transition probability function,

$$\begin{aligned} P^n(x,A) &= P[X_{t+n} \in A \mid X_t = x] \\ &= \int \cdots \int k(x,y_1)k(y_1,y_2)\cdots k(y_{n-1},y)\cdots dy_1\cdots dy_{n-1} \end{aligned}$$

For example, Let $\phi(A)$ be defined as the Lebesgue measure of A. If F has a bounded domain, then $\phi(X) < \infty$. If the transition density function is bounded, say $k(x, y) \leq K$, then Condition D is satisfied.

Another version of the Doeblin condition will be used in order to state a central limit theorem for the local Lyapunov exponent process.

Condition 2 (D_0) *Condition D is satisfied and there is only one ergodic set which has no cyclically moving subsets.*

In addition to the D_0 condition on the transition probability function, the function g in equation (9) is assumed to be uniformly square integrable, that is, $E|g(X)|^2 < C < \infty$.

The mean of $\lambda_n(t)$ (as $n \to \infty$) converges to the global Lyapunov exponent, λ, due to the results of Furstenberg and Kesten [7] that

$$\lambda = \lim_{n\to\infty} \frac{1}{n} E\left\{\ln \|J_n J_{n-1} \ldots J_1\|\right\} \tag{10}$$

A Central Limit Theorem for Local Lyapunov Exponents based on a Central Limit Theorem for Markov processes due to Rosenblatt can now be applied to (9) [13].

Theorem 1 *Let $\{W_k\}$ be a stationary Markov process. Assume that the process W_k satisfies condition D_0. If the function g is uniformly square integrable then*

$$\frac{\sqrt{n}\,(\lambda_n - \lambda)}{\sqrt{Var\,(\lambda_n)}} \to N(0,1) \quad as \quad n \to \infty.$$

This result can make it possible to investigate the variance of the mean of the local Lyapunov exponents about the global Lyapunov exponent. To characterize the pattern of fluctuation in local Lyapunov exponents, their variance, frequency distribution, and autocorrelation may be useful.

4 Examples

In this section, the local Lyapunov exponents of two systems will be examined. The Rossler system is an example of a chaotic system and is a set of three first order differential equations. Reconstruction in three dimensional space using time-delays will make it possible to graph the local regions in space that are most unpredictable. Surface pressure data is from a higher dimensional nonchaotic system. However, the local Lyapunov exponents still provide information about the predictability of the system as related to the seasons.

4.1 Rossler System

The Rossler system is a coupled, nonlinear system of three first order differential equations:

$$\begin{aligned} \frac{da}{dt} &= -(b+c) \\ \frac{db}{dt} &= a + 0.5b \\ \frac{dc}{dt} &= 0.2 + c(a-10). \end{aligned}$$

The data $\{x_t\}$ are a series of 400 points generated by sampling one of the variables, say $a(t)$ at equally spaced time steps. The Rossler equations were numerically integrated with a fixed time step, $\Delta t = 0.5$ and a_t was sampled at every time step. A noise of five percent was added to this system at each sampling time. Since the maximum of the range of $a(t)$ is about 20, a $\sigma = 0.1$ was chosen for the variance of $\{e_t\}$. The Rossler time series $\{x_t\}$ is in the form of equation (1).

The time series of the Rossler system with five percent noise can be seen in Figure 1. The Rossler system is an example of a very simple nonlinear system that produces very complicated dynamics. A three dimensional phase space is needed to reconstruct the dynamics of the system. Figure 2 is a two dimensional phase space portrait of the system, where the third dimension is projected onto a plane. The system evolves in a clockwise direction.

In this example, the 5-step ahead local Lyapunov exponents were chosen for further analysis. There are 396 products of five Jacobians possible. Figure 3a shows the range of these exponents divided into five equally spaced groups. The smallest local exponents have been labeled 1 and the largest have been labeled 5. The largest local exponents are of interest because they represent the times when the series is most unpredictable. The size of the exponents are labeled on the time series in Figure 3b. The 5's often appear just before a large peak in the series. In Figure 4 the largest exponents have been labeled 5 in their two dimensional phase space location. Here it can be noticed that the largest exponents are found in the areas where complicated dynamics occur, that is, it is not certain whether nearby trajectories will continue to evolve in a smooth elliptical motion in the $X(t) - X(t-1)$ plane or bend into the third dimension, the $X(t-2)$ plane. This example demonstrates that local Lyapunov exponents may be able to detect unpredictable behavior locally in a system.

Figure 5 shows the distribution of local Lyapunov exponents as products of Jacobians increase from 5, 10, 20, 50, to 100. As n increases, the mean of the local Lyapunov exponents appear to converge to the global Lyapunov exponent, $\lambda = 0.045$. This is consistent with the asymptotics of the mean of the local Lyapunov exponents in equation (10).

4.2 Surface Pressure Data

Ten years of daily surface pressure data from four European cities, Postdam, Fanoe, Uccle, and Prague . A Principal Component Analysis showed the largest eigenvalue to be three orders of magnitude larger than the remaining ones so the Principal Component was used as the data series $\{x_t\}$.

Figure 6 is the distribution of local Lyapunov exponents for the pressure data. The global exponent for the system is $\lambda = -0.38$. Since $\lambda > 0$, this is not a chaotic system. However, the examination of the largest 5-step ahead local Lyapunov exponents reveals that they occur primarily in the winter months. Figure 7 shows the 100 largest of the 5-step ahead local Lyapunov exponents on a yearly cycle between 0 and 1. The largest exponents are concentrated between $0 - 0.2$ and $0.8 - 1$. This finding is consistent with the observation of meteorologists that winter is inherently more variable than the other seasons.

5 Conclusion

This paper proposes the definition of the local Lyapunov exponent as a finite time version of the global Lyapunov exponent. The global exponent measures the long term average response to perturbations of the system, in contrast to the local exponent that measures short term responses to perturbations at different locations in the state space. Instead of an invariant quantity of the system, the local Lyapunov exponent is a time dependent process. Convergence in distribution results presented for the process make the characterization of the fluctuations of these local Lyapunov exponents about their mean possible.

The two examples presented, the Rossler system and surface pressure data show that local Lyapunov exponents can be used to identify regions in the state space that are most unpredictable and the places in the times series where n-step ahead prediction may be difficult.

Systems that are identified as being near the transition to chaos, i.e., a global Lyapunov exponent near 0, such as measles [5], may vary between periods where there is finite-time sensitive dependence on initial conditions and there is not. Local Lyapunov exponents are one way to quantify this behavior. This fluctuation of the local exponents about the global exponent may provide important information in understanding the heterogeneity of the system. The characterization of the patterns of fluctuation in local Lyapunov exponents such as their variance and autocorrelation is important area for future research.

Local Lyapunov exponents can be used to determine the parts of the state space that may be more predictable or less predictable than others. They may also detect the places in the time series where short term predictability is highest and lowest. Thus the concept of predictability of a time series can be extended to the local behavior of a system and may depend on "where in the time series" or "where in the state space" is the area of interest.

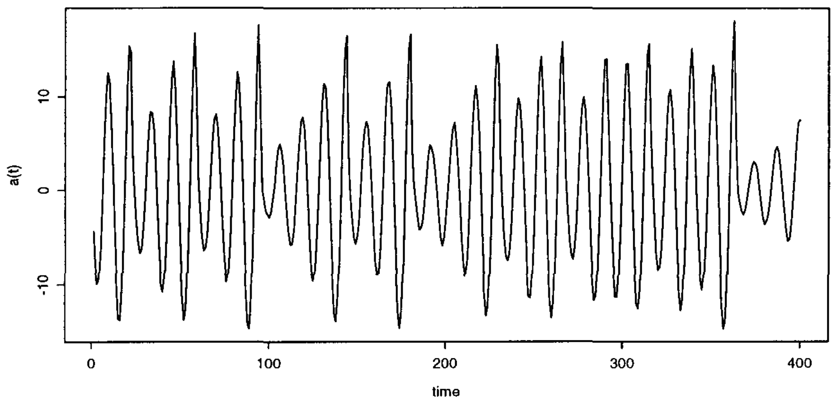

Figure 1: Rossler series: 5% noise.

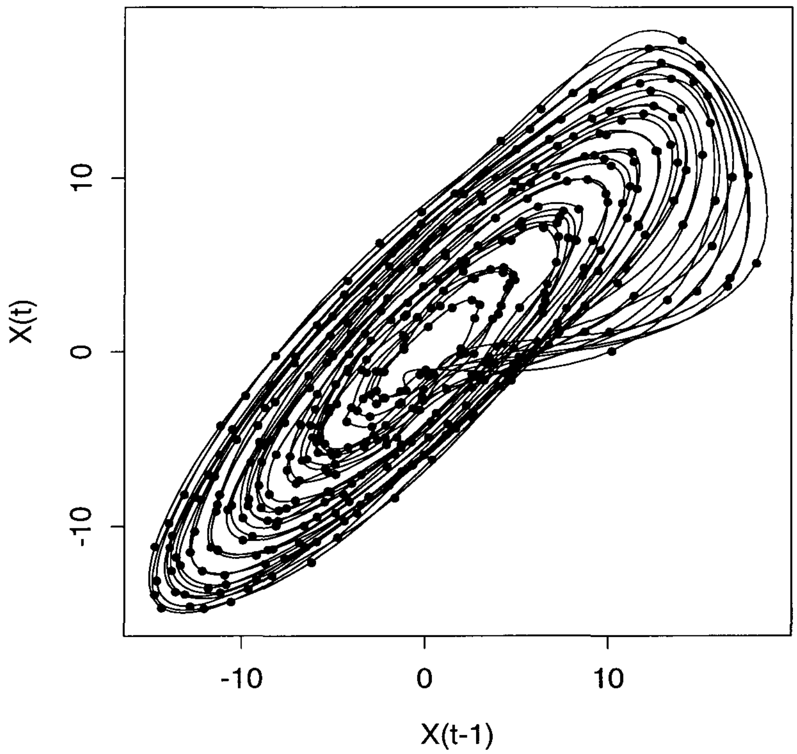

Figure 2: 2-D phase space for the 5% noise Rossler system.

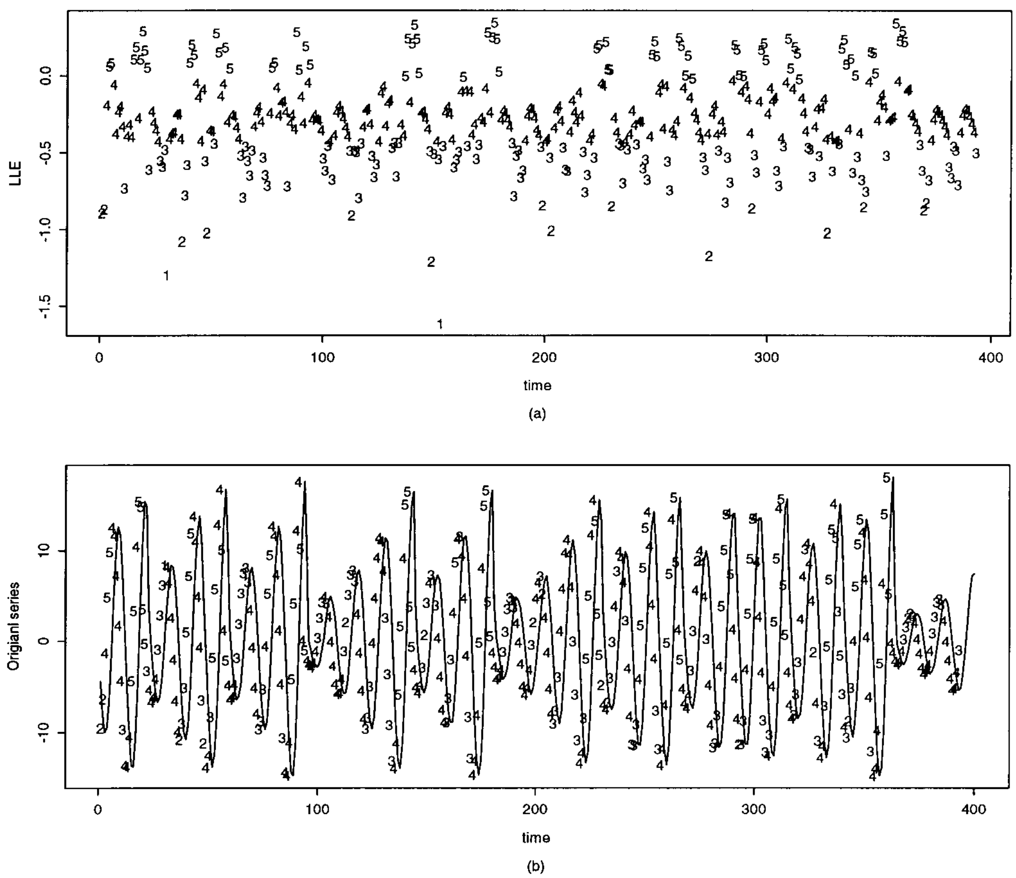

Figure 3: (a) 5-step ahead local Lyapunov exponents for the 5% noise Rossler system. (b) Series coded by size of the 5-step ahead local Lyapunov exponents.

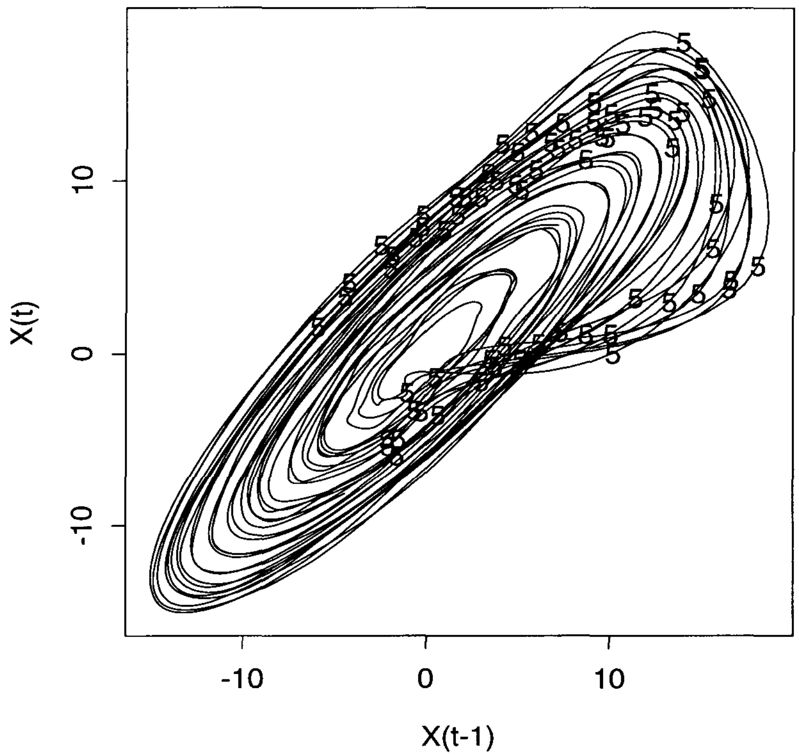

Figure 4: 5-step ahead local Lyapunov exponents for the 5% noise Rossler system in phase space.

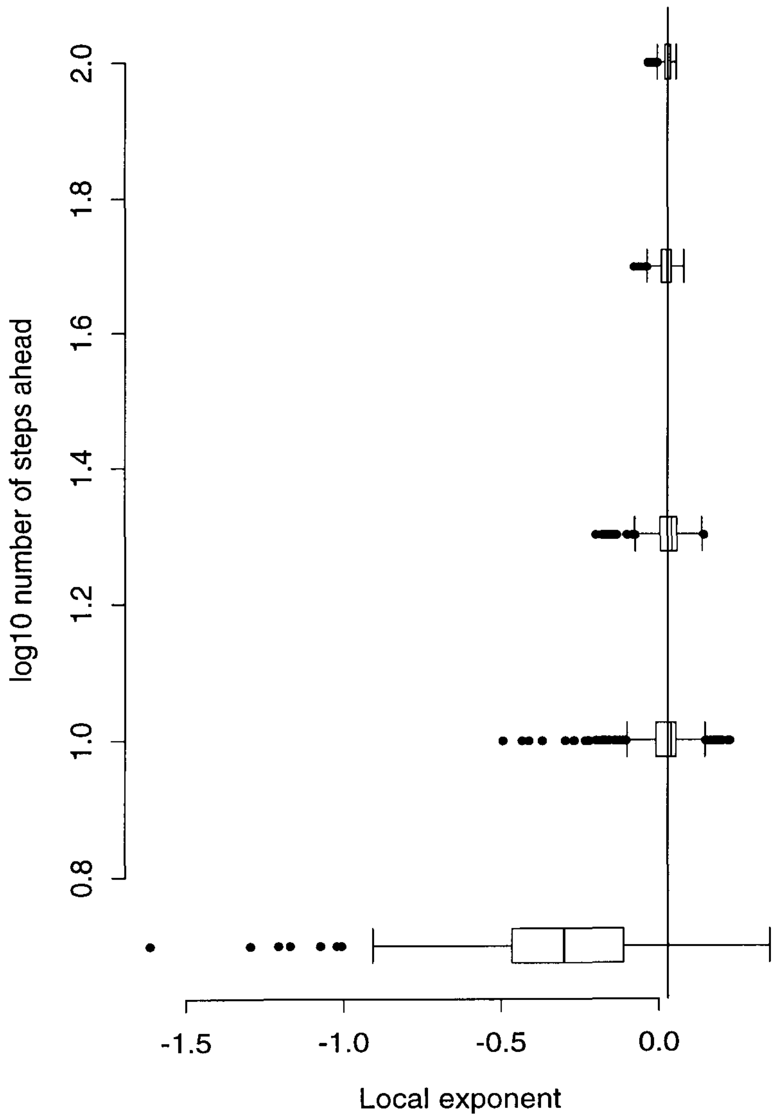

Figure 5: Distribution of the 5-step ahead local Lyapunov exponents for the 5% noise Rossler system.

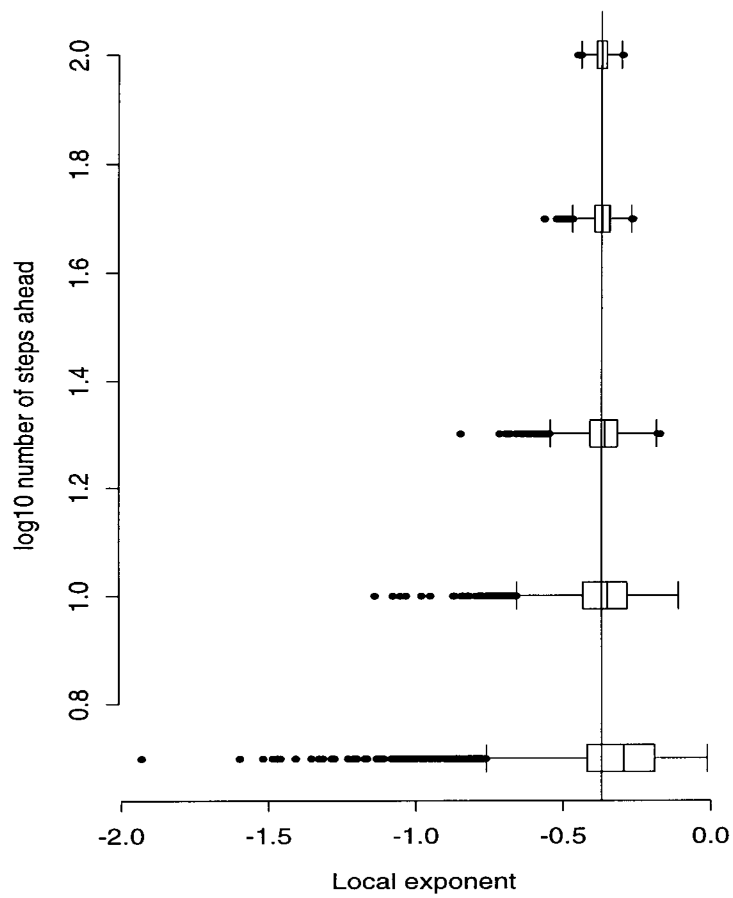

Figure 6: Distribution of the 5-step ahead local Lyapunov exponents for the pressure data.

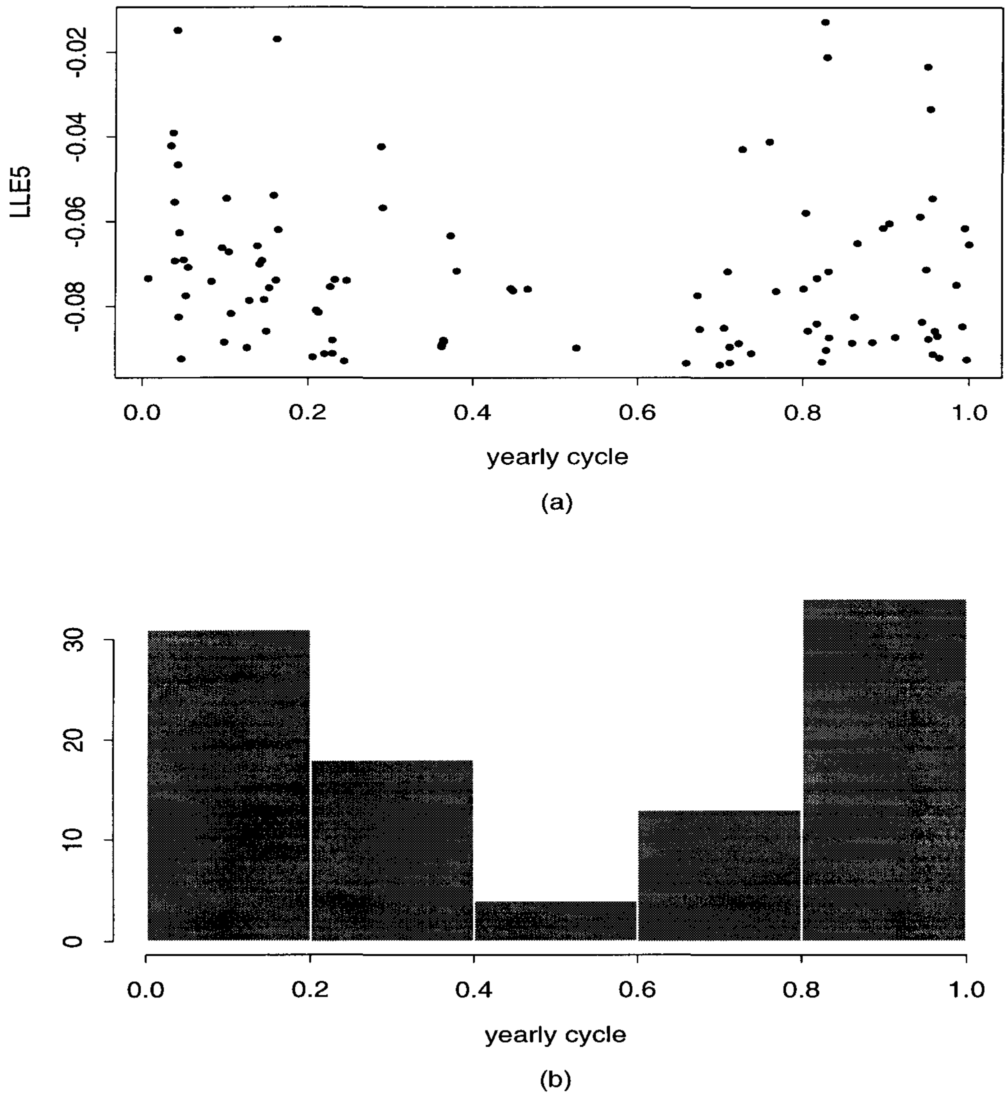

Figure 7: (a) The 100 largest 5-step ahead local Lyapunov exponents for the pressure data over the yearly cycle. (b) The histogram of (a).

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