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ABSTRACT. Use of comparison theorems for solutions of stochastic differential equations is made to prove sufficient conditions for almost sure or stochastic ordering of first passage times of one-dimensional diffusion processes through an assigned boundary. Conditions on boundary and parameters that make orded the first passage time distributions of two different diffusion processes are determined. Some examples are discussed in detail in order to show the use of ordering notion in the comparison of the features of different models used in various branches of science ranging from biology to finance or physics.

1. INTRODUCTION

The study of the first passage time through boundaries for one-dimensional diffusion processes is of importance in a variety of problems varying from biology to finance, statistics and queuing theory (cf. in example, [2],[3],[21],[25],[27]). Since analytical results are seldom obtained numerical or simulation approaches are largely employed in these contexts to determine features of the considered models (cf. [8],[9] and references quoted therein). Alternatively, suitable techniques for obtaining asymptotic behaviors of first passage times (FPT) distributions are available (cf. [17]) and in various instances at least first moments of FPT are known in analytical form (cf. [22],[20]). Unfortunately, the extremely complex expression of these quantities makes their use hard,

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when one wishes to compare features of different stochastic models used in the literature to describe a specific phenomena (cf. [13]) in order to decide which one is more realistic. However in many instances it would be hopeful to understand the dependence of the model features from the single parameters involved in the description through the stochastic differential equation (SDE) or to determine which SDE gives rise to more realistic features.

In this paper we propose a new approach to these problems. In order to compare features of different diffusion models we investigate on order relationships between their FPTs, proving, in Section 3, suitable sufficient conditions for these orderings.

While comparison of random variables via stochastic ordering are largely discussed in the literature (cf.[24] and references cited therein), particularly with reference of reliability problems, their use for FPTs comparison seems limited to state discrete instances (cf. [5],[23],[1],[16],[15],[14]) to very specific models (cf. [4]).

The main purpose of this paper is to compare FPTs corresponding to two time-homogeneous diffusion processes $X_1(t), X_2(t)$ originated in $X_1(0) = X_2(0) = x_0$, crossing a boundary $S > x_0$, and characterized by drift and diffusion coefficient $\mu_i(x)$ and $\sigma_i^2(x), i = 1, 2$ whose SDE can be written as usual:

$$dX_i(t) = \mu_i(x)dt + \sigma_i(x)dW(t) \tag{1.1}$$

where $W(t)$ indicates a standard Wiener process.

In order to determine conditions to order FPTs we make use of well-known comparison theorems between sample paths of the considered processes. Since Skorohod comparison theorem (cf. [26]) various improvements to this theorem have been proposed in the literature (cf. [11], [6], [19], [18], [29], [10], [28]) mainly directed to relax some of the hypotheses introduced in the former version. Here we limit ourself to report in Section 2 the version due to Yamada (cf. [28]) that will be used in Section 3 to prove theorems about FPTs. We explicitly note how comparison theorems for SDE have been mainly used in the literature to prove existence and unicity theorems for solutions of SDE while applications seems to focus on optimal control problems (cf. i.e.[10], [11] and references quoted therein). Their use in comparison features of different stochastic models seems to be an useful alternative approach not yet sufficiently investigated.

Further, in Section 2 we briefly introduce the necessary mathematical background on stochastic ordering limiting ourselves to results and notations in use in this paper while we refer to Shaked (cf. [24]) for a complete description.

Section 3 is devoted to the presentation of our main results, i.e. sufficient conditions that guarantee suitable ordering between FPTs corresponding to diffusion processes. A corollary of this theorems enable us to pinpoint how different diffusion processes can give rise to FPTs characterized by identical distribution for particular choices of parameters values. This result seems of interest when the choice between models is not motivated theoretically and estimations are made by means of FPTs measures.

Finally in Section 4 we discuss some examples of interest for applications, making use of the theorems of Section 3. In particular we show how use of our method can help to understand the role of single model parameters when the complexity of the model could blind such properties. Furthermore we pinpoint how the existence of order relationships between two FPTs arising from two different diffusion processes can help to establish which one determine more realistic features in specific instances.

2. Mathematical background

Throughout this paper $(\Omega, \mathcal{A}, P)$ will be the background probability space and we assume that it is large enough to hold the processes we will discuss. We consider two diffusion processes defined by eq.1.1 on the diffusion intervals $(\alpha_i, \beta_i), i = 1,2$ not necessarily coinciding. We chose x_0 and S belonging to the intersection of the two diffusion intervals and we indicate with

(2.1a) $$T_{X_i}(S\,|x_0) = \inf\{t > 0 : X_i(t) \geq S; \quad X_i(t_0) = x_0\}$$

the first passage time random variable of $X_i(t)$ through S.

For the considered processes, if pathwise uniquess of solutions holds, Yamada (cf. [28]) proved the following version of Skorohod theorem to which we will refer as Skorohod-Yamada:

Theorem 1. *Let $\sigma_1(x) = \sigma_2(x)$ and $\mu_1(x) \leq \mu_2(x)$ then, for every $t \in (0, \infty)$, one has*

(2.1b) $$X_1(t) \leq X_2(t).$$

with probability one.

Remark 1. *Yamada considered non-homogeneous diffusion processes but we will limit ourselves to homogeneous ones to avoid heavier notations. However our results can be extended to non-homogeneous processes when uniqueness of solution is guaranteed.*

As further comment we note that this theorem deals with almost sure ordering notion between two stochastic processes driven by the same Wiener process. However if we consider the two SDE as driven by two independent Wiener processes the theorem assert the existence of stochastic order between the two process. Indeed recall that (cf. [24]):

Definition 1. *Two random variables Y and Z such that*

$$P(Y > u) \leq P(Z > u), \forall u \in \Re \tag{2.3}$$

are ordered in the (usual) stochastic order and we indicate that Y is smaller of Z in this order writing

$$Y \leq^{st} Z. \tag{2.4}$$

Than it can be proven (cf. [24]) that almost sure order implies stochastic order but viceversa is not true.

3. Main results

Our goal here is to determine suitable conditions on the coefficients characterizing 1.1 to prove almost sure or stochastic ordering for their FPTs through assigned boundaries. To pursue this objective we need to state some auxiliary results.

Lemma 1. *Consider the FPTs of any diffusion process X, originating at x_0, through the boundaries S' and S , with $S' \geq S$. It holds with probability one:*

$$T_X(S\,|x_0) \leq T_X(S'\,|x_0) \tag{3.1}$$

Proof. Due to continuity property of diffusion processes, one has

$$T_X(S'\,|x_0) = T_X(S\,|x_0) + T_X(S'\,|S), \tag{3.2}$$

and the thesis follows after remarking that $T_X(S'\,|S) \geq 0$. ∎

Lemma 2. *Consider the FPTs through S of the diffusion processes Y_1, Y_2 characterized by equals drift and diffusion coefficient but originated with probability one in two points $y_1 < y_2$ respectively. One has:*

$$T_{Y_1}(S\,|y_1) \geq^{st} T_{Y_2}(S'\,|y_2). \tag{3.3}$$

Proof. Due to the strong Markov property of the considered processes one has:

$$\begin{aligned} T_{Y_1}(S\,|y_1) = T_{Y_1}(y_2\,|y_1) + T_{Y_1}(S\,|y_2) =^{st} \\ T_{Y_1}(y_2\,|y_1) + T_{Y_2}(S\,|y_2) \geq^{st} T_{Y_2}(S\,|y_2) \end{aligned} \tag{3.4}$$

∎

Lemma 3. *Let $T_X(S\,|x_0)$ and $T_{X^*}(S\,|x_0)$ be the FPTs of two processes obeying to the same SDE with the same initial condition through the same boundary S. Suppose the process X defined on (α,β) while X^* is constrained by a reflecting boundary in $a \in (\alpha, x_0)$. It holds:*

$$T_X(S\,|x_0) \geq T_{X^*}(S\,|x_0) \tag{3.5}$$

with probability one.

Proof. We partition the sample paths of X^* in two groups. To the first one belongs sample paths that do not touch a before crossing S and coincide with paths of X, while to the second group belongs paths reflected in a. Clearly these paths cross S at times less or equal to times at which cross S the paths of X that have pursued their excursion under a and 3.5 follows. ∎

Let us introduce now for each process X_i a space transformation

$$y = g_i(x) = \int \frac{dx}{\sqrt[2]{\sigma_i^2(x)}}, \qquad i = 1,2 \tag{3.6}$$

that changes the process X_i into a process Y_i having unit infinitesimal variance and with drift (cf. [12])

$$\mu_{Y_i}(y) = \frac{1}{\sqrt[2]{\sigma_i^2(x)}}\left(-\frac{1}{4}\frac{d\sigma_i^2}{dx} + b_i(x)\right)\Bigg|_{x=g_i^{-1}(y)}. \tag{3.7}$$

We can now prove the following

Lemma 4. *Consider the process Y_2, let $g_i(x), i = 1,2$ be defined from 3.6 and suppose:*

$$\sigma_1^2(x) \geq \sigma_2^2(x) \tag{3.8}$$

for $x \in (\max(\alpha_1, \alpha_2), S))$ and

$$\begin{aligned} \mu_{Y_2}(y) &\geq \mu_{Y_2}(y - (g_1(S) - g_1(x_0)), \\ y &\in [g_2(\alpha_2), g_2(S) - (g_2(x_0) - g_1(x_0))] \end{aligned} \tag{3.9}$$

then it holds:

$$T_{Y_2}(g_2(S)\,|g_2(x_0)) \geq T_{Y_2}(g_1(S)\,|g_1(x_0)) \tag{3.10}$$

with probability one.

Proof. Consider the points $g_1(S), g_1(x_0), g_2(S), g_2(x_0)$, due to (3.8) and (3.6) they can be ordered in two different ways depending upon

$$a.\ (g_2(x_0), g_2(S)) \cap (g_1(x_0), g_1(S)) = (g_2(x_0), g_1(S)) \tag{3.11}$$

or

$$b.\ \ (g_2(x_0), g_2(S)) \cap (g_1(x_0), g_1(S)) = \emptyset. \tag{3.12}$$

We discuss explicitly the case a. since case b. can be studied in a similar way. Consider first the process Y_2^*,:

$$Y_2^* = Y_2 - (g_2(x_0) - g_1(x_0)), \tag{3.13}$$

defined on $[g_2(\alpha_2) - (g_2(x_0) - g_1(x_0)), g_2(S) - (g_2(x_0) - g_1(x_0))]$,and then the process Y_2^{**} obtained from Y_2^* introducing a reflecting boundary in $g_2(\alpha_2)$; their drifts, for y belonging to the respective intervals, are:

$$\mu_{Y_2^*}(y) = \mu_{Y_2^{**}}(y) = \mu_{Y_2}(y - (g_2(x_0) - g_1(x_0)) \tag{3.14}$$

Due to (3.9), it holds:

$$\mu_{Y_2}(y) \geq \mu_{Y_2^{**}}(y). \tag{3.15}$$

for $y \in [g_2(\alpha_2), g_2(S) - (g_2(x_0) - g_1(x_0))]$. Making use of Skorohod-Yamada comparison theorem we have with probability 1:

$$Y_2 \geq Y_2^{**} \tag{3.16}$$

and hence, with probability one it holds:

$$T_{Y_2^{**}}(g_1(S)\,|g_1(x_0)) \geq T_{Y_2}(g_1(S)\,|g_1(x_0)). \tag{3.17}$$

Since, due to Lemma 3, the presence of a reflecting boundary has the effect to decrease the FPT we can translate the boundary in $g_2(\alpha_2)$ to $g_2(\alpha_2) - (g_2(x_0) - g_1(x_0))$ extending the validity of (3.17) to the FPT of the process Y_2^* for which we have, with probability 1:

$$T_{Y_2^*}(g_1(S)\,|g_1(x_0)) \geq T_{Y_2^{**}}(g_1(S)\,|g_1(x_0)) \geq T_{Y_2}(g_1(S)\,|g_1(x_0)). \tag{3.18}$$

Let us now remark that due to (3.8) and (3.6) we have $g_1(S) \leq g_2(S) - (g_2(x_0) - g_1(x_0))$, hence, due to Lemma 1:

$$\begin{aligned} & T_{Y_2^*}(g_2(S) - (g_2(x_0) - g_1(x_0))\,|g_1(x_0)) \\ = \;& T_{Y_2}(g_2(S)\,|g_2(x_0)) \geq T_{Y_2^*}(g_1(S)\,|g_1(x_0)) \end{aligned} \tag{3.19}$$

and the theses follows from (3.19) and (3.18). ∎

Remark 2. *If $\mu_{Y_2}(y)$ is differentiable and if*

$$\frac{d\mu_{Y_2}(y)}{dy} \leq 0, \quad y \in [g_2(\alpha_2), g_2(S) - (g_1(S) - g_1(x_0))]. \tag{3.20}$$

than hypothesis (3.9) is verified.

We can now prove the following

Theorem 2. *If $g_1(\alpha_1) = g_2(\alpha_2)$ and if the drifts (3.7) verify the inequalities:*

$$\begin{aligned} &\mu_{Y_1}(y) \geq \mu_{Y_2}(y) \qquad \forall y \in [g_1(\alpha_1), g_2(S)] \\ &\frac{d\mu_{Y_2}(y)}{dy} \leq 0 \qquad \forall y \in [g_1(\alpha_1), g_2(S)] \end{aligned} \tag{3.21}$$

and the infinitesimal variances are such that

$$\sigma_1^2(x) \geq \sigma_2^2(x) \tag{3.22}$$

then for any $x_0 \in (\max(\alpha_1, \alpha_2), S))$, $S \in (\max(\alpha_1, \alpha_2), \min(\beta_1, \beta_2))$, $x_0 < S$ one has

$$T_{X_1}(S\,|x_0) \leq T_{X_2}(S\,|x_0) \tag{3.23}$$

Proof. Making use of the comparison theorem of Skorohod-Yamada and of the first of our hypothesis about drifts we have with probability 1

$$Y_1 \geq Y_2, \tag{3.24}$$

hence for $\forall t \geq 0$ and $\forall a, b \in [g_1(\alpha_1), g_1(\beta_1)]$, $a < b$ we have, with probability one:

$$T_{Y_1}(b\,|a) \leq T_{Y_2}(b\,|a). \tag{3.25}$$

Let us choose $a = g_1(x_0), b = g_1(S)$. Observe now that (3.6) is a space transformation, not involving the time and is monotone non-decreasing. Hence from (3.25) follows:

$$T_{X_1}(S\,|x_0) = T_{Y_1}(g_1(S)\,|g_1(x_0)) \leq T_{Y_2}(g_1(S)\,|g_1(x_0)) \tag{3.26}$$

Hence, making use of Lemma 2 we obtain:

$$\begin{aligned} T_{X_1}(S\,|x_0) &\leq T_{Y_2}(g_1(S)\,|g_1(x_0)) \leq T_{Y_2}(g_2(S)\,|g_2(x_0)) \\ &= T_{X_2}(S\,|x_0) \end{aligned} \tag{3.27}$$

∎

Corollary 1. *If $g_1(\alpha_1) = g_2(\alpha_2)$ and if the drifts (3.7) verify the equality:*

$$\mu_{Y_1}(y) = \mu_{Y_2}(y) \qquad \forall y \in [g_1(\alpha_1), g_2(S)]$$

and the infinitesimal variances are such that (3.22) holds then for any $y_0 \in [g_1(\alpha_1), g_2(S)]$, $\Sigma \in [g_1(\alpha_1), g_2(S)]$, $y_0 < \Sigma$ one has

$$T_{X_1}(g_1^{-1}(\Sigma)\,|g_1^{-1}(y_0)) = T_{X_2}(g_2^{-1}(\Sigma)\,|g_2^{-1}(y_0)) \tag{3.28}$$

Proof. It is an immediate consequence of the equality of FPTs of the transformed processes ∎

Note that here the only property of transformations 3.6 used in the proof of the theorem is its monotonicity due to hypothesis 3.22. Hence we can generalize this theorem to more general transformations:

Theorem 3. *Let $\gamma_1(x)$ and $\gamma_2(x)$ be two monotone nondecreasing transformations changing the processes X_1 and X_2 into two processes Y_1' and Y_2' respectively, characterized by equal infinitesimal variance, and such that*

$$\begin{aligned} \gamma_1(x_0) &\leq \gamma_2(x_0) \\ \gamma_1(S) &\leq \gamma_2(S) \\ \gamma_1(S) - \gamma_1(x_0) &\leq \gamma_2(S) - \gamma_2(x_0) \end{aligned} \tag{3.29}$$

If $\gamma_1(\alpha_1) = \gamma_2(\alpha_2)$ and if the drifts (3.7) verify the inequalities:

$$\begin{aligned} &\mu_{Y_1'}(y) \geq \mu_{Y_2'}(y) \qquad \forall y \in [\gamma_1(\alpha_1), \gamma_2(S)] \\ &\frac{d\mu_{Y_2'}(y)}{dy'} \leq 0 \qquad \forall y \in [\gamma_1(\alpha_1), \gamma_2(S)] \end{aligned} \tag{3.30}$$

then for any $x_0 \in (\max(\alpha_1, \alpha_2), S))$, $S \in (\max(\alpha_1, \alpha_2), \min(\beta_1, \beta_2))$, $x_0 < S$ one has

$$T_{X_1}(S\,|x_0) \leq T_{X_2}(S\,|x_0) \tag{3.31}$$

Proof. Making use of the considered transformations the drifts and infinitesimal variances of the processes X_i are transformed into (cf. [12])

$$\mu'_{Y_i'}(y) = \left(\frac{1}{2}\sigma_i^2(x)\frac{d^2\gamma_i(x)}{dx^2} + b_i(x)\frac{d\gamma_i(x)}{dx}\right)\Bigg|_{x=\gamma_i^{-1}(y)} \tag{3.32}$$

and

$$\sigma_1^2(y) = \sigma_2^2(y) \tag{3.33}$$

for the processes $Y_i', i = 1, 2$. Here

$$\sigma^2_{Y_i'}(y) = \sigma^2_{X_i}(x)\left[\frac{d\gamma_i(x)}{dx}\right]^2\Bigg|_{x=\gamma_i^{-1}(y)} . \tag{3.34}$$

One proceed then with the technique used to prove Theorem 2. Note that in Lemma 4 the hypothesis $g_1(x) \leq g_2(x)$ has not been entirely used. Indeed in the proof of that Lemma we could simply assume hypothesis 3.29 . ∎

Corollary 2. *Consider the transformations $\gamma_i(x), i = 1,2$ defined in the Theorem. If $\gamma_1(\alpha_1) = \gamma_2(\alpha_2)$ and if the drifts (3.7) verify the equality:*

$$\mu_{Y_1'}(y) = \mu_{Y_2'}(y) \qquad \forall y \in [\gamma_1(\alpha_1), \gamma_2(S)] \tag{3.35}$$

then for any $x_0 \in (\max(\alpha_1, \alpha_2), S))$, $S \in (\max(\alpha_1, \alpha_2), \min(\beta_1, \beta_2))$, $x_0 < S$ one has with probability one:1

$$T_{X_1}(\gamma_1^{-1}(S) \,|\gamma_1^{-1}(x_0)) = T_{X_2}(\gamma_2^{-1}(S) \,|\gamma_2^{-1}(x_0)) \tag{3.36}$$

Proof. It is an immediate consequence of the equality of FPTs of the transformed processes. ∎

Remark 3. *If $g_2(\alpha_2) < g_1(\alpha_1)$ one can introduce a reflecting boundary to constrain the processes X_2 on a suitable interval in order that for the new transformed process Y_2^*, $g_2(\alpha_2) = g_1(\alpha_1)$. We can then apply the theorem to this new process extending finally the result to the original process by noting that, as proved in Lemma 3 the introduction of a reflecting boundary can only increase the FPT. The case $g_2(\alpha_2) > g_1(\alpha_1)$ can be studied in a similar way introducing a reflecting boundary to constrain the processes X_1 on a suitable interval in order to obtain $g_2(\alpha_2) = g_1(\alpha_1)$. However in this case the reflecting boundary cannot be removed to obtain an order relationship for the original processes since in this case the inequality does not hold if we increase the FPT. An analogous procedure can be carried out when transformations γ_i are used.*

Remark 4. *When it results possible to determine a choice for the boundary, the initial point and the processes parameters such that stochastic ordering holds between FPTs and the two FPTs means are equals than (cf.* [24]*) for this choice of values the two random variable are equals. Note that, if $g_2(\alpha_2) = g_1(\alpha_1)$ the only instance for which this happens corresponds to Corollaries 1 and 2 but if $g_2(\alpha_2) < g_1(\alpha_1)$ it can be verified in other instances.*

Remark 5. *If in equations (1.1) we introduce independent Wiener processes in correspondence to each $X_i(t)$, it is easy to verify that Lemmas 1 and 3 and Theorems 2 and 3 keep their validity if we substitute the usual stochastic order to the order with probability one in their thesis..*

Remark 6. *Conditions of Theorems 2 and 3 are sufficient conditions. FPTs can be ordered under less restrictive conditions than those of the Theorems. Indeed, since the order FPTs is connected to the ordering of the processes sample paths in any strip $[S - \varepsilon, S]$ with ε arbitrary,* *necessary conditions should deal with sample paths order on this strip. However comparison theorems for processes sample paths hold on the entire diffusion interval and it seems hard determine general conditions for sample path comparisons on strips.*

4. Examples

Here we make use of FPTs ordering to pursue different objectives. First we show how use of theorems 2-3 can help to investigate the dependence of FPTs distributions from the process parameters. In a second example we consider different diffusion processes that could be used to model the same behavior and we compare the different features determining an order between respective FPTs.

Example 1. Let us consider a Feller process, i.e. a diffusion process on $(0, \infty)$ characterized by linear drift and linear diffusion coefficient

$$\begin{aligned} \mu(x) &= px + q \\ \sigma^2(x) &= 2rx. \end{aligned} \tag{4.1}$$

Different boundary condition can be considered in 0 (cf. [12]) we consider here a reflecting boundary in 0. We explicitly underline that FPTs distribution through $S > 0$ is not analytically known while numerical computations can be found for example in [13]. The expression of FPTs mean and variance have been determined (cf. [7]) but their expressions are very complex and it is hard to disclose the dependences from the parameters. In order to determine the dependence of FPT distribution from the parameters p, q, r we compare the FPTs of two Feller processes of parameters p_i, q_i, r_i, $i = 1, 2$ originated in $x_0 \geq 0$ through a boundary $S > x_0$.

Consider the processes Y_i obtained from the processes X_i via transformations:

$$\begin{aligned} \gamma_1(x) &= x \\ \gamma_2(x) &= \tfrac{r_1}{r_2} x. \end{aligned} \tag{4.2}$$

Their infinitesimal moments are:

$$\begin{aligned} \mu_{Y_1}(y) &= p_1 y + q_1 \qquad \sigma^2_{Y_1}(y) = 2r_1 y \\ \mu_{Y_2}(y) &= p_2 y + q_2 \tfrac{r_1}{r_2} \quad \sigma^2_{Y_2}(y) = 2r_1 y. \end{aligned} \tag{4.3}$$

Hence, hypothesis of Theorem 2 are verified in the instances considered in Table I.

$T_{X_1} \leq T_{X_2}$

$r_1 \geq r_2$	$p_1 \geq p_2; p_2 < 0$	$\frac{q_1}{r_1} \geq \frac{q_2}{r_2}$	$\forall x$
	$p_1 \geq p_2; p_2 < 0$	$\frac{q_1}{r_1} \leq \frac{q_2}{r_2}$	$x \geq q_2 - q_1 \frac{r_2}{r_1}$
	$p_1 \leq p_2; p_2 < 0$	$\frac{q_1}{r_1} \geq \frac{q_2}{r_2}$	$x \leq \frac{(r_2 q_1 - q_2 r_1)}{r_1 (p_2 - p_1)}$

Table I

Interchanging the role of the variables X_1 and X_2 one immediately verifies that hypothesis of Theorem 2 are fulfilled in the cases considered in Table II.

$T_{X_2} \leq T_{X_1}$	$r_2 \geq r_1$	$p_2 \geq p_1; p_1 < 0$	$\frac{q_2}{r_2} \geq \frac{q_1}{r_1}$	$\forall x$
		$p_2 \geq p_1; p_1 < 0$	$\frac{q_2}{r_2} \leq \frac{q_1}{r_1}$	$x \geq q_1 - q_2 \frac{r_1}{r_2}$
		$p_2 \leq p_1; p_1 < 0$	$\frac{q_2}{r_2} \geq \frac{q_1}{r_1}$	$x \leq \frac{(r_1 q_2 - q_1 r_2)}{r_2(p_1 - p_2)}$

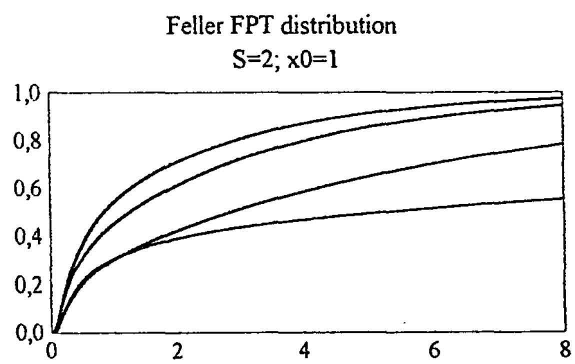
Table II

The cases considered in the second lines of Tables I and II request to move the reflecting boundary from 0 to $q_2 - q_1 \frac{r_2}{r_1}$ and $q_1 - q_2 \frac{r_1}{r_2}$ respectively while the cases considered in the third lines correspond to absorbing boundaries that respect the constrains on x. Furthermore it seems interesting to remark the dependence of the FPTs order from the ratios $\frac{q_i}{r_i}$.

Furthermore if $p_1 = p_2$ and $\frac{q_1}{r_1} = \frac{q_2}{r_2}$ from Corollary 2 we have:

$$T_{X_1}\left(\frac{r_1 S}{r_2} \middle| \frac{r_1 x_0}{r_2}\right) = T_{X_2}(S \,|x_0). \tag{4.4}$$

In Fig. 1 we compare FPT distribution of a Feller process X_1 originated in $x_0 = 1$ through $S = 2$ and characterized by $p = -1, q = 1, r = 1$ with the analogous for Feller processes, that corresponds to X_2 in Theorem 2, determined by different choices of parameters.



	From the	top to the	bottom
	$p = -0.2$	$q = 0.7$	$r = 0.8$
FIGURE 1.	$p = -1$	$q = 1$	$r = 1$
	$p = -1.2$	$q = 0.8$	$r = 0.8$
	$p = -0.8$	$q = 0.4$	$r = 0.8$

The last two choices for the parameters are examples for the first and third line of Table I and we from Fig. 1 we see that $T_{X_1} \leq^{st} T_{X_2}$.

When we compare the FPT of a Feller process with $p = -0.2, q = 0.7, r = 0.8$, considered as process X_1, with the FPTs of the other processes of Fig. 1, we cannot establish an order relation between T_{X_1} and T_{X_2} by means of Theorems 2 or 3. Indeed the parameters values verify conditions of the third line of Table II but the boundary and the initial value do not verify $x \le \frac{(r_1 q_2 - q_1 r_2)}{r_2(p_1 - p_2)}$. However in this case, from Fig. 1, we see that $T_{X_2} \ge^{st} T_{X_1}$.

Example 2. Let us now compare FPTs of a Feller process (4.1) with the analogous FPT of an Ornstein-Uhlenbeck process, having infinitesimal moments:

$$\mu_2(x) = -\beta x; \qquad \sigma_2^2(x) = \sigma^2. \tag{4.5}$$

These two models are used in the literature to model the membrane potential behavior and the neuron spike activity and in order to chose the most appropriate it is useful have the possibility to compare their features (cf. [13]).

Let the two processes be originated in x_0 and consider a boundary S. Here the two diffusion intervals do not coincide since the Ornstein-Uhlenbeck is defined on $(-\infty, +\infty)$, however one could wish to compare the FPTs of the two processes and this can be done if x_0 and S are positive but some new difficulty can arise when we make use of theorems 2 and 3.

To deal with this comparison we first make use of Theorem 2 pinpointing difficulties that limit its application, then making use of Theorem 3,we establish order relationships between the FPTs of the two processes for certain choices of parameters values.

Let

$$g_1(x) = \sigma \sqrt[2]{\frac{2x}{r}} \tag{4.6}$$

$$g_2(x) = x. \tag{4.7}$$

two transformations on the processes X_1 and X_2 respectively.

The processes X_1 and X_2 are transformed via (4.6) and (4.7) into the processes Y_1 and Y_2 characterized by infinitesimal variance

$$\sigma^2_{Y_1}(y) = \sigma^2_{Y_2}(y) = \sigma^2 \tag{4.8}$$

and drifts

$$\mu_{Y_1}(y) = \frac{p}{2} y + \frac{\sigma^2}{y}\left(\frac{q}{r} - \frac{1}{2}\right) \tag{4.9}$$

$$\mu_{Y_2}(y) = \beta y. \tag{4.10}$$

Note that (3.22) holds for

$$x \geq \frac{\sigma^2}{2r}, \tag{4.11}$$

hence, in order to make use of Theorem 2, we can consider the two new processes X_1' and X_2' obtained from X_1 and X_2 introducing a reflecting boundary in $x = \frac{\sigma^2}{2r}$. However this boundary cannot be moved to obtain relations about original processes. Alternatively we can use Theorem 3 whose hypothesis (3.29) are verified for transformations (4.6) and (4.7) if $x_0 \geq \frac{\sigma^2}{2r}$. The hypothesis of the Theorem are verified in each of the following instances if a reflecting boundary in 0 is introduced for the Ornstein-Uhlenbeck process and $\beta < 0$:

$$\begin{aligned} &a.\ \frac{q}{r} \geq \frac{1}{2}; \quad \frac{p}{2} \geq \beta \\ &b.\ \frac{q}{r} \geq \frac{1}{2}; \quad \beta > \frac{p}{2} \quad if\ \ x < \frac{r}{2}\sqrt[2]{\frac{\frac{q}{r}-\frac{1}{2}}{\beta-\frac{p}{2}}} \\ &c.\ \frac{q}{r} \leq \frac{1}{2}; \quad \beta < \frac{p}{2} \quad if\ \ x > \frac{r}{2}\sqrt[2]{\frac{\frac{1}{2}-\frac{q}{r}}{\frac{p}{2}-\beta}} \end{aligned} \tag{4.12}$$

in these cases we conclude:

$$T_{X_1}(S\,|x_0) \leq T_{X_2'}(S\,|x_0) \tag{4.13}$$

Where X_2' is the Ornstein-Uhlenbeck process with reflecting boundary in the origin. Making use of Lemma 3 we can now remove the reflecting boundary and obtain, when (??) are verified:

$$T_{X_1}(S\,|x_0) \leq T_{X_2}(S\,|x_0). \tag{4.14}$$

In Fig. 2 two examples illustrate (4.14) when the usual stochastic order is considered.The second and the forth shapes correspond to a case were (4.14) holds since condition b in (4.12) is verified. Indeed for the Feller process (shape on the top) we have:

$$p = -3\ q = 1\ r = 1\ S = 0.5\ x_0 = 0.25 \tag{4.15}$$

while for the Ornstein-Uhlenbeck process (second shape from the bottom) we have:

$$\beta = -1\ \sigma^2 = 0.5\ S = 0.5\ x_0 = 0.25. \tag{4.16}$$

The first and third shapes illustrate a case were (4.14) holds since condition a in (4.12) is verified.

Indeed in these case we have:

$$p = -1\ q = 1\ r = 1\ S = 2\ x_0 = 1 \tag{4.17}$$

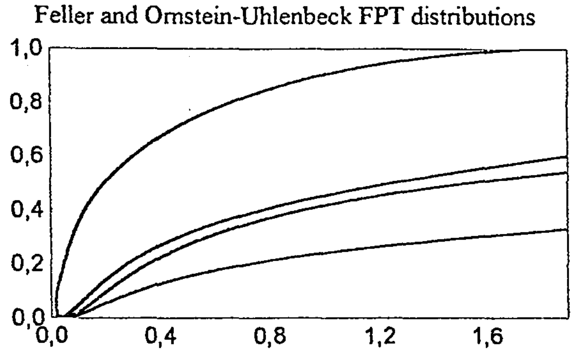


FIGURE 2. The parameter values are specified in the text

for the Feller process (second shape from the top) and

$$\beta = -1 \; \sigma^2 = 2 \; S = 2 \; x_0 = 1 \tag{4.18}$$

for the Ornstein-Uhlenbeck process (first shape from the bottom).

Let us now interchange the roles of X_1 and X_2 considering the Ornstein-Uhlenbeck process as process X_1 and the Feller process as process X_2 in Theorem 3. Now (3.29) holds for

$$S \leq \frac{\sigma^2}{2r}. \tag{4.19}$$

The hypothesis (3.21a) of the Theorem 3 are verified in each of the following instances if a reflecting boundary in 0 is introduced for the Ornstein-Uhlenbeck process:

$$\begin{aligned} &a. \ \frac{q}{r} \leq \frac{1}{2}; \quad \frac{p}{2} \leq \beta \\ &b. \ \frac{q}{r} \geq \frac{1}{2}; \quad \beta \geq \frac{p}{2} \quad if \ x \geq \frac{r}{2} \sqrt[2]{\frac{\frac{q}{r} - \frac{1}{2}}{\beta - \frac{p}{2}}} \\ &c. \ \frac{q}{r} \leq \frac{1}{2}; \quad \beta \leq \frac{p}{2} \quad if \ x \leq \frac{r}{2} \sqrt[2]{\frac{\frac{1}{2} - \frac{q}{r}}{\frac{p}{2} - \beta}} \end{aligned} \tag{4.20}$$

while (3.21b) is verified if

$$\frac{p}{2} - \frac{\sigma^2}{y^2}\left(\frac{q}{r} - \frac{1}{2}\right) \leq 0 \tag{4.21}$$

in these cases we conclude:

$$T_{X_1}(S \,|x_0) \geq T_{X_2'}(S \,|x_0) \tag{4.22}$$

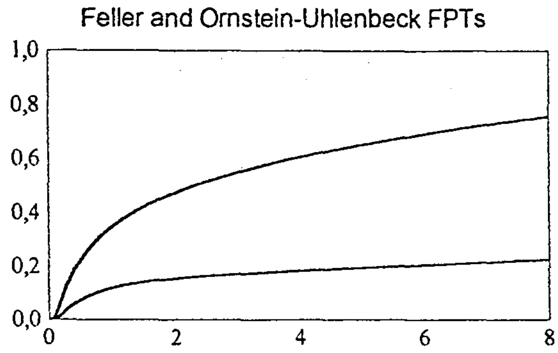


FIGURE 3. Form the top to the bottom, for the choices of the parameters discussed in the text: Ornstein-Uhlenbeck and Feller

Where X_2' is the Ornstein-Uhlenbeck process with reflecting boundary in the origin. Note that in this case we cannot use Lemma 3 to remove the reflecting boundary. However in some instances the order relationship holds for the process X_2 , at least in usual stochastic order,even if we cannot prove it by means of the proved theorems. An example of this type is illustrated in Fig. 3 where we compare FPTs distribution of a Feller process of parameters:

$$p = -1 \; q = 0.2 \; r = 0.5 \; S = 2 \; x_0 = 1 \tag{4.23}$$

with the FPTs distribution of an Ornstein-Uhlenbeck process on $(-\infty, \infty)$ of parameters:

$$\beta = -0.5 \; \sigma^2 = 2 \; S = 2 \; x_0 = 1. \tag{4.24}$$

These choices for the processes parameters correspond to case a of (4.20), and the order relation (4.22) holds for the Ornstein-Uhlenbeck process constrained by a reflecting boundary in 0. However, as shown in Fig. 3, the order relation is true even for the process without reflecting boundary.

We explicitly note that the instances analyzed via theorem 2 and 3 do not cover all possible ranges of values for the parameters. For different choices it can happen that FPT process results greater than the analogous for the other. In particular this becomes true if we consider the less strict condition of ordering in distribution in spite of the almost sure ordering considered in the theorems. Unfortunately it is not yet clear which conditions imply this weaker order. However when stochastic order is concerned one can make use of numerical methods to check possible ordering, extending then the results to other parameters choices by means of comparison theorems 3 and 4.

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