THE ASYMPTOTIC COVARIANCE MATRIX OF ESTIMATES OF FACTOR LOADINGS WITH NORMALIZED VARIMAX ROTATION

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ABSTRACT

We derive the asymptotic covariance matrix of the estimates of normalized varimax-rotated factor loadings. We partition the process of obtaining the estimates of normalized varimax-rotated factor loadings into the three stages: (i) normalization, (ii) raw varimax rotation, and (iii) denormalization, and use the chain rule to combine the matrix of partial derivatives from each of the three stages. For the stage (ii) we make use of the existing formulas for the asymptotic covariance matrix of the estimates of raw varimax-rotated factor loadings.

Key words and phrases: asymptotic covariance matrix, factor analysis, factor loadings, Kronecker product, vec operator, normalized varimax rotation.

1. Introduction

We derive the asymptotic covariance matrix of the estimates of normalized varimax-rotated factor loadings (Kaiser (1958), see also Neudecker (1981) and Sherin (1966)), which is an extension of work by Archer and Jennrich (1973) and Jennrich (1974) who derived the asymptotic covariance matrix of the estimates of raw varimax-rotated factor loadings. The normalized varimax rotation is by far the most frequently used method of rotation in practice of applied research using factor analysis. Thus it is important to report the asymptotic covariance matrix of the estimates of normalized varimax-rotated factor loadings.

Let $\Lambda^* = (\lambda_{ir}^*)$ be the p x m matrix of normalized varimax-rotated factor loadings (i.e., after the nomalized varimax rotation); $\Lambda = (\lambda_{ir})$ be the matrix of unrotated factor loadings (i.e., before the normalized varimax rotation); $T = (t_{sr})$ be the m x m orthogonal rotation (i.e., transformation) matrix; Ψ be the p x p positive definite diagonal matrix of residual variances. Then the factor analysis model is given by the following decomposition of the covariance matrix Σ:

$$(1) \qquad \Sigma = \Lambda\Lambda' + \Psi \qquad = (\Lambda T)(T'\Lambda') + \Psi,$$

where Λ' is the transpose of Λ. The equation (1) indicates that the matrix of factor loadings in the first term of the decomposition is not unique (called the indeterminacy regarding the orthogonal rotation). Thus we need the extra conditions regarding the rotation to make the model uniquely defined. The normal varimax rotation is an orthogonal rotation

$$\Lambda^* = \Lambda T \quad \text{and} \quad T'T = I_m$$

such that the variance of squared normalized factor loadings

$$(\frac{1}{p})\sum_{j=1}^{m}\{\sum_{i=1}^{p}(\frac{\lambda_{ij}^*}{h_i})^4-(\frac{1}{p})(\sum_{i=1}^{p}(\frac{\lambda_{ij}^*}{h_i})^2)^2\}$$

is maximized, where h_i is the square-root of the i-th diagonal element of the p x p diagonal normalization matrix $H = I_p\#(\Lambda\Lambda') = (h_i^2)$, with the Hadamard (i.e., elementwise) product # and the p-dimensional identity matrix I_p. (Note that H is unchanged by the orthogonal rotation. Thus the * sign which indicates "after the normalized varimax rotation" is not with H.)

2. The asymptotic covariance matrix

Let $\hat{\lambda} = \mathrm{vec}(\hat{\Lambda})$ and $\hat{\lambda}^* = \mathrm{vec}(\hat{\Lambda}^*)$, where $\mathrm{vec}(\hat{\Lambda})$ denotes the pm x 1 vector listing m columns of the p x m matrix $\hat{\Lambda}$ starting from the first column. Then the asymptotic covariance matrix of the estimates of normalized varimax-rotated factor loadings is expressed as

$$\mathrm{Cov}(\hat{\lambda}^*) \quad = (\frac{\partial\lambda^*}{\partial\lambda'})\mathrm{Cov}(\hat{\lambda})(\frac{\partial\lambda^*}{\partial\lambda'})',$$

where $\mathrm{Cov}(\hat{\lambda})$ is the asymptotic covariance matrix of the MLEs of unrotated factor loadings which was given by Jennrich and Thayer (1973) (See also Lawley and Maxwell (1971) and Hayashi and Sen (1996a).); $\frac{\partial\lambda^*}{\partial\lambda'}$ is the pm x pm matrix consisting of the partial derivatives which maps the differentials of $\hat{\lambda}$ to the differentials of $\hat{\lambda}^*$ evaluated at $\hat{\lambda} = \lambda$.

The following four matrices of factor loadings are involved in the mapping of the normalized varimax rotation:

(i) Unrotated factor loadings: Λ,

(ii) Normalized unrotated factor loadings: $\Lambda^{\#} = H^{-1/2}\Lambda$,

(iii) Normalized raw varimax-rotated factor loadings: $\Lambda^{b} = \Lambda^{\#}T$ $(= H^{-1/2}\Lambda T = H^{-1/2}\Lambda^{*})$,

(iv) Normalized varimax-rotated factor loadings: $\Lambda^{*} = \Lambda T$ $(= H^{1/2}\Lambda^{b})$.

(Note that the mapping in the stage (iii) is the raw varimax rotation.) Therefore, by the chain rule,

$$\frac{\partial\lambda^{*}}{\partial\lambda'} = \left(\frac{\partial\lambda^{*}}{\partial\lambda^{b'}}\right)\left(\frac{\partial\lambda^{b}}{\partial\lambda^{\#'}}\right)\left(\frac{\partial\lambda^{\#}}{\partial\lambda'}\right), \tag{2}$$

where $\frac{\partial\lambda^{*}}{\partial\lambda^{b'}}$ is the matrix of partial derivatives which maps the differentials of $\hat{\lambda}^{b}$ to the differentials of $\hat{\lambda}^{*}$ evaluated at $\hat{\lambda}^{b} = \lambda^{b}$; $\frac{\partial\lambda^{b}}{\partial\lambda^{\#'}}$ is the matrix of partial derivatives which maps the differentials of $\hat{\lambda}^{\#}$ to the differentials of $\hat{\lambda}^{b}$ evaluated at $\hat{\lambda}^{\#} = \lambda^{\#}$; $\frac{\partial\lambda^{\#}}{\partial\lambda'}$ is the matrix of partial derivatives which maps the differentials of $\hat{\lambda}$ to the differentials of $\hat{\lambda}^{\#}$ evaluated at $\hat{\lambda} = \lambda$.

Thus our task is to obtain these three matrices of partial derivatives on the RHS of the above equation. The elementwise expressions for the matrix of partial derivatives of λ^{b} with respect to $\lambda^{\#}$ in the middle of the RHS have already been given by Archer and Jennrich (1973), and we will present the matrix of partial derivatives of λ^{b} with respect to $\lambda^{\#}$ using a matrix approach. (See also Hayashi and Sen (1996b).)

3. The matrices of partial derivatives

Now, we give the expressions for the three matrices of partial derivatives necessary to compute the asymptotic covariance matrix of the estimates of normalized varimax-rotated factor loadings.

3.1 Normalization: $\Lambda \dashrightarrow \Lambda^{\#}$

The first process is to normalize the matrix of unrotated factor loadings Λ by premultiplication by $H^{-1/2}$. The matrix of partial derivatives that maps λ to $\lambda^{\#}$ is given by

$$(3) \quad \frac{\partial\lambda^{\#}}{\partial\lambda'} = -(1/2)(\Lambda'\otimes H^{-3/2})\{((\mathrm{vec}(I_p))1_{pm}')\#((I_{p2}+K_{pp})(\Lambda\otimes I_p))\} + I_m\otimes H^{-1/2},$$

where $\otimes$ is the Kronecker product; 1_{pm} is the pm-dimensional column vector whose elements are all 1's; K_{pp} is a unit matrix defined such that $K_{pp}\mathrm{vec}(A) = \mathrm{vec}(A')$ for any p x p matrix A (See, e.g., Magnus and Neudecker (1988)).

3.2 Raw varimax rotation: $\Lambda^{\#} \dashrightarrow \Lambda^{b}$

Once the initial unrotated factor loadings are normalized, the rotation is exactly identical to the raw varimax rotation. The elements of the matrix of partial derivatives for the raw varimax rotation were obtained by Archer and Jennrich (1973). We give a matrix version of the identical results, which are as follows:

$$(4) \quad \frac{\partial\lambda^{b}}{\partial\lambda^{\#'}} = \left[I_{pm} - (I_m\otimes\Lambda^{b})K_m\left\{\left(\frac{\partial\xi}{\partial\lambda^{b'}}\right)(I_m\otimes\Lambda^{b})(J_m^{(2)} - J_m^{(1)})\right\}^{-1}\left(\frac{\partial\xi}{\partial\lambda^{b'}}\right)\right](T'\otimes I_p),$$

where

$$(5) \quad \frac{\partial\xi}{\partial\lambda^{b'}} = J_m^{(2)'}(K_{mm} - I_{m2})\sum_{i=1}^{m}\left\{(J_{m,i}J_{m,i}'\otimes I_m)K_{mm}\left(\frac{\partial\xi_{(+)}}{\partial\lambda^{b'}}\right)(J_{m,i}J_{m,i}'\otimes I_p)\right\}$$

with

$$(6) \quad \frac{\partial\xi_{(+)}}{\partial\lambda^{b'}} = 3(1_m1_m'\otimes(\Lambda^{b}\#\Lambda^{b})')\#(1_m'\otimes\Lambda^{b'}\otimes 1_m) - 1_m'\otimes(\Lambda^{b}\#\Lambda^{b}\#\Lambda^{b})'\otimes 1_m$$

$$- (1/p)\left[(1_m'\otimes\Lambda^{b'}\otimes 1_m)\#\{1_m\otimes J_m^{(3)}(\mathrm{vec}(\Lambda^{b'}\Lambda^{b}))1_{pm}'\right.$$

$$- J_m^{(3)}(\mathrm{vec}(\Lambda^{b\prime}\Lambda^b))\otimes 1_m 1_{pm}'\} + 2(1_m 1_m'\otimes\Lambda^{b\prime})\#(\mathrm{vec}(\Lambda^{b\prime}\Lambda^b))1_{pm}'],$$

where $K_m = (K_{(1)}', K_{(2)}', ..., K_{(m)}')'$ is the m^2 x (1/2)m(m - 1) matrix with the m x (1/2)m(m - 1) unit matrix $K_{(r)}$ which has 1's in the (i, I(i,r)) elements, $1 \le i \le r - 1$; -1's in the (r + i, I(r, r + i)) elements, $1 \le i \le m - r$; 0's in the rest of the elements; and I(i, r) is defined such that I(i, r) = (1/2)(i - 1)(2m - i) + (r - i), $1 \le i < r \le m$; $J_{m,i} = (0, ..., 0, 1, 0, ..., 0)'$ is a m-dimensional unit vector whose i-th element is 1 and the rest are 0's; $J_m^{(1)}$ is the m^2 x (1/2)m(m - 1) unit matrix which consists of m(m - 1) submatrices with the (i, i) submatrix (of order m x (m - i)) being of the form $(0_{(m-i)\times i}, I_{m-i})'$, $1 \le i \le m - 1$ (The last m rows of $J_m^{(1)}$ are all 0's.); $J_m^{(2)}$ is the m^2 x (1/2)m(m - 1) unit matrix which consists of m(m - 1) submatrices with the (i + 1, i) submatrix (of order m x i) being of the form $(I_i, 0_{i\times(m-i)})'$, $1 \le i \le m - 1$ (The first m rows of $J_m^{(2)}$ are all 0's.) (See Tables 1 and 2 for the components of $J_m^{(1)}$ and $J_m^{(2)}$, respectively.); $J_m^{(3)} = (\mathrm{vec}(J_{m,1}J_{m,1}')', ..., \mathrm{vec}(J_{m,m}J_{m,m}')')'$ is the m x m^2 unit matrix which has 1's in the (i, (i - 1)m + i) elements, $1 \le i \le m$, and 0's in the rest of the elements.

3.3 Denormalization: $\Lambda^b \dashrightarrow \Lambda^*$

After performing the raw-varimax rotation on the matrix of normalized unrotated factor loadings, the matrix of normalized rotated factor loadings need to be denormalized to cancel out the effects of initial normalization. The matrix of partial derivatives that maps the differentials of λ^b to the differentials of λ^* is given by

$$(7)\quad \frac{\partial\lambda^*}{\partial\lambda^{b\prime}} = (1/2)(\Lambda^{b\prime}\otimes H^{1/2})\{((\mathrm{vec}(I_p))1_{pm}')\#((I_{p^2} + K_{pp})(\Lambda\otimes I_p)(\frac{\partial\lambda^b}{\partial\lambda'})^{-1})\} + I_m\otimes H^{1/2},$$

where

(8)

$$\frac{\partial\lambda^b}{\partial\lambda'} = -(1/2)((T'\Lambda')\otimes H^{-3/2})\{((vec(I_p))1_{pm}')\#((I_{p^2}+K_{pp})(\Lambda\otimes I_p))\} + T'\otimes H^{-1/2}$$

$$-(I_m\otimes(H^{-1/2}\Lambda))\Big[(I_m\otimes T)K_m\{(\frac{\partial\xi}{\partial\lambda^{b'}})(I_m\otimes\Lambda^b)(J_m^{(2)}-J_m^{(1)})\}^{-1}$$

$$(\frac{\partial\xi}{\partial\lambda^{b'}})(T'\otimes I_p)(\frac{\partial\lambda^\#}{\partial\lambda'})\Big],$$

where $\frac{\partial\lambda^\#}{\partial\lambda'}$, $\frac{\partial\xi}{\partial\lambda^{b'}}$, $J_m^{(1)}$, $J_m^{(2)}$, and K_m have already been defined in sections 3.1 and 3.2.

4. Conclusion

In sections 2 and 3, we derived the asymptotic covariance matrix of the estimates of normalized varimax-rotated factor loadings. The normalized varimax rotation is by far the most frequently used method of rotation in practice of factor analysis. Thus we think that it is important to report the analytical formulas for the asymptotic covariance matrix of the estimates of normalized varimax-rotated factor loadings.

In section 2, we presented the process of obtaining the estimates of normalized varimax-rotated factor loading as consisting of the three stages: (i) normalization, (ii) raw varimax rotation, and (iii) denormalization. Each stage has its corresponding matrix of partial derivatives, which was obtained in section 3. The three matrices of partial derivatives from the three stages were then combined by the chain rule to derive the matrix of partial derivatives corresponding to the normalized varimax rotation given in equation (2). For the stage (ii) we used a matrix version of the existing elementwise formulas for the matrix of partial derivatives corresponding to the raw varimax rotation. It substantially reduced the burden of the process of derivation.

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Table 1. Components of unit matrix $J_m^{(1)}$

$0_{1\times(m-1)}$	$0_{1\times(m-2)}$	0	0	0
I_{m-1}	$0_{(m-1)\times(m-2)}$	0	0	0
$0_{2\times(m-1)}$	$0_{2\times(m-2)}$	0	0	0
$0_{(m-2)\times(m-1)}$	I_{m-2}	0	0	0
0	$0_{3\times(m-2)}$	0	0	0
.....	.....	.....	.....	.....
0	0	0	$0_{(m-2)\times2}$	$0_{(m-2)\times1}$
0	0	0	I_2	$0_{2\times1}$
0	0	0	$0_{(m-1)\times2}$	$0_{(m-1)\times1}$
0	0	0	$0_{1\times2}$	I_1
0	0	0	$0_{m\times2}$	$0_{m\times1}$

Note: **I**'s are the identity (sub)matrices and 0's are the null (sub)matrices. The subscripts indicate the orders of the submatrices.

Table 2. Components of unit matrix $J_m^{(2)}$

$0_{m\times1}$	$0_{m\times2}$	0	0	0
I_1	$0_{1\times2}$	0	0	0
$0_{(m-1)\times1}$	$0_{(m-1)\times2}$	0	0	0
$0_{2\times1}$	I_2	0	0	0
$0_{(m-2)\times1}$	$0_{(m-2)\times2}$	0	0	0
.....	.....	.....	.....	.....
0	0	0	$0_{3\times(m-2)}$	0
0	0	0	I_{m-2}	$0_{(m-2)\times(m-1)}$
0	0	0	$0_{2\times(m-2)}$	$0_{2\times(m-1)}$
0	0	0	$0_{(m-1)\times(m-2)}$	I_{m-1}
0	0	0	$0_{1\times(m-2)}$	$0_{1\times(m-1)}$

Note: **I**'s are the identity (sub)matrices and 0's are the null (sub)matrices. The subscripts indicate the orders of the submatrices.

(1) Proof for the matrix of partial derivatives of $\lambda^{\#}$ with respect to λ in (3)

(i) Express $\frac{\partial\lambda^{\#}}{\partial\lambda'}$ as a matrix function involving $\frac{\partial h^{-1/2}}{\partial\lambda'}$

$$\Lambda^{\#} = H^{-1/2}\Lambda$$

$$\Rightarrow \quad d\Lambda^{\#} = (dH^{-1/2})\Lambda + H^{-1/2}(d\Lambda)$$

$$\begin{aligned}\Rightarrow \quad d\lambda^{\#} &= \mathrm{vec}(d\Lambda^{\#})\\ &= \mathrm{vec}((dH^{-1/2})\Lambda) + \mathrm{vec}(H^{-1/2}(d\Lambda))\\ &= (\Lambda'\otimes I_p)\mathrm{vec}(dH^{-1/2}) + (I_m\otimes H^{-1/2})\mathrm{vec}(d\Lambda)\\ &= (\Lambda'\otimes I_p)(dh^{-1/2}) + (I_m\otimes H^{-1/2})(d\lambda)\end{aligned}$$

$$\Rightarrow \quad \frac{\partial\lambda^{\#}}{\partial\lambda'} = (\Lambda'\otimes I_p)\left(\frac{\partial h^{-1/2}}{\partial\lambda'}\right) + I_m\otimes H^{-1/2}$$

(ii) Obtain the expression for $\frac{\partial h}{\partial\lambda'}$

$$H = I_p\#(\Lambda\Lambda')$$

$$\begin{aligned}\Rightarrow \quad dH &= I_p\#(d(\Lambda\Lambda'))\\ &= I_p\#((d\Lambda)\Lambda' + \Lambda(d\Lambda'))\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad dh &= \mathrm{vec}(dH)\\ &= \mathrm{vec}(I_p\#((d\Lambda)\Lambda' + \Lambda(d\Lambda')))\\ &= \mathrm{vec}(I_p)\#\mathrm{vec}((d\Lambda)\Lambda' + \Lambda(d\Lambda'))\\ &= \mathrm{vec}(I_p)\#(\mathrm{vec}((d\Lambda)\Lambda') + \mathrm{vec}(\Lambda(d\Lambda')))\\ &= \mathrm{vec}(I_p)\#((\Lambda\otimes I_p)\mathrm{vec}(d\Lambda) + (I_p\otimes\Lambda)\mathrm{vec}(d\Lambda'))\\ &= \mathrm{vec}(I_p)\#((\Lambda\otimes I_p)(d\lambda) + (I_p\otimes\Lambda)K_{pm}(d\lambda))\end{aligned}$$

$$= vec(I_p)\#((\Lambda \otimes I_p + (I_p \otimes \Lambda)K_{pm})(d\lambda))$$

$$= vec(I_p)\#((\Lambda \otimes I_p + K_{pp}(\Lambda \otimes I_p))(d\lambda))$$

$$= vec(I_p)\#((I_{p^2} + K_{pp})(\Lambda \otimes I_p)(d\lambda))$$

$$\Rightarrow \quad \frac{\partial h}{\partial \lambda'} = ((vec(I_p))1_{pm}')\#((I_{p^2} + K_{pp})(\Lambda \otimes I_p))$$

(iii) Obtain the expression for $\frac{\partial h^{-1/2}}{\partial \lambda'}$

$$dH^{-1/2} = -(1/2)H^{-3/2}(dH) \qquad \text{(since H is diagonal)}$$

$$\Rightarrow \quad dh^{-1/2} = vec(dH^{-1/2})$$

$$= -(1/2)vec(H^{-3/2}(dH))$$

$$= -(1/2)(I_p \otimes H^{-3/2})vec(dH)$$

$$= -(1/2)(I_p \otimes H^{-3/2})(dh)$$

$$\Rightarrow \quad \frac{\partial h^{-1/2}}{\partial \lambda'} = -(1/2)(I_p \otimes H^{-3/2})(\frac{\partial h}{\partial \lambda'})$$

$$\Rightarrow \quad \frac{\partial h^{-1/2}}{\partial \lambda'} = -(1/2)(I_p \otimes H^{-3/2})(((vec(I_p))1_{pm}')\#((I_{p^2} + K_{pp})(\Lambda \otimes I_p)))$$

(We used the formula for $\frac{\partial h}{\partial \lambda'}$ in (ii).)

(iv) Combine (i) and (iii) to obtain (3)

$$\frac{\partial \lambda^\#}{\partial \lambda'} = -(1/2)(\Lambda' \otimes I_p)(I_p \otimes H^{-3/2})(((vec(I_p))1_{pm}')\#((I_{p^2} + K_{pp})(\Lambda \otimes I_p)))$$

$$+ I_m \otimes H^{-1/2}$$

$$= -(1/2)(\Lambda' \otimes H^{-3/2})(((vec(I_p))1_{pm}')\#((I_{p^2} + K_{pp})(\Lambda \otimes I_p))) + I_m \otimes H^{-1/2}$$

(2) Outline of proof for the expression for the matrix of partial derivatives of λ^b with respect to $\lambda^\#$ given in (4) - (6)

Although the matrix expression for the partial derivatives necessary for the raw varimax rotation has already been given in Hayashi and Sen (1996b), we give the outline of proof here for completeness. First, the coordinatewise expression for the elements of the matrix of partial derivatives given by Archer and Jennrich (1973) are

$$(9) \qquad \frac{\partial \lambda_{ir}^{b}}{\partial \lambda_{js}^{\#}} = \delta_{ij} t_{sr} - \sum_{u=1}^{m-1} \sum_{v=u+1}^{m} \sum_{t=1}^{m} \left(e_{iruv} \left(\frac{\partial \xi_{uv}}{\partial \lambda_{it}^{b}} \right) t_{st} \right).$$

Here, $E = (e_{iruv})$ has the elements

$$(10) \qquad e_{iruv} = \sum_{t=1}^{r-1} \lambda_{it}^{b} L^{l(t,r),l(u,v)} - \sum_{t=r+1}^{m} \lambda_{it}^{b} L^{l(r,t),l(u,v)},$$

where L^{ij} is the (i, j) element of the inverse of the matrix L, and the (l(r,s), l(u,v)) element of L is

$$(11) \qquad L_{l(r,s),l(u,v)} = \sum_{i=1}^{p} \left(\lambda_{iu}^{b} \left(\frac{\partial \xi_{rs}}{\partial \lambda_{iv}^{b}} \right) - \lambda_{iv}^{b} \left(\frac{\partial \xi_{rs}}{\partial \lambda_{iu}^{b}} \right) \right),$$

with $l(r, s) = (1/2)(r - 1)(2m - r) + (s - r)$, $1 \le r < s \le m$, and $l(u, v) = (1/2)(u - 1)(2m - u) + (v - u)$, $1 \le u < v \le m$. For $1 \le r < s \le m$,

$$(12) \qquad \frac{\partial \xi_{rs}}{\partial \lambda_{ir}^{b}} = 3(\lambda_{ir}^{b})^2 \lambda_{is}^{b} - (\lambda_{is}^{b})^3 - \left(\frac{1}{p}\right) \left\{ \lambda_{is}^{b} \sum_{j=1}^{p} ((\lambda_{jr}^{b})^2 - (\lambda_{js}^{b})^2) + 2 \lambda_{ir}^{b} \sum_{j=1}^{p} (\lambda_{jr}^{b} \lambda_{js}^{b}) \right\},$$

$$\frac{\partial \xi_{rs}}{\partial \lambda_{is}^{b}} = - \frac{\partial \xi_{sr}}{\partial \lambda_{is}^{b}},$$

$$\frac{\partial \xi_{rs}}{\partial \lambda_{it}^{b}} = 0, \text{ for all } t \neq s.$$

Now, we can easily verify that (5) and (6) are the matrix expression corresponding to (12). (Note that $1_m 1_m' \otimes \Lambda^{b\prime}$ and $1_m 1_m' \otimes (\Lambda^b \# \Lambda^b)'$ correspond to λ_{ir}^b and $(\lambda_{ir}^b)^2$; $\Lambda^{b'}\otimes 1_m 1_m'$ and $(\Lambda^b \# \Lambda^b \# \Lambda^b)'\otimes 1_m 1_m'$ corresponds to λ_{is}^b and $(\lambda_{is}^b)^3$, respectively. Note also that $1_m \otimes J_m^{(3)}(\mathrm{vec}(\Lambda^{b'}\Lambda^b))1_{pm}'$ and $J_m^{(3)}(\mathrm{vec}(\Lambda^{b'}\Lambda^b))\otimes 1_m 1_{pm}'$ correspond to $(\Lambda^{b'}\Lambda^b)_{rr}$ and $(\Lambda^{b'}\Lambda^b)_{ss}$, respectively.)

Next, the matrix expressions for (10) and (11) are

(13) $$E = (I_m \otimes \Lambda^b)K_m L^{-1},$$

(14) $$L = (\frac{\partial \xi}{\partial \lambda^{b'}})(I_m \otimes \Lambda^b)(J_m^{(2)} - J_m^{(1)}),$$

respectively, where K_m, $J_m^{(1)}$, and $J_m^{(2)}$ have already been defined in the text. Thus, using (13) and (14), we obtain the matrix expression for (9) as

$$\begin{aligned}\frac{\partial \lambda^b}{\partial \lambda^{\#}} &= (I_{pm} - E(\frac{\partial \xi}{\partial \lambda^{b'}}))(T'\otimes I_p) \\ &= (I_{pm} - (I_m \otimes \Lambda^b)K_m L^{-1}(\frac{\partial \xi}{\partial \lambda^{b'}}))(T'\otimes I_p) \\ &= \Big[I_{pm} - (I_m \otimes \Lambda^b)K_m \Big\{(\frac{\partial \xi}{\partial \lambda^{b'}})(I_m \otimes \Lambda^b)(J_m^{(2)} - J_m^{(1)})\Big\}^{-1}(\frac{\partial \xi}{\partial \lambda^{b'}})\Big] \\ &\qquad (T'\otimes I_p).\end{aligned}$$

(3) Proof for the matrix of partial derivatives of λ^* with respect to λ^b in (7)

(i) Express $\frac{\partial \lambda^*}{\partial \lambda^{b'}}$ as a matrix function involving $\frac{\partial h^{1/2}}{\partial \lambda'}$

$$\begin{aligned}\Lambda^* &= H^{1/2}\Lambda^b \\ \Rightarrow \quad \frac{\partial \lambda^*}{\partial \lambda^{b'}} &= (\Lambda^{b'}\otimes I_p)(\frac{\partial h^{1/2}}{\partial \lambda^{b'}}) + I_m \otimes H^{1/2}\end{aligned}$$

(ii) Obtain the expression for $\frac{\partial h^{1/2}}{\partial \lambda'}$

$$\begin{aligned}dH^{1/2} &= (1/2)H^{1/2}(dH) \\ \Rightarrow \quad dh^{1/2} &= \mathrm{vec}(dH^{1/2})\end{aligned}$$

$$= (1/2)\text{vec}(H^{1/2}(dH))$$

$$= (1/2)(I_p \otimes H^{1/2})(dh)$$

$$\Rightarrow \quad \frac{\partial h^{1/2}}{\partial \lambda^{b'}} = (1/2)(I_p \otimes H^{1/2})\Big\{((\text{vec}(I_p))1_{pm}')\#((I_{p^2} + K_{pp})(\Lambda \otimes I_p)(\frac{\partial \lambda}{\partial \lambda^{b'}}))\Big\}$$

$$= (1/2)(I_p \otimes H^{1/2})\Big\{((\text{vec}(I_p))1_{pm}')\#((I_{p^2} + K_{pp})(\Lambda \otimes I_p)(\frac{\partial \lambda^b}{\partial \lambda'})^{-1})\Big\}$$

(We used the expression for the differential dh in section (1) in the proofs.)

(iii) Combine (i) and (ii) to obtain (7)

(4) Proof for the matrix of partial derivatives of λ^b with respect to λ in (8)

(i) Express $\frac{\partial \lambda^b}{\partial \lambda'}$ as a matrix function involving $\frac{\partial h}{\partial \lambda'}$ and $\frac{\partial t}{\partial \lambda'}$, and insert the expression for $\frac{\partial h}{\partial \lambda'}$ in section (1) in the proofs

$$\Lambda^b = \Lambda^{\#}T = H^{-1/2}\Lambda T$$

$$\Rightarrow \quad d\Lambda^b = (dH^{-1/2})\Lambda T + H^{-1/2}(d\Lambda)T + H^{-1/2}\Lambda(dT)$$

$$= -\,(1/2)H^{-3/2}(dH)\Lambda T + H^{-1/2}(d\Lambda)T + H^{-1/2}\Lambda(dT)$$

$$\Rightarrow \quad d\lambda^b = \text{vec}(d\Lambda^b)$$

$$= -\,(1/2)\text{vec}(H^{-3/2}(dH)\Lambda T) + \text{vec}(H^{-1/2}(d\Lambda)T) + \text{vec}(H^{-1/2}\Lambda(dT))$$

$$= -\,(1/2)((T'\Lambda')\otimes H^{-3/2})\text{vec}(dH) + (T'\otimes H^{-1/2})\text{vec}(d\Lambda)$$

$$+ (I_m \otimes (H^{-1/2}\Lambda))\text{vec}(dT)$$

$$= -\,(1/2)((T'\Lambda')\otimes H^{-3/2})(dh) + (T'\otimes H^{-1/2})(d\lambda) + (I_m \otimes (H^{-1/2}\Lambda))(dt)$$

$$\Rightarrow \quad \frac{\partial \lambda^b}{\partial \lambda'} = -\,(1/2)((T'\Lambda')\otimes H^{-3/2})(\frac{\partial h}{\partial \lambda'}) + T'\otimes H^{-1/2} + (I_m \otimes (H^{-1/2}\Lambda))(\frac{\partial t}{\partial \lambda'})$$

$$\Rightarrow \quad \frac{\partial \lambda^b}{\partial \lambda'} = -\,(1/2)((T'\Lambda')\otimes H^{-3/2})\Big\{((\text{vec}(I_p))1_{pm}')\#((I_{p^2} + K_{pp})(\Lambda \otimes I_p))\Big\}$$

$$+ T' \otimes H^{-1/2} + (I_m \otimes (H^{-1/2}\Lambda))(\frac{\partial t}{\partial \lambda'})$$

(ii) Obtain the matrix expression for $\frac{\partial t}{\partial \lambda'}$

We can obtain the matrix expression for $\frac{\partial t}{\partial \lambda'}$ in the identical way to the expression for the matrix of partial derivatives of t with respect to the sample covariance matrix arranged as a vector given in Hayashi and Sen (1996b). First, equation (12) in Archer and Jennrich (1973) gives

$$T'dT = -L^{-1}[d\xi((d\Lambda^{\#})T)].$$

$$\Rightarrow \qquad dT = -(TT')^{-1}TL^{-1}(d\xi((d\Lambda^{\#})T))$$
$$= -TL^{-1}(d\xi((d\Lambda^{\#})T)),$$

which is, in our notation, equivalent to

$$dt_{ir} = -\sum_{u=1}^{m-1}\sum_{v=u+1}^{m}\sum_{j=1}^{p}\sum_{s=1}^{m}\sum_{t=1}^{m}(g_{iruv})(\frac{\partial \xi_{uv}}{\partial \lambda_{jt}{}^{b}})(d\lambda_{js}{}^{\#})(t_{st}),$$

where

$$g_{iruv} = \sum_{t=1}^{r-1} t_{it}L^{l(t,r),l(u,v)} - \sum_{t=r+1}^{m} t_{it}L^{l(r,t),l(u,v)},$$

with $l(r,s) = (1/2)(r-1)(2m-r) + (s-r)$; $1 \le i, r, s \le m$, $1 \le j \le p$, $1 \le u < v \le m$. Now, arrange dt_{ir} as

(15) $$dt_{ir} = -\sum_{w=1}^{(1/2)m(m-1)}\sum_{j=1}^{p}\sum_{t=1}^{m}(g_{ir,w})(\frac{\partial \xi_{w}}{\partial \lambda_{jt}{}^{b}})\{\sum_{s=1}^{m}(d\lambda_{js}{}^{\#})(t_{st})\}$$

and construct the matrix expression for g_{iruv}:

(16) $$G_m = (I_m \otimes T)K_m(L^b)^{-1}$$

$$= (I_m \otimes T)K_m\left\{\left(\frac{\partial \xi}{\partial \lambda^{b\prime}}\right)(I_m \otimes \Lambda^b)(J_m^{(2)} - J_m^{(1)})\right\}^{-1},$$

where the order of G_m is m^2 x (1/2)m(m - 1), and $\frac{\partial \xi}{\partial \lambda^{b\prime}}$, $J_m^{(1)}$, $J_m^{(2)}$, and K_m are all defined in the text. Then the above coordinatewise expression in (15) can be written in matrix form as

$$\begin{aligned} dt &= d(\text{vec}(T)) \\ &= -G_m\left(\frac{\partial \xi}{\partial \lambda^{*\prime}}\right)\text{vec}((d\Lambda^{\#})T) \\ &= -(I_m \otimes T)K_m\left\{\left(\frac{\partial \xi}{\partial \lambda^{b\prime}}\right)(I_m \otimes \Lambda^b)(J_m^{(2)} - J_m^{(1)})\right\}^{-1}\left(\frac{\partial \xi}{\partial \lambda^{*\prime}}\right)(T' \otimes I_p)(d\lambda^{\#}). \end{aligned}$$

(We used the expression for G_m in (16).) Thus

$$\frac{\partial t}{\partial \lambda'} = -(I_m \otimes T)K_m\left\{\left(\frac{\partial \xi}{\partial \lambda^{b\prime}}\right)(I_m \otimes \Lambda^b)(J_m^{(2)} - J_m^{(1)})\right\}^{-1}\left(\frac{\partial \xi}{\partial \lambda^{*\prime}}\right)(T' \otimes I_p)\left(\frac{\partial \lambda^{\#}}{\partial \lambda'}\right)$$

(iii) Combine (i) and (ii) to obtain (8)

Appendix: Some Properties of Matrix Operations Used in Proofs Section (which are given in Magnus and Neudecker (1988))

(i)	$(A \otimes B)(C \otimes D)$	$= (AC) \otimes (BD)$	
(ii)	$(A \otimes B)'$	$= A' \otimes B'$	
(iii)	$K_{mn}\text{vec}(A)$	$= \text{vec}(A')$	(for m x n matrix A)
(iv)	$K_{pm}(A \otimes B)$	$= (B \otimes A)K_{qn}$	(for m x n matrix A and p x q matrix B)
(v)	$\text{vec}(ABC)$	$= (C' \otimes A)\text{vec}(B)$	
(via)	$\text{vec}(AB)$	$= (B' \otimes I_m)\text{vec}(A)$	
(vib)	$\text{vec}(AB)$	$= (I_q \otimes A)\text{vec}(B)$	
(vii)	$\text{vec}(A \# B)$	$= \text{vec}(A) \# \text{vec}(B)$	
(viii)	$d(AB)$	$= (dA)B + A(dB)$	