A NOTE ON THE ARITHMETIC-GEOMETRIC-HARMONIC MEAN INEQUALITIES

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ABSTRACT

The familiar inequalities relating the arithmetic, geometric and harmonic means are derived as corollaries to likelihood ratio tests.

KEY WORDS: Exponential distribution; exponential family; inverted exponential distribution; likelihood ratio test.

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1 LIKELIHOOD RATIO TEST INEQUALITIES

Likelihood ratio test construction and the inequalities relating the arithmetic, geometric and harmonic means, are standard topics in mathematical statistics texts; see for example, Casella and Berger (1990, pp 346–350 and p 183). In this note the familiar inequalities are derived as corollaries to likelihood ratio tests for testing equality of multiple parameters versus general alternatives.

1.1 THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY

Suppose that $Y_1, \ldots, Y_n$ are independent random variables with probability density functions

$$f_{Y_i}(y) = \lambda_i \exp(-\lambda_i y), \qquad y > 0, \quad \lambda_i > 0, \quad i = 1, \ldots, n. \tag{1}$$

Consider testing

$$\mathrm{H_0}\colon\ \lambda_1 = \cdots = \lambda_n, \qquad \text{versus} \qquad \mathrm{H_A}\colon\ \lambda_i \text{ are not all equal.}$$

The maximized likelihood under $\mathrm{H_0}$ is

$$L_0 = (\bar{Y})^{-n} e^{-n},$$

and the maximized likelihood under $\mathrm{H_A}$ is

$$L_{\mathrm{A}} = \left(\prod_{i=1}^{n} Y_i\right)^{-1} e^{-n}.$$

Because the likelihood ratio test statistic is bounded above by 1, it follows that

$$1 \geq \frac{(\bar{Y})^{-n} e^{-n}}{\left(\prod_{i=1}^{n} Y_i\right)^{-1} e^{-n}}. \tag{2}$$

The inequality above is equivalent to

$$\bar{Y} \geq \left(\prod_{i=1}^{n} Y_i\right)^{1/n}, \tag{3}$$

i.e., the arithmetic-geometric mean inequality.

1.2 THE GEOMETRIC-HARMONIC MEAN INEQUALITY

The geometric-harmonic mean inequality can be derived from the arithmetic-geometric mean inequality by replacing Y_i with $1/Y_i$. Accordingly, the geometric-harmonic inequality can also be deduced from a likelihood ratio statistic if we first make the transformation of variables $X_i =$

$1/Y_i$, $i = 1, \ldots, n$, where $Y_1, \ldots, Y_n$ are distributed as in (1). Then $X_1, \ldots, X_n$ are independent random variables with densities

$$f_{X_i}(x) = \frac{\lambda_i}{x^2} \exp(-\lambda_i/x), \qquad x > 0, \quad \lambda_i > 0, \quad i = 1, \ldots, n. \tag{4}$$

Consider the same null and alternative hypotheses considered previously. The maximized likelihood under $\mathrm{H_0}$ is

$$L_0 = \left\{ \frac{n}{\sum_{i-1}^n (1/X_i)} \right\}^n \left(\prod_{i=1}^n X_i^2 \right)^{-1} e^{-n},$$

and the maximized likelihood under $\mathrm{H_A}$ is

$$L_{\mathrm{A}} = \left(\prod_{i=1}^n X_i \right) \left(\prod_{i=1}^n X_i^2 \right)^{-1} e^{-n}.$$

Because the likelihood ratio test statistic is bounded above by 1, it follows that

$$1 \geq \frac{\left\{ \frac{n}{\sum_{i-1}^n (1/X_i)} \right\}^n \left(\prod_{i=1}^n X_i^2 \right)^{-1} e^{-n},}{\left(\prod_{i=1}^n X_i \right) \left(\prod_{i=1}^n X_i^2 \right)^{-1} e^{-n}.} \tag{5}$$

The inequality above is equivalent to

$$\left(\prod_{i=1}^n X_i \right)^{1/n} \geq \frac{n}{\sum_{i=1}^n (1/X_i)}, \tag{6}$$

i.e., the geometric-harmonic mean inequality.

2 SUMMARY AND GENERALIZATIONS

Derivation of the arithmetic-geometric-harmonic mean inequalities as corollaries to likelihood ratio tests is a good exercise for introductory mathematical statistics courses as the required calculus is minimal. It reinforces the relationships between the arithmetic, geometric and harmonic means, provides practice in likelihood ratio test construction, and emphasizes the fact that likelihood ratio test statistics are bounded above by 1.

The exercise in Section (1.1) can be generalized although the generalization is of lesser interest. Suppose instead of (1) it is assumed that $Y_1, \ldots, Y_n$ are independent random variables with probability density or mass function of the exponential family form

$$f_{Y_i}(y) = \exp\left\{ \frac{y\theta_i - b(\theta_i)}{\tau} + c(y, \tau) \right\} \quad i = 1, \ldots, n. \tag{7}$$

In (7), $\tau > 0$ is assumed known, $b(\cdot)$ is a known, twice differentiable, convex function, and $c(\cdot, \cdot)$ is a known function.

Consider testing

$$\text{H}_0\text{: } \theta_1 = \cdots = \theta_n, \qquad \text{versus} \qquad \text{H}_\text{A}\text{: } \theta_i \text{ are not all equal.}$$

For these hypotheses the likelihood ratio test statistic leads to the inequality

$$\bar{Y}b'^{-1}(\bar{Y}) - b\left(b'^{-1}(\bar{Y})\right) \leq n^{-1}\sum_{i=1}^{n}\left\{Y_i b'^{-1}(Y_i) - b\left(b'^{-1}(Y_i)\right)\right\}, \tag{8}$$

where $b'^{-1}(y)$ is the inverse function to $b'(t) = db(t)/dt$.

The exponential distributions of Section (1.1) have the form (7) with $\theta_i = -\lambda_i$, $b(t) = \ln(-1/t)$, $b'(t) = -1/t$, and $b'^{-1}(y) = -1/y$. Accordingly the inequality in (3) can be deduced from that in (8) by making the appropriate substitutions for b and b'^{-1}.

Normal densities with means θ_i and variances τ^2 have the form (7) with $b(t) = t^2/2$, $b'(t) = t$, and $b'^{-1}(y) = y$. In this case (8) leads to the inequality $(\bar{Y})^2 \leq n^{-1}\sum_{i=1}^{n} Y_i^2$, or in its more familiar form, $\sum_{i=1}^{n}(Y_i - \bar{Y})^2 > 0$.

Other inequalities can be derived by consideration of other exponential family densities.

3 REFERENCES

Casella, G. and Berger, R. L. (1990), *Statistical Inference*, Pacific Grove, CA: Wadsworth & Brooks/Cole.