THE INSTITUTE OF STATISTICS

ALTERNATIVE METHOD FOR TESTING THE UNIT ROOT NULL HYPOTHESIS IN THE PRESENCE OF A BREAK

by

Seungho Huh and David A Dickey
Department of Statistics, North Carolina State University

Institute of Statistics Mimeograph Series No. 2529
February 2001

NORTH CAROLINA STATE UNIVERSITY
Raleigh, North Carolina

ALTERNATIVE METHOD FOR TESTING THE UNIT ROOT NULL HYPOTHESIS IN THE PRESENCE OF A BREAK

BY
SEUNGHO HUH AND DAVID A DICKEY
DEPT. OF STATISTICS, NCSU
MIMEO NO..2529. Feb. 2001

Alternative Method for Testing the Unit Root Null Hypothesis in the Presence of a Break

Seungho Huh and David A. Dickey
Department of Statistics, North Carolina State University, Raleigh, NC27695-8203, USA

Abstract In this paper, we suggest alternative test statistics for testing the unit root null hypothesis in the presence of a trend-break. Our new test procedure which we call the "bisection" method is based on the idea of subgrouping. The idea here is to split the data in half and look at the minimum of the resulting two unit root test statistics. This avoids the necessity of searching for the break. It uses all the data in the sense that the minimum is chosen, but clearly is not efficient in its use of the data. We anticipate paying a price in power for a gain in simplicity. Considering some data generating processes, we display empirical size and power results from simulation. We also apply our bisection method to the well-known Nelson and Plosser (1982) data set and compare the results with those of other researchers. The simple bisection method rejects unit roots in several, but not all, of the series for which the more complicated search methods reject.

1 Introduction

Since the pioneering work by Perron (1989), many researchers have been interested in testing for a unit root in time series with a trend-break. It is known that power decreases in finite samples as the trend-break becomes larger when the usual Dickey-Fuller (1979) test is used.

Perron (1989) suggested formal statistical tests of the null hypothesis of a unit root which can distinguish the unit root hypothesis from that of stationarity around a trend with a single trend-break (either in the intercept or the slope). His original approach assumed the trend-break is known a priori and treated as exogenous. Perron (1997) and Vogelsang and Perron (1998), on the other hand, considered unit root tests treating the date of a possible trend-break as unknown. They introduced various methods of choosing the break date. A variation of Perron's (1989) test in which the break point is estimated rather than fixed was also studied by Zivot and Andrews (1992). They checked all possible break points and took the minimum t statistic.

In this paper, we present an alternative test statistic for testing the unit root hypothesis allowing for a possible trend break. Although our method assumes that the break time is unknown, it is ignored rather than estimated. Using the new test statistic, we perform a Monte Carlo simulation to obtain its empirical powers which are rather invariant to the break size.

Section 2 describes the idea of subgrouping and simulation results to find the optimal number of subgroups. As a result, we suggest new test statistics in section 3. Section 4 presents the data generating processes and some empirical power results from a Monte Carlo simulation. In section 5, we apply our test procedure to the real data analyzed originally by Nelson and Plosser (1982). We finally make concluding remarks in section 6.

2 Subgrouping of data

Our new test procedure is based on the idea of dividing the whole data into some subgroups of the same size. For each subgroup, a certain t type statistic is calculated. Then the minimum of all these statistics is defined as a new test statistic for the unit root null hypothesis in time series with a break.

Suppose we have stationary series around a broken trend. After dividing the data into subgroups, we expect to have a smaller value of the test statistic from a subgroup without a break than from another subgroup with a break. Therefore taking the minimum among all the statistics might give us reasonable power as we are using left tailed tests. This is a key motivation for our subgrouping idea.

The approach by Perron (1989) assumes the break point is known a priori. Using dummy variables, he combines the data before the break with the data after the break. We do not have to assume a known break point in our new procedure.

With the break point assumed unknown, Zivot and Andrews (1992) choose the break point that gives the least favorable result for the unit root null hypothesis.

Then they take the t type statistic giving that break point. Our procedure is simpler because it does not consider estimating the unknown break point.

We perform some simulations to decide the optimal number of subgroups. The data generating process given by (1) in section 4 is used for various numbers of subgroups. We consider $n(\hat{\rho}-1)$ type statistics from the ordinary least squares (OLS) estimator, the simple symmetric (SS) estimator and the weighted symmetric (WS) estimator. The number of replications is 5,000 and the sample size per replication is $n = 100$. We consider the break point $c = 36$ (for OLS) or $c = 37$ (for SS and WS) and the break size θ = 1, 3, 5 and 7. In Tables 1 - 3 and Figures 1 - 3, we show empirical size and power results from simulation.

Clearly the optimal number of subgroups is 2. We expect similar results for t type statistics. The next section describes our new test procedure which we call the "bisection" method.

3 Bisection method

From the previous work by Leybourne et al (1998) and Huh and Dickey (1999), we know that, in the presence of an early break, the conventional Dickey-Fuller (1979) test based on the least squares estimator can be subject to serious size distortion but tests based on the weighted symmetric estimator cure this problem. Therefore our bisection method in this paper is based on the WS estimator of ρ, $\tilde{\rho}_w$, and the associated pivotal statistic τ_w in the non-zero mean AR(1) process

$$Y_t = \mu + \rho Y_{t-1} + e_t.$$

Our new bisection test statistic is defined as

$$\tau_w^* \equiv \min(\tau_{w,1}, \tau_{w,2})$$

where $\tau_{w,k}$ is τ_w for subgroup k for $k = 1, 2$. Each subgroup is supposed to have the same number of observations. Note that we are dealing with the mean adjusted case. See Appendix for the limiting distribution of τ_w^*. Figure 4 shows empirical distributions of τ_w^*, $\tau_{w,1}$, $\tau_{w,2}$ and the usual τ_w associated with the data generating process (1) in section 4. We assume $n = 100$, $c = 75$, $\theta = 5$ and $\rho = .7$. Notice that the empirical distributions of $\tau_{w,1}$ and τ_w^* are very close to each other since the break is in the second subgroup ($c = 75$). That does not necessarily mean that they have the same power for a certain ρ because the critical values are different as shown in Figure 4. In the figure, the vertical line on the left denotes the critical value for τ_w^*, so the power for τ_w^* is lower than that for $\tau_{w,1}$ as can be seen in Figure 5. This generally holds good for other values of ρ.

As to the linear trend adjusted case, we consider the WS estimator of ρ and the associated pivotal statistic $\tau_{w,\tau}$ in the model

$$Y_t = \mu + \beta t + \rho Y_{t-1} + e_t.$$

$\tau_{w,\tau}$ can be regarded as the linear trend adjusted version of τ_w. We then use $\tau_{w,\tau}^*$, the minimum of two $\tau_{w,\tau}$ statistics, as a new test statistic. See Huh and Dickey (1999) for more details about τ_w and $\tau_{w,\tau}$.

4 Empirical size and power results

In this section, we display some size and power simulation results using bisection test statistics.

We consider two data generating processes (DGPs) given by

$$Y_t = \theta\sigma I(t > c) + W_t, \quad W_t = \rho W_{t-1} + e_t, \qquad t = 1, 2, ..., n \qquad (1)$$

$$Y_t = \gamma\sigma(t - c)I(t > c) + W_t, \quad W_t = \rho W_{t-1} + e_t, \quad t = 1, 2, ..., n \qquad (2)$$

where the e_t are normal independent $(0,\sigma^2)$ random variables and c is the break point. We can assume $\sigma = 1$ since $\tilde{\rho}_w$ is the ratio of two quadratic forms so it is invariant to σ. Notice that DGP (1) corresponds to a break-in-level model with levels 0 and $\theta\sigma$

and DGP (2) to a break-in-slope model with slopes 0 and $\gamma\sigma$. Figures 6 - 9 present some typical time series data generated from DGPs (1) and (2).

We perform a Monte Carlo simulation for some values of θ, γ, c and ρ. The number of replications is 5,000 and the sample size per replication is $n = 100$. The critical values for τ_w^* and $\tau_{w,\tau}^*$ are also calculated by simulation.

4.1 Data with a break in level

4.1.1 Mean adjusted case

For DGP (1), we first use the mean adjusted statistics τ_w and τ_w^*. As might be expected from the results in Huh and Dickey (1999), neither τ_w nor τ_w^* show size distortion for DGP (1).

If we use τ_w^* as a test statistic, the powers are in general higher than those for the usual τ_w when there is a break. Some exceptions are for the early break ($c = 1$) or the late break ($c = 99$). For given ρ, the powers for τ_w^* are relatively invariant whereas those for the usual τ_w decrease dramatically as θ becomes larger.

If there is no break ($\theta = 0$), on the other hand, the usual τ_w is more powerful than τ_w^*. This is so especially for large values of ρ. Notice that our power results for τ_w in Table 4 agree closely with those of Pantula et al (1994).

Table 4 and Figures 10 - 12 present the empirical size and power results for DGP (1) using τ_w and τ_w^*. As might be expected, τ_w outperforms τ_w^* for small breaks and τ_w^* outperforms τ_w for larger breaks. In all cases, τ_w^* is more uniform with respect to λ.

We compare the size and power properties of our test procedure with those of Vogelsang and Perron (1998). They consider a DGP

$$
\begin{aligned}
Y_t &= \theta I(t > T_b^c) + \gamma(t - T_b^c) I(t > T_b^c) + Z_t, \\
Z_t &= \rho Z_{t-1} + \alpha \Delta Z_{t-1} + e_t + \psi e_{t-1}, \qquad t = 1, \cdots, n \qquad (3)
\end{aligned}
$$

where $e_t \sim NI(0,1)$ and T_b^c stands for the true break date. With $\alpha = \psi = \gamma = 0$, DGP (3) is exactly the same as our DGP (1). They introduce 4 different test statistics ($T_b(t_{\hat{\alpha}})$, $T_b(|t_{\hat{\gamma}}|)$, $T_b(t_{\hat{\gamma}})$ and $T_b(F_{\hat{\theta},\hat{\gamma}})$) according to the methods of estimating the true break date. For their simulation, $n = 100$ and $T_b^c = 50$. When $\rho = .8$, our τ_w^* shows higher power than most of their statistics regardless of the break size θ. Although one statistic ($T_b(t_{\hat{\alpha}})$) with $\theta = 10$ has higher power than our τ_w^*, that can be attributed to its size distortion problem. In Table 5, we show empirical powers for DGP (1) using various statistics.

4.1.2 Linear trend adjusted case

We next use the linear trend adjusted statistics $\tau_{w,\tau}$ and $\tau_{w,\tau}^*$ for DGP (1). Both $\tau_{w,\tau}$ and $\tau_{w,\tau}^*$ retain empirical sizes ($\rho = 1$) close to the nominal 5% significance level.

When there is no break, $\tau_{w,\tau}$ shows higher power than $\tau_{w,\tau}^*$ for given ρ. Notice also that $\tau_{w,\tau}$ and $\tau_{w,\tau}^*$ generate lower power than τ_w and τ_w^* respectively.

When the break is fairly big ($\theta = 10$), the powers for $\tau_{w,\tau}^*$ are generally higher than those for $\tau_{w,\tau}$. As in the mean adjusted case, the powers for $\tau_{w,\tau}^*$ are relatively invariant compared with those for $\tau_{w,\tau}$.

The empirical size and power results using $\tau_{w,\tau}$ and $\tau_{w,\tau}^*$ are presented in Table 6 and Figures 13 - 15.

4.2 Data with a break in slope

For DGP (2) which represents a break-in-slope model, we adopt the linear trend adjusted statistics $\tau_{w,\tau}$ and $\tau_{w,\tau}^*$. Even though both $\tau_{w,\tau}$ and $\tau_{w,\tau}^*$ retain the nominal 5% significance level, $\tau_{w,\tau}$ becomes severely under-sized as γ grows larger.

As expected, $\tau_{w,\tau}^*$ gives smaller powers than $\tau_{w,\tau}$ if there is no break. As γ becomes larger, the powers for $\tau_{w,\tau}$ decrease dramatically except for the early or late breaks whereas $\tau_{w,\tau}^*$ maintains reasonable powers. $\tau_{w,\tau}^*$ is generally more powerful than $\tau_{w,\tau}$ for $\gamma > 0$.

The simulation results for DGP (2) using $\tau_{w,\tau}$ and $\tau^*_{w,\tau}$ are displayed in Table 7 and Figures 16 - 18.

To compare with the results of Vogelsang and Perron (1998), we consider DGP (3) with $\alpha = \psi = \theta = 0$ which is exactly the same as our DGP (2). When $\rho = .8$, the powers of our $\tau^*_{w,\tau}$ are as good as those of their statistics for some values of γ. Power comparisons between $\tau^*_{w,\tau}$ and statistics of Vogelsang and Perron (1998) are displayed in Table 8.

5 Empirical applications

We now apply our bisection method to the data set analyzed originally by Nelson and Plosser (1982). The data set consists of 14 major macroeconomic time series which include measures of output, spending, money, prices and interest rates. The data are annual, generally averages for the year, with starting dates from 1860 to 1909 and ending in 1970 in all cases. We analyze the natural logarithm of all the data except for the interest rate series, which is analyzed in levels form. Many researchers have referred to this data set. See Nelson and Plosser (1982) for more details about the data set.

In Table 9, we compare our unit root test results with those of Nelson and Plosser (1982), Perron (1989), Zivot and Andrews (1992) and Perron (1997). Numbers are the values of the test statistics.

Nelson and Plosser (1982) apply the usual Dickey-Fuller test with extra lags of the first differences of the data (augmented Dickey-Fuller test). Methods of Perron (1989) and Zivot and Andrews (1992) are briefly described in section 2. Perron (1997) is closely related to and complements Zivot and Andrews (1992) in that similar procedures are analyzed. See the original articles for more details.

Our method adopts the number of augmenting terms, k, producing the minimum test statistic for each series. The other methods also use various values of k according to some selection criteria. Nelson and Plosser (1982) use critical values from Fuller (1996) whereas the other methods, including ours, use their own critical values generated from simulation. Our bisection method using $\tau^*_{w,\tau}$ rejects the unit root null hypothesis at $\alpha = .05$ for the series "Real GNP", "Real per capita GNP", "Industrial production", "Unemployment rate", "Real wages" and "Money stock".

6 Summary

In this work, we suggest new test statistics τ^*_w and $\tau^*_{w,\tau}$ for testing the unit root null hypothesis in the presence of a trend break. These statistics are based on the idea of dividing the data into subgroups of the same size. The optimal number of subgroups turns out to be 2.

Our bisection method can be used without assuming a known break point unlike Perron's (1989) original approach. It is simpler than the methods of Zivot and Andrews (1992) and Perron (1997). When there is a trend-break and the break size is fairly big, the empirical powers of the new statistics are in general higher than the usual τ_w and $\tau_{w,\tau}$ respectively. Based on simulation studies, the power properties of our test procedure are as good as those of Vogelsang and Perron's (1998).

We also apply our procedure to the well-known Nelson and Plosser (1982) data set and compare the results with those of other researchers.

Table 1: Empirical size and power for DGP (1) using various subgroups (OLS, $n = 100$, $c = 36$)

No. of subgroups	1	2	5	10	20
$\theta = 1$, $\rho = .1$	1.0000	1.0000	0.9994	0.6938	0.0994
.2	1.0000	1.0000	0.9936	0.5424	0.0884
.3	1.0000	1.0000	0.9600	0.4026	0.0826
.4	1.0000	1.0000	0.8554	0.2948	0.0752
.5	1.0000	0.9984	0.6788	0.2096	0.0682
.6	1.0000	0.9780	0.4740	0.1434	0.0634
.7	0.9974	0.8388	0.2898	0.0974	0.0598
.8	0.9132	0.5042	0.1626	0.0744	0.0544
.9	0.4288	0.1870	0.0962	0.0528	0.0524
1	0.0474	0.0470	0.0530	0.0504	0.0488
$\theta = 3$, $\rho = .1$	1.0000	1.0000	0.9986	0.6764	0.0946
.2	1.0000	1.0000	0.9888	0.5246	0.0848
.3	0.9996	1.0000	0.9416	0.3856	0.0798
.4	0.9976	0.9992	0.8126	0.2806	0.0728
.5	0.9806	0.9852	0.6260	0.1980	0.0658
.6	0.9144	0.9052	0.4298	0.1342	0.0606
.7	0.7458	0.6786	0.2564	0.0930	0.0576
.8	0.4824	0.3650	0.1472	0.0710	0.0530
.9	0.2382	0.1464	0.0886	0.0476	0.0504
1	0.0460	0.0478	0.0514	0.0498	0.0476

Table 1: *continued*

No. of subgroups	1	2	5	10	20
$\theta = 5$, $\rho = .1$	0.7650	1.0000	0.9986	0.6760	0.0946
.2	0.5990	1.0000	0.9886	0.5240	0.0844
.3	0.4212	1.0000	0.9408	0.3846	0.0792
.4	0.2654	0.9990	0.8104	0.2792	0.0726
.5	0.1612	0.9818	0.6224	0.1966	0.0656
.6	0.0976	0.8920	0.4230	0.1330	0.0604
.7	0.0712	0.6440	0.2482	0.0908	0.0576
.8	0.0626	0.3116	0.1394	0.0694	0.0530
.9	0.0686	0.1114	0.0808	0.0466	0.0504
1	0.0386	0.0418	0.0464	0.0470	0.0476
$\theta = 7$, $\rho = .1$	0.0064	1.0000	0.9986	0.6760	0.0946
.2	0.0018	1.0000	0.9886	0.5240	0.0900
.3	0.0004	1.0000	0.9408	0.3846	0.0768
.4	0	0.9990	0.8104	0.2792	0.0724
.5	0	0.9818	0.6224	0.1966	0.0708
.6	0	0.8920	0.4230	0.1330	0.0678
.7	0.0004	0.6424	0.2480	0.0908	0.0632
.8	0.0008	0.3060	0.1382	0.0692	0.0504
.9	0.0128	0.0978	0.0790	0.0464	0.0546
1	0.0334	0.0358	0.0442	0.0468	0.0520

Table 2: Empirical size and power for DGP (1) using various subgroups (SS, $n = 100$, $c = 37$)

No. of subgroups	1	2	5	10	20
$\theta = 1$, $\rho = .1$	1.0000	1.0000	0.9968	0.6278	0.1916
.2	1.0000	1.0000	0.9806	0.4670	0.1602
.3	1.0000	1.0000	0.9178	0.3584	0.1266
.4	1.0000	0.9998	0.7978	0.2696	0.1144
.5	1.0000	0.9992	0.5980	0.1874	0.1058
.6	1.0000	0.9754	0.3856	0.1336	0.0802
.7	0.9982	0.8314	0.2360	0.1024	0.0766
.8	0.9384	0.4970	0.1326	0.0730	0.0698
.9	0.4960	0.1856	0.0800	0.0644	0.0542
1	0.0536	0.0470	0.0512	0.0482	0.0494
$\theta = 3$, $\rho = .1$	1.0000	1.0000	0.9934	0.6062	0.1872
.2	1.0000	1.0000	0.9684	0.4490	0.1562
.3	0.9998	1.0000	0.8896	0.3418	0.1242
.4	0.9980	0.9990	0.7462	0.2516	0.1106
.5	0.9866	0.9886	0.5396	0.1736	0.1010
.6	0.9324	0.9036	0.3392	0.1258	0.0778
.7	0.7828	0.6590	0.2152	0.0946	0.0732
.8	0.5340	0.3600	0.1242	0.0682	0.0682
.9	0.2770	0.1548	0.0734	0.0600	0.0522
1	0.0524	0.0436	0.0472	0.0470	0.0478

Table 2: *continued*

No. of subgroups	1	2	5	10	20
$\theta = 5$, $\rho = .1$	0.8080	1.0000	0.9932	0.6060	0.1854
.2	0.6590	1.0000	0.9682	0.4482	0.1544
.3	0.4718	1.0000	0.8888	0.3416	0.1224
.4	0.3176	0.9986	0.7420	0.2510	0.1092
.5	0.1868	0.9848	0.5340	0.1724	0.1002
.6	0.1244	0.8860	0.3294	0.1246	0.0770
.7	0.0832	0.6200	0.2018	0.0930	0.0722
.8	0.0680	0.3078	0.1142	0.0666	0.0672
.9	0.0804	0.1162	0.0672	0.0586	0.0506
1	0.0436	0.0406	0.0448	0.0444	0.0474
$\theta = 7$, $\rho = .1$	0.0110	1.0000	0.9932	0.6060	0.1852
.2	0.0024	1.0000	0.9682	0.4482	0.1544
.3	0.0014	1.0000	0.8888	0.3416	0.1224
.4	0	0.9986	0.7420	0.2510	0.1092
.5	0	0.9848	0.5338	0.1724	0.1002
.6	0.0002	0.8860	0.3292	0.1246	0.0768
.7	0	0.6184	0.2004	0.0930	0.0722
.8	0.0014	0.3012	0.1132	0.0666	0.0672
.9	0.0154	0.1008	0.0654	0.0584	0.0504
1	0.0394	0.0334	0.0428	0.0440	0.0474

Table 3: Empirical size and power for DGP (1) using various subgroups (WS, $n = 100$, $c = 37$)

No. of subgroups	1	2	5	10	20
$\theta = 1,\ \rho = .1$	1.0000	1.0000	0.9988	0.5346	0.1500
.2	1.0000	1.0000	0.9842	0.4218	0.1246
.3	1.0000	1.0000	0.9312	0.3052	0.1012
.4	1.0000	0.9998	0.8060	0.2262	0.0958
.5	1.0000	0.9992	0.6092	0.1666	0.0904
.6	1.0000	0.9844	0.4166	0.1218	0.0698
.7	0.9992	0.8750	0.2438	0.0932	0.0634
.8	0.9592	0.5470	0.1486	0.0708	0.0600
.9	0.5518	0.1942	0.0738	0.0560	0.0498
1	0.0526	0.0488	0.0514	0.0484	0.0484
$\theta = 3,\ \rho = .1$	1.0000	1.0000	0.9966	0.5144	0.1456
.2	1.0000	1.0000	0.9748	0.4044	0.1204
.3	0.9996	1.0000	0.9050	0.2880	0.0996
.4	0.9994	0.9992	0.7600	0.2122	0.0928
.5	0.9924	0.9914	0.5554	0.1562	0.0866
.6	0.9576	0.9318	0.3722	0.1148	0.0678
.7	0.8338	0.7214	0.2154	0.0882	0.0622
.8	0.5912	0.4088	0.1344	0.0658	0.0590
.9	0.3070	0.1544	0.0688	0.0522	0.0468
1	0.0538	0.0440	0.0476	0.0472	0.0474

Table 3: *continued*

No. of subgroups	1	2	5	10	20
$\theta = 5$, $\rho = .1$	0.8564	1.0000	0.9966	0.5142	0.1432
.2	0.7314	1.0000	0.9746	0.4042	0.1182
.3	0.5670	0.9998	0.9014	0.2872	0.0964
.4	0.4028	0.9978	0.7582	0.2118	0.0904
.5	0.2524	0.9874	0.5490	0.1556	0.0844
.6	0.1656	0.9174	0.3654	0.1144	0.0664
.7	0.1088	0.6748	0.2084	0.0872	0.0610
.8	0.0846	0.3446	0.1248	0.0640	0.0574
.9	0.0876	0.1172	0.0630	0.0512	0.0462
1	0.0452	0.0410	0.0452	0.0446	0.0466
$\theta = 7$, $\rho = .1$	0.0232	1.0000	0.9966	0.5142	0.1428
.2	0.0064	1.0000	0.9746	0.4042	0.1178
.3	0.0020	0.9998	0.9014	0.2872	0.0962
.4	0.0004	0.9978	0.7582	0.2118	0.0906
.5	0	0.9874	0.5490	0.1556	0.0842
.6	0.0002	0.9172	0.3648	0.1144	0.0662
.7	0.0002	0.6736	0.2076	0.0872	0.0604
.8	0.0016	0.3366	0.1240	0.0640	0.0570
.9	0.0142	0.1024	0.0612	0.0512	0.0460
1	0.0412	0.0360	0.0428	0.0444	0.0464

Table 4: Empirical size and power for DGP (1) using τ_w and τ_w^*

		$\theta = 0$		$\theta = 5$		$\theta = 10$	
ρ	c	τ_w	τ_w^*	τ_w	τ_w^*	τ_w	τ_w^*
0.5	1	1.0000	1.0000	0.9998	0.9968	0.9810	0.9900
0.5	20	1.0000	1.0000	0.6532	0.9878	0.0000	0.9884
0.5	50	1.0000	1.0000	0.1898	1.0000	0.0000	0.9996
0.5	80	1.0000	0.9996	0.6576	0.9884	0.0000	0.9902
0.5	99	1.0000	0.9996	1.0000	0.9970	0.9774	0.9940
0.8	1	0.9834	0.6006	0.8782	0.4746	0.5392	0.3858
0.8	20	0.9824	0.5890	0.2132	0.3476	0.0000	0.3416
0.8	50	0.9854	0.5994	0.0780	0.5918	0.0000	0.5960
0.8	80	0.9840	0.6020	0.2594	0.4016	0.0000	0.4056
0.8	99	0.9816	0.5918	0.8532	0.5094	0.5614	0.4784
0.9	1	0.6090	0.2256	0.4218	0.1690	0.1904	0.1362
0.9	20	0.6236	0.2244	0.1214	0.1210	0.0006	0.0952
0.9	50	0.6220	0.2316	0.0912	0.2332	0.0002	0.2234
0.9	80	0.6294	0.2188	0.1746	0.1496	0.0056	0.1380
0.9	99	0.6256	0.2258	0.4632	0.2020	0.2708	0.1790
1	1	0.0588	0.0516	0.0504	0.0494	0.0480	0.0502
1	20	0.0538	0.0524	0.0470	0.0398	0.0358	0.0298
1	50	0.0516	0.0496	0.0518	0.0484	0.0328	0.0498
1	80	0.0508	0.0466	0.0496	0.0434	0.0404	0.0310
1	99	0.0584	0.0488	0.0520	0.0518	0.0472	0.0488

Table 5: Empirical size and power for DGP (1) using various statistics ($n = 100$, $c = 50$)

ρ			1			0.8
θ	0	5	10	0	5	10
$T_b(t_{\hat{\alpha}})^\dagger$	0.040	0.108	0.507	0.295	0.435	0.861
$T_b(\|t_{\hat{\gamma}}\|)^\dagger$	0.049	0.048	0.032	0.301	0.098	0.042
$T_b(t_{\hat{\gamma}})^\dagger$	0.050	0.050	0.040	0.350	0.133	0.055
$T_b(F_{\hat{\theta},\hat{\gamma}})^\dagger$	0.052	0.050	0.020	0.339	0.194	0.163
τ_w^*	0.050	0.048	0.050	0.599	0.592	0.596

†Values are from Vogelsang and Perron (1998)

Table 6: Empirical size and power for DGP (1) using $\tau_{w,\tau}$ and $\tau^*_{w,\tau}$

		$\theta = 0$		$\theta = 5$		$\theta = 10$	
ρ	c	$\tau_{w,\tau}$	$\tau^*_{w,\tau}$	$\tau_{w,\tau}$	$\tau^*_{w,\tau}$	$\tau_{w,\tau}$	$\tau^*_{w,\tau}$
0.5	1	1.0000	0.9880	0.9986	0.9498	0.9270	0.9088
0.5	20	1.0000	0.9848	0.8418	0.8906	0.0018	0.8802
0.5	50	1.0000	0.9850	0.9720	0.9850	0.0992	0.9854
0.5	80	1.0000	0.9844	0.8406	0.9044	0.0012	0.8854
0.5	99	1.0000	0.9892	0.9984	0.9516	0.9258	0.9170
0.8	1	0.8292	0.2970	0.6226	0.2494	0.3082	0.1912
0.8	20	0.8386	0.2954	0.2842	0.2016	0.0050	0.1486
0.8	50	0.8346	0.3040	0.4354	0.3068	0.0484	0.3016
0.8	80	0.8312	0.2986	0.2946	0.2152	0.0060	0.1848
0.8	99	0.8238	0.2938	0.6160	0.2542	0.3454	0.2372
0.9	1	0.3066	0.1084	0.2150	0.0936	0.0982	0.0816
0.9	20	0.2968	0.1126	0.1298	0.0932	0.0162	0.0594
0.9	50	0.2974	0.1152	0.1874	0.1130	0.0528	0.1090
0.9	80	0.2906	0.1108	0.1426	0.0936	0.0232	0.0654
0.9	99	0.2982	0.1082	0.2306	0.1048	0.1318	0.0902
1	1	0.0574	0.0558	0.0544	0.0524	0.0426	0.0458
1	20	0.0620	0.0496	0.0496	0.0470	0.0342	0.0316
1	50	0.0540	0.0498	0.0528	0.0466	0.0342	0.0480
1	80	0.0498	0.0502	0.0536	0.0448	0.0362	0.0314
1	99	0.0494	0.0476	0.0456	0.0512	0.0430	0.0494

Table 7: Empirical size and power for DGP (2) using $\tau_{w,\tau}$ and $\tau^*_{w,\tau}$

		$\gamma=0$		$\gamma=1$		$\gamma=2$	
ρ	c	$\tau_{w,\tau}$	$\tau^*_{w,\tau}$	$\tau_{w,\tau}$	$\tau^*_{w,\tau}$	$\tau_{w,\tau}$	$\tau^*_{w,\tau}$
0.5	1	1.0000	0.9858	1.0000	0.9872	1.0000	0.9860
0.5	20	1.0000	0.9862	0.0000	0.8884	0.0000	0.8804
0.5	50	1.0000	0.9860	0.0000	0.9852	0.0000	0.9892
0.5	80	1.0000	0.9862	0.0000	0.8848	0.0000	0.8852
0.5	99	1.0000	0.9868	1.0000	0.9860	1.0000	0.9822
0.8	1	0.8296	0.3098	0.8172	0.2974	0.8322	0.3084
0.8	20	0.8170	0.2934	0.0000	0.1544	0.0000	0.1572
0.8	50	0.8256	0.3060	0.0000	0.2980	0.0000	0.3020
0.8	80	0.8330	0.3126	0.0000	0.1704	0.0000	0.1672
0.8	99	0.8170	0.3016	0.8138	0.2940	0.7774	0.3064
0.9	1	0.2962	0.1128	0.2994	0.1006	0.3002	0.1088
0.9	20	0.3050	0.1138	0.0000	0.0524	0.0000	0.0538
0.9	50	0.2968	0.1106	0.0000	0.1118	0.0000	0.1206
0.9	80	0.2974	0.1078	0.0000	0.0624	0.0000	0.0582
0.9	99	0.3026	0.1130	0.2894	0.1114	0.0552	0.1180
1	1	0.0572	0.0510	0.0556	0.0568	0.0566	0.0470
1	20	0.0526	0.0498	0.0000	0.0302	0.0000	0.0282
1	50	0.0552	0.0504	0.0000	0.0496	0.0000	0.0516
1	80	0.0622	0.0514	0.0000	0.0260	0.0000	0.0270
1	99	0.0574	0.0482	0.0522	0.0520	0.0552	0.0524

Table 8: Empirical size and power for DGP (2) using various statistics ($n = 100$, $c = 50$)

ρ			1			0.8
γ	0	1	2	0	1	2
$T_b(t_{\hat{\alpha}})^{\dagger}$	0.040	0.044	0.076	0.295	0.236	0.386
$T_b(\lvert t_{\hat{\gamma}}\rvert)^{\dagger}$	0.049	0.036	0.032	0.301	0.239	0.230
$T_b(t_{\hat{\gamma}})^{\dagger}$	0.050	0.072	0.061	0.350	0.376	0.371
$T_b(F_{\hat{\theta},\hat{\gamma}})^{\dagger}$	0.052	0.027	0.021	0.339	0.193	0.184
$\tau^*_{w,\tau}$	<u>0.050</u>	<u>0.050</u>	<u>0.052</u>	<u>0.306</u>	<u>0.298</u>	<u>0.302</u>

†Values are from Vogelsang and Perron (1998)

Table 9: Test results for the unit root null hypothesis

Series	n	N&P‡	P1§	Z&A¶	P2‖	Bisection**
Real GNP	62	-2.99	-5.03*	-5.58*	-5.93*	-4.09†
Nominal GNP	62	-2.32	-5.42*	-5.82*	-8.16*	-2.28
Real per capita GNP	62	-3.04	-4.09†	-4.61	-4.81	-3.71†
Industrial production	111	-2.53	-5.47*	-5.95*	-6.01*	-5.72†
Employment	81	-2.66	-4.51*	-4.95†	-5.14†	-3.46
Unemployment rate	81	-3.55†	N/A	N/A	N/A	-4.06†
GNP deflator	82	-2.52	-4.04†	-4.12	-4.14	-2.67
Consumer prices	111	-1.97	-1.28	-2.76	-3.09	-3.18
Wages	71	-2.09	-5.41*	-5.30†	-5.41†	-2.93
Real wages	71	-3.04	-4.28†	-4.74	-5.41	-4.35†
Money stock	82	-3.08	-4.29†	-4.34	-4.69	-4.48†
Velocity	102	-1.66	-1.66	-3.39	-2.81	-2.44
Interest rate	71	0.686	-0.45	-0.98	-1.35	-0.80
Common stock prices	100	-2.05	-4.87†	-5.61*	-5.50†	-3.34

*statistical significance at the 1% level

†statistical significance at the 5% level

‡Nelson and Plosser (1982)

§Perron (1989)

¶Zivot and Andrews (1992)

‖Perron (1997)

**critical value from simulation : -3.62 for $n = 100$

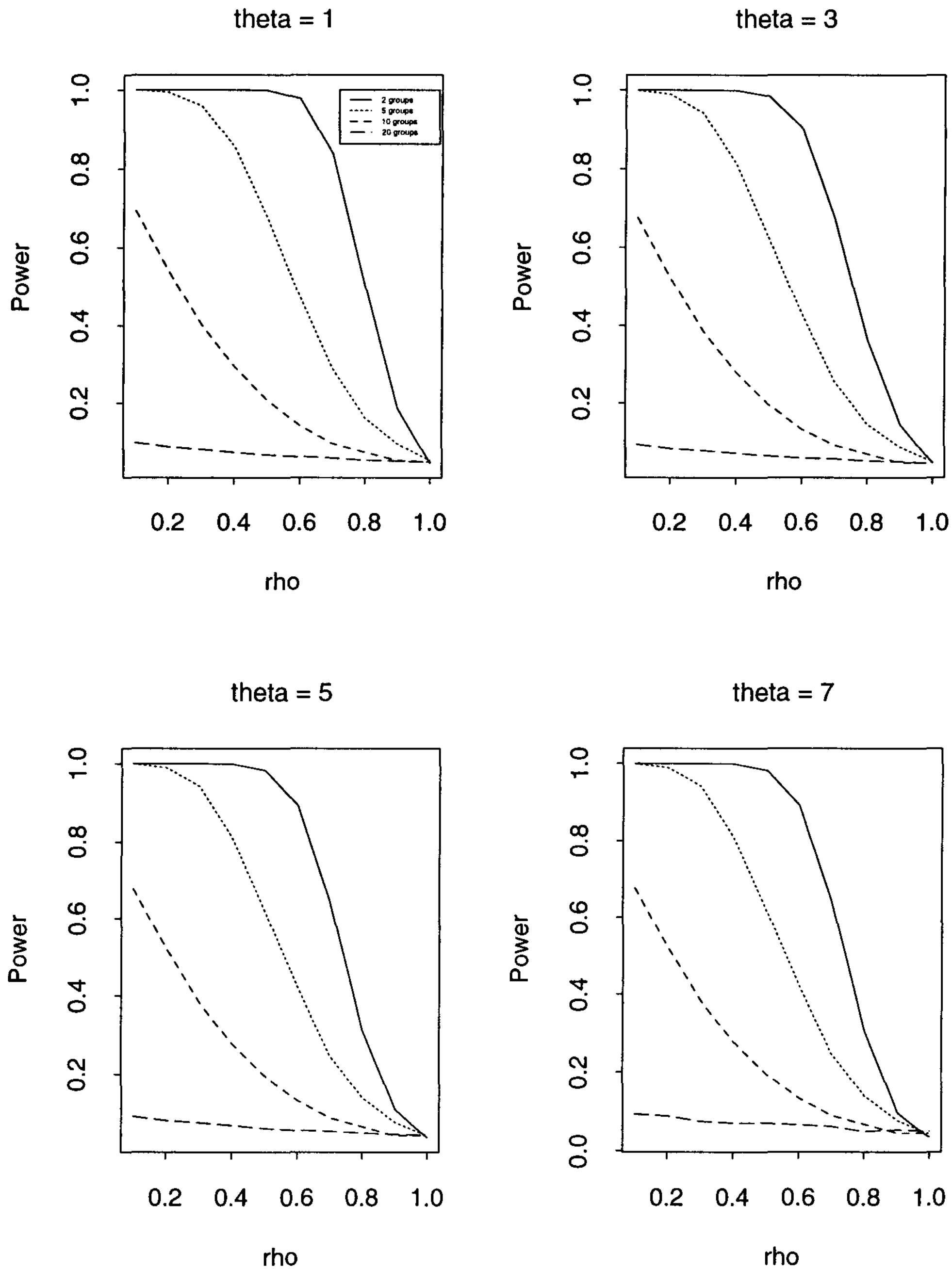

Figure 1: Empirical size and power for DGP (1) using various subgroups (OLS, $n = 100$, $c = 36$)

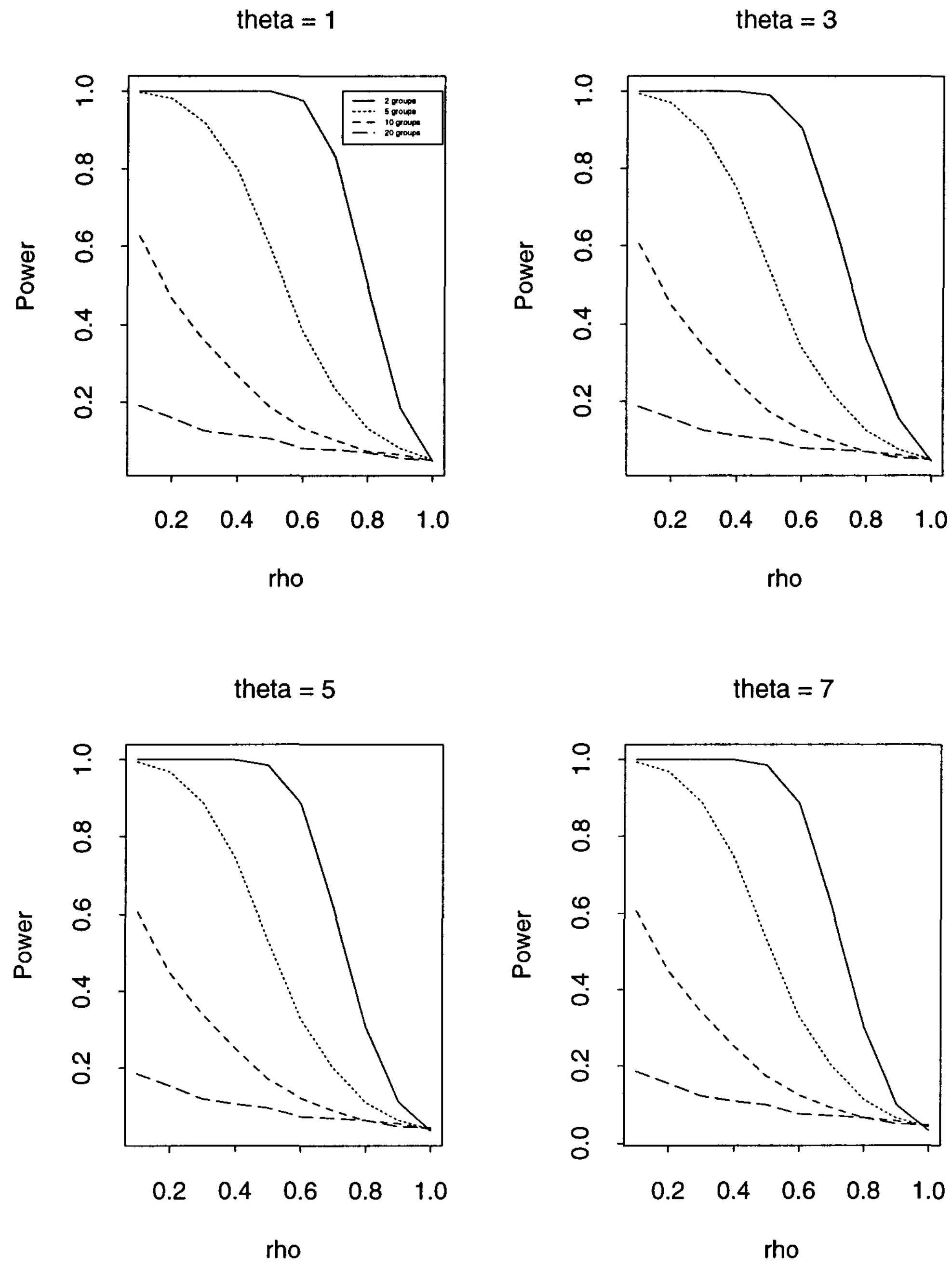

Figure 2: Empirical size and power for DGP (1) using various subgroups (SS, $n = 100$, $c = 37$)

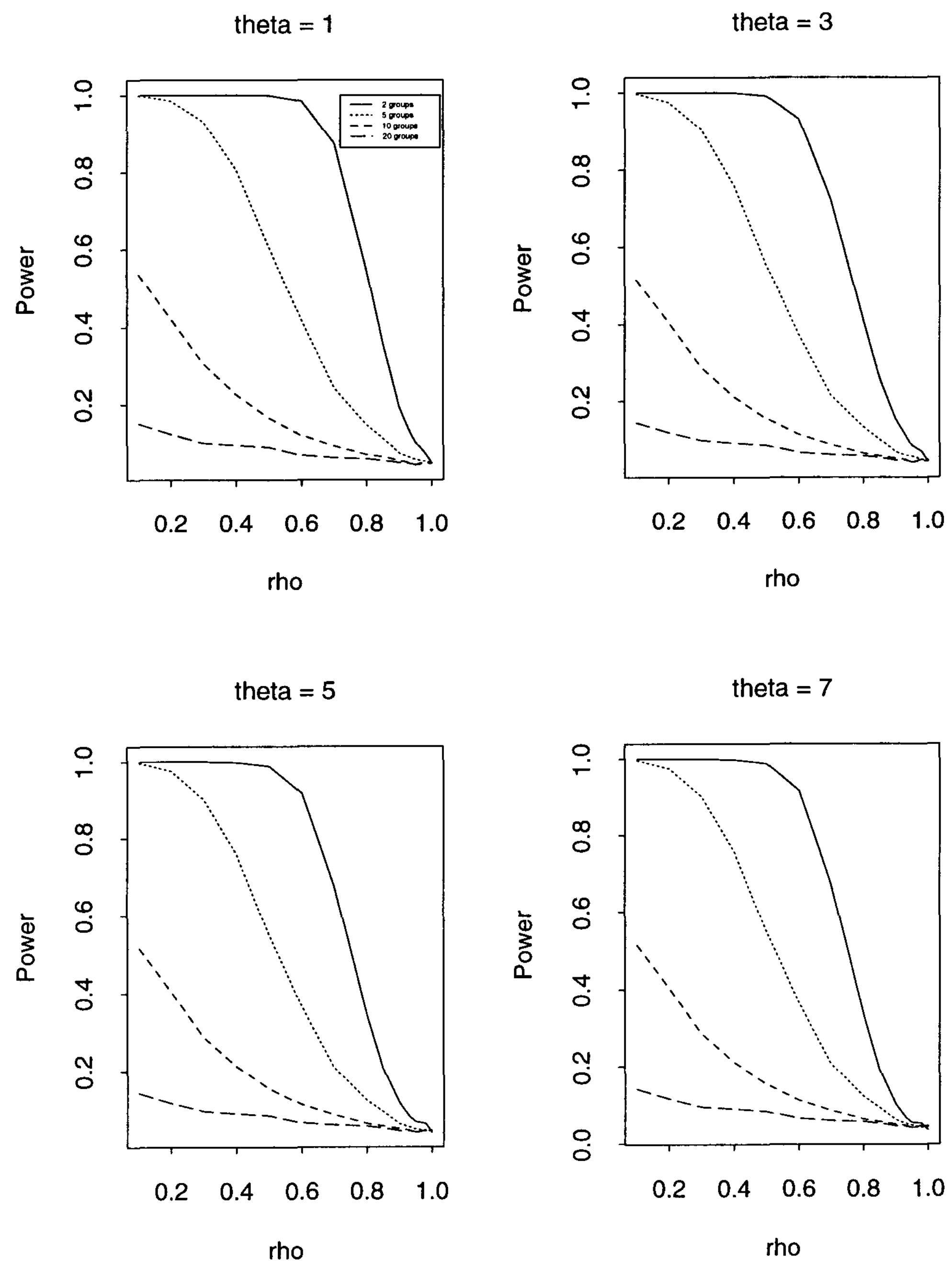

Figure 3: Empirical size and power for DGP (1) using various subgroups (WS, $n = 100$, $c = 37$)

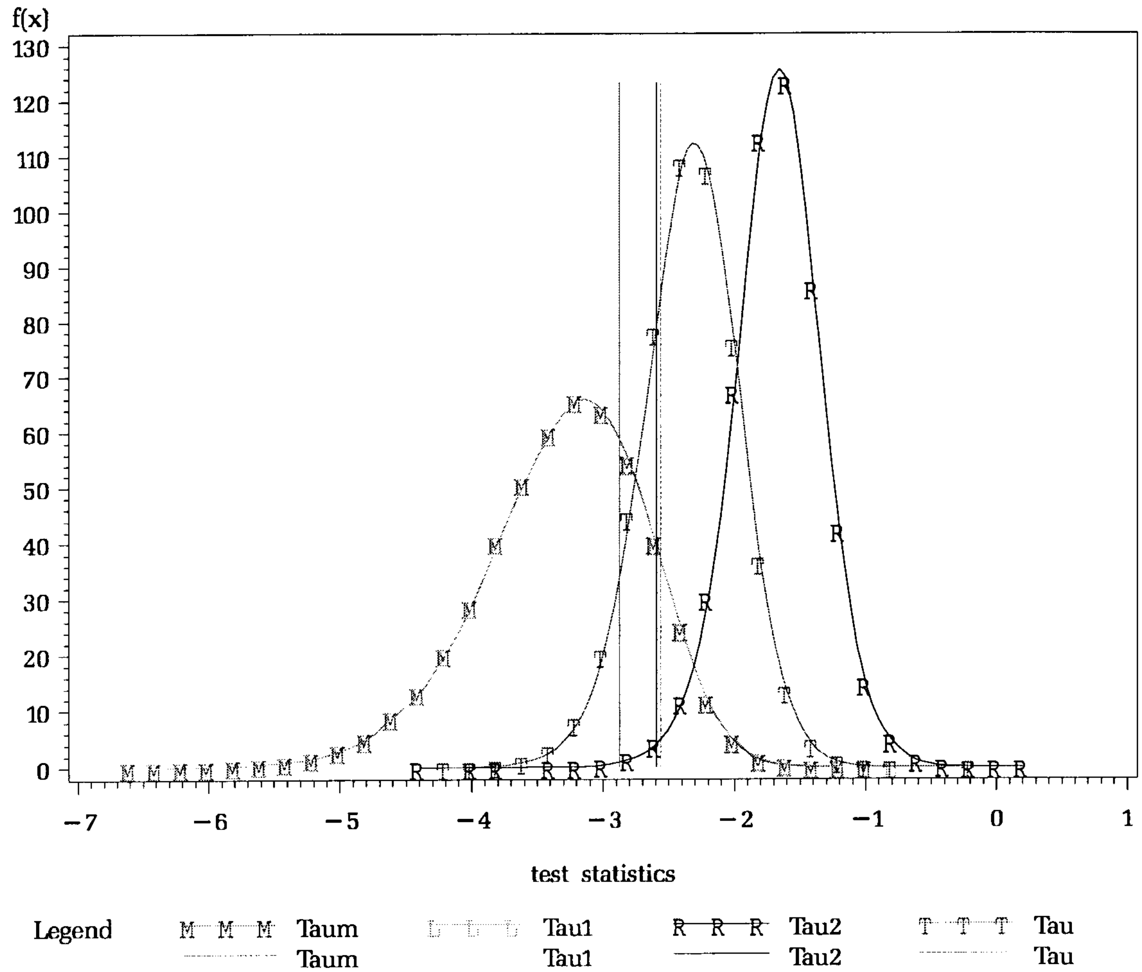

Figure 4: Empirical distributions of test statistics (WS, $n = 100$, $c = 75$, $\theta = 5$ and $\rho = .7$; T for τ_w, M for τ_w^*, L for $\tau_{w,1}$ and R for $\tau_{w,2}$; M overlays L.)

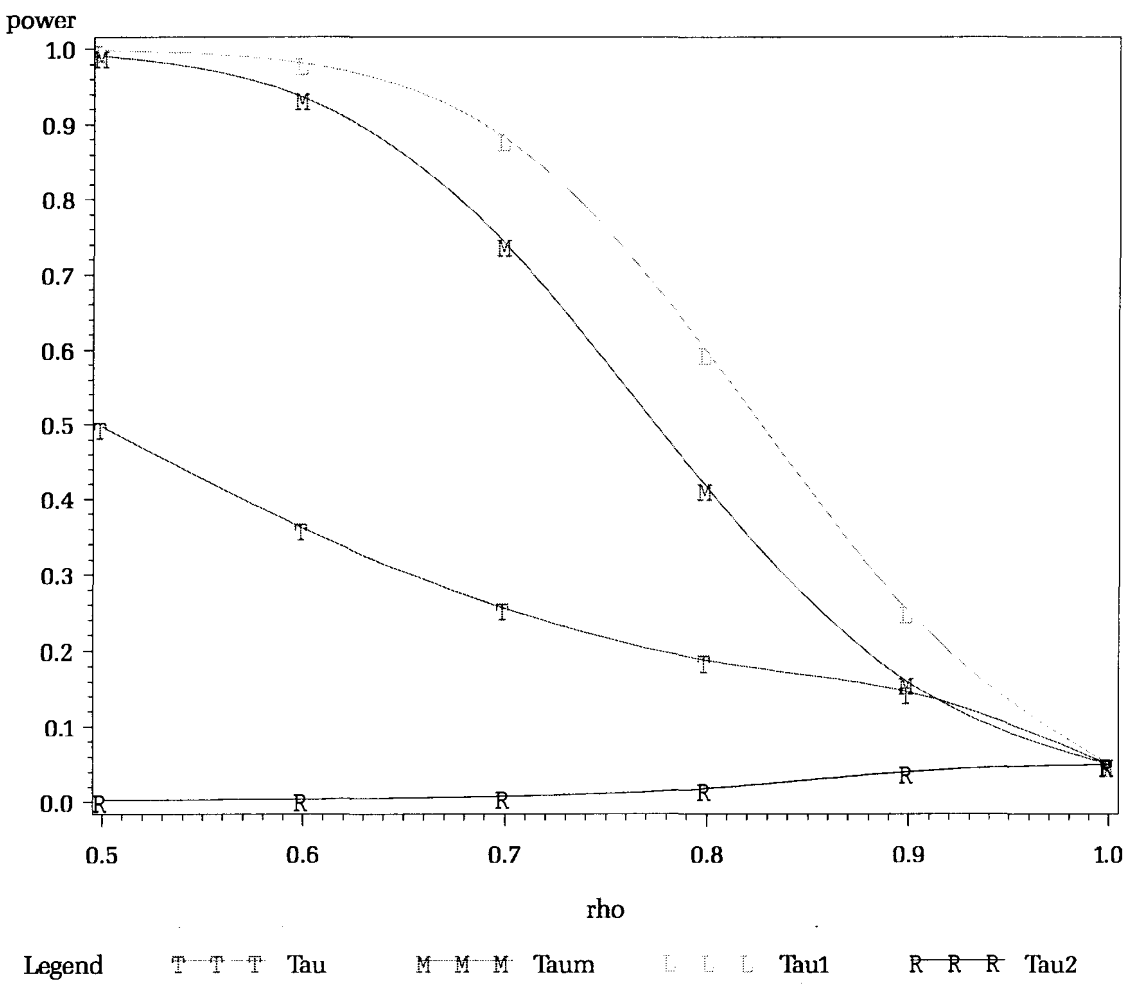

Figure 5: Empirical size and power of test statistics (WS, $n = 100$, $c = 75$ and $\theta = 5$; T for τ_w, M for τ_w^*, L for $\tau_{w,1}$ and R for $\tau_{w,2}$)

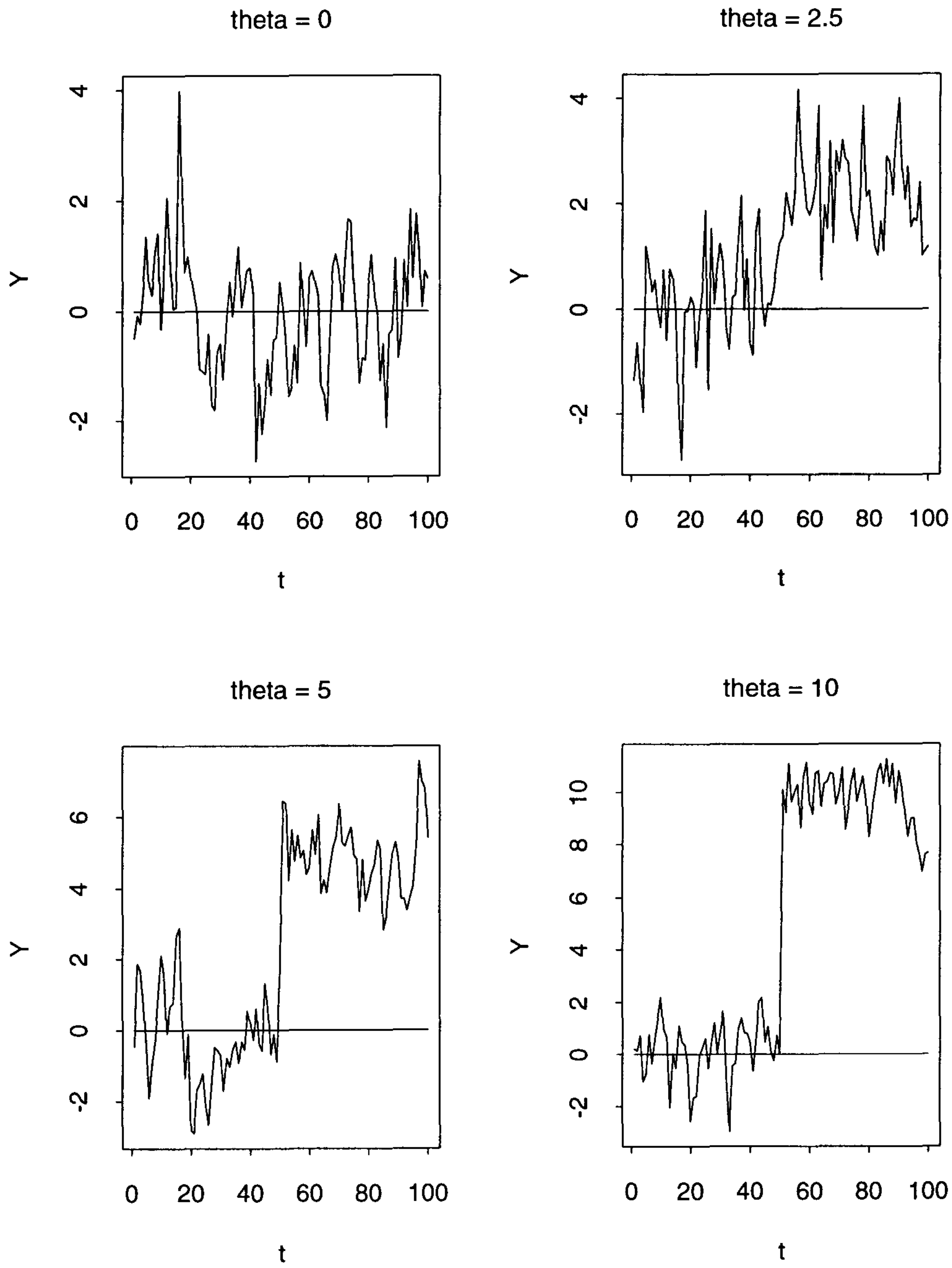

Figure 6: Data from a break-in-level model (DGP (1), $\rho = 0.5$, $c = 50$)

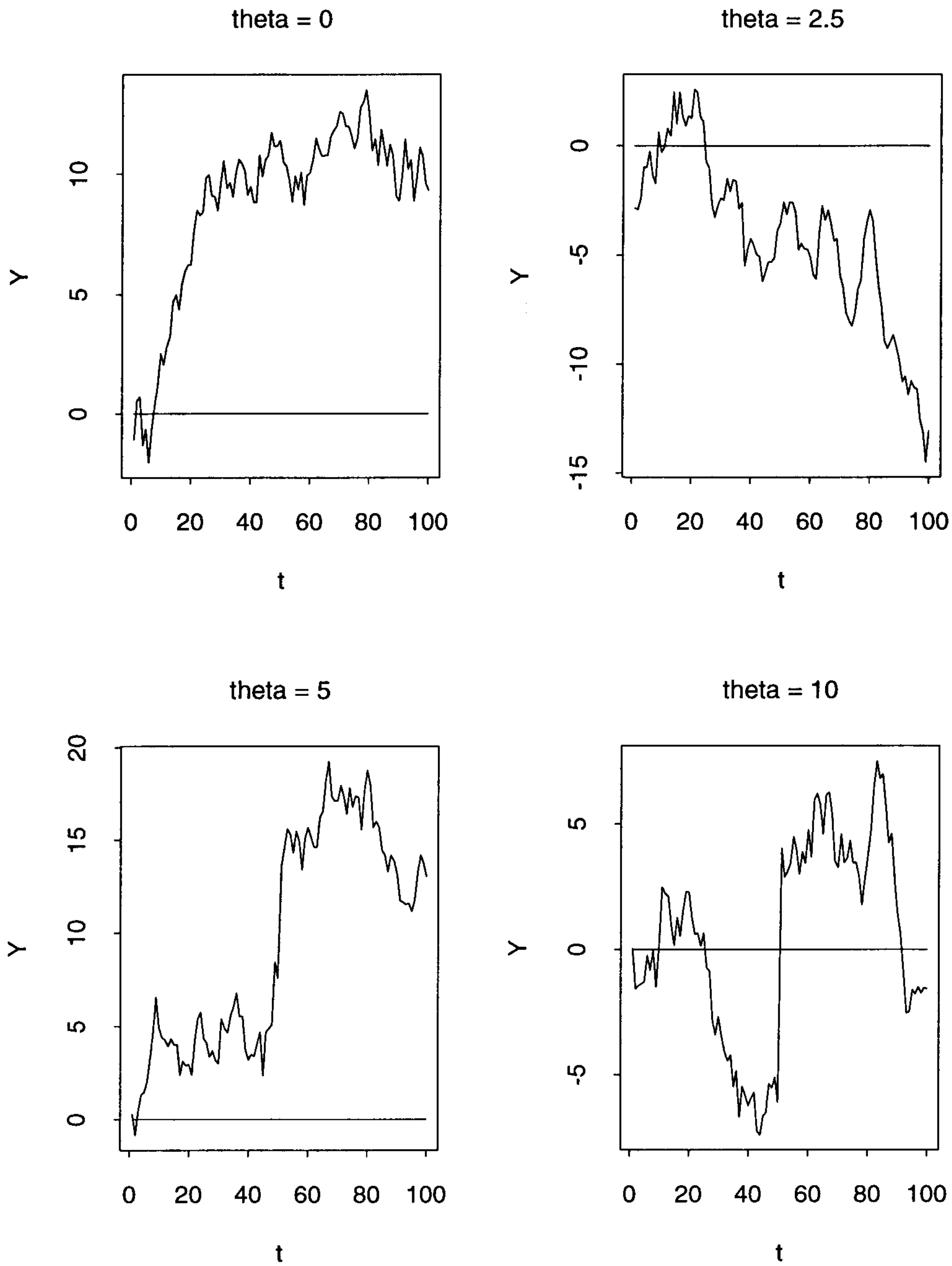

Figure 7: Data from a break-in-level model (DGP (1), $\rho = 1$, $c = 50$)

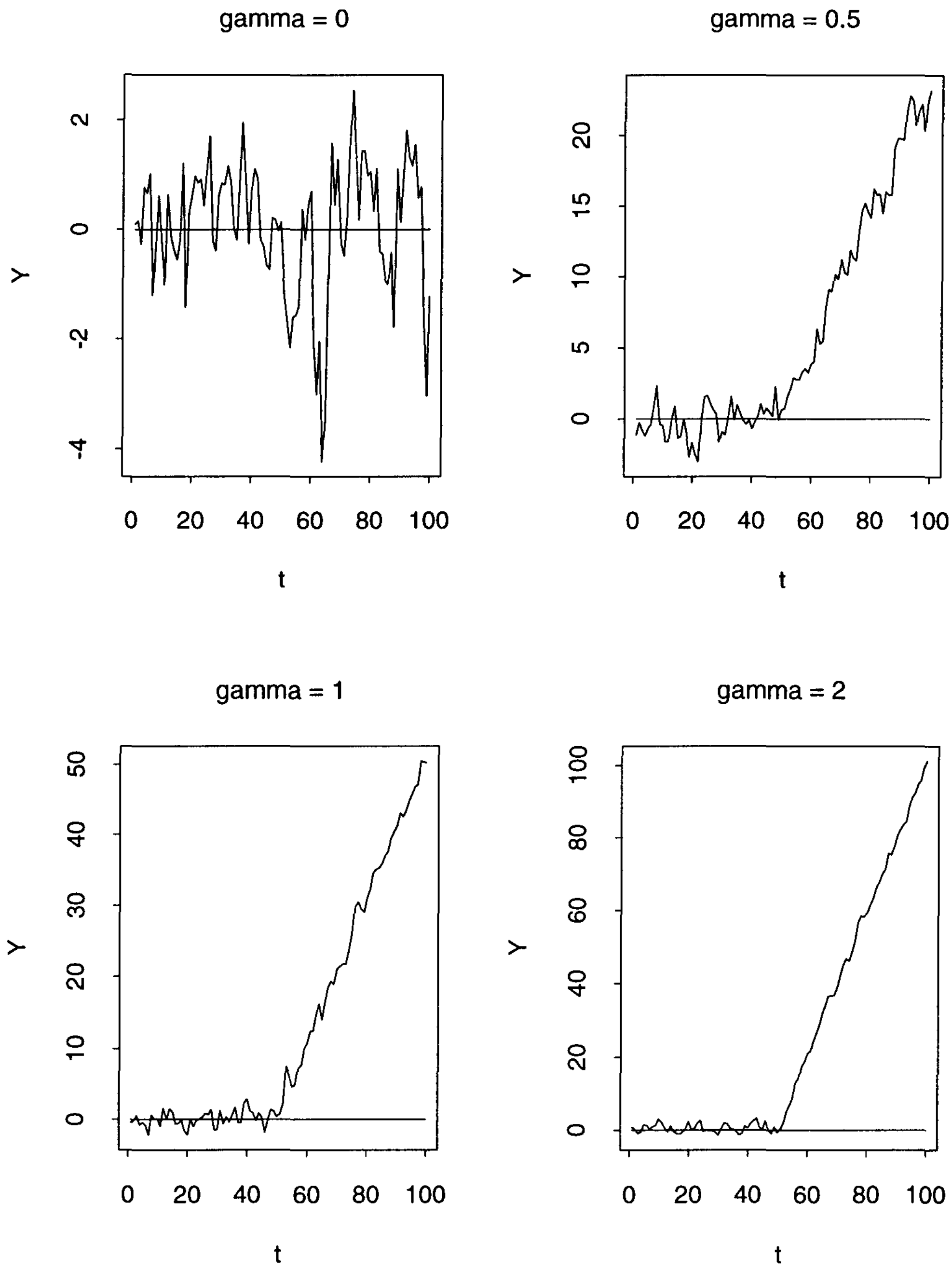

Figure 8: Data from a break-in-slope model (DGP (2), $\rho = 0.5$, $c = 50$)

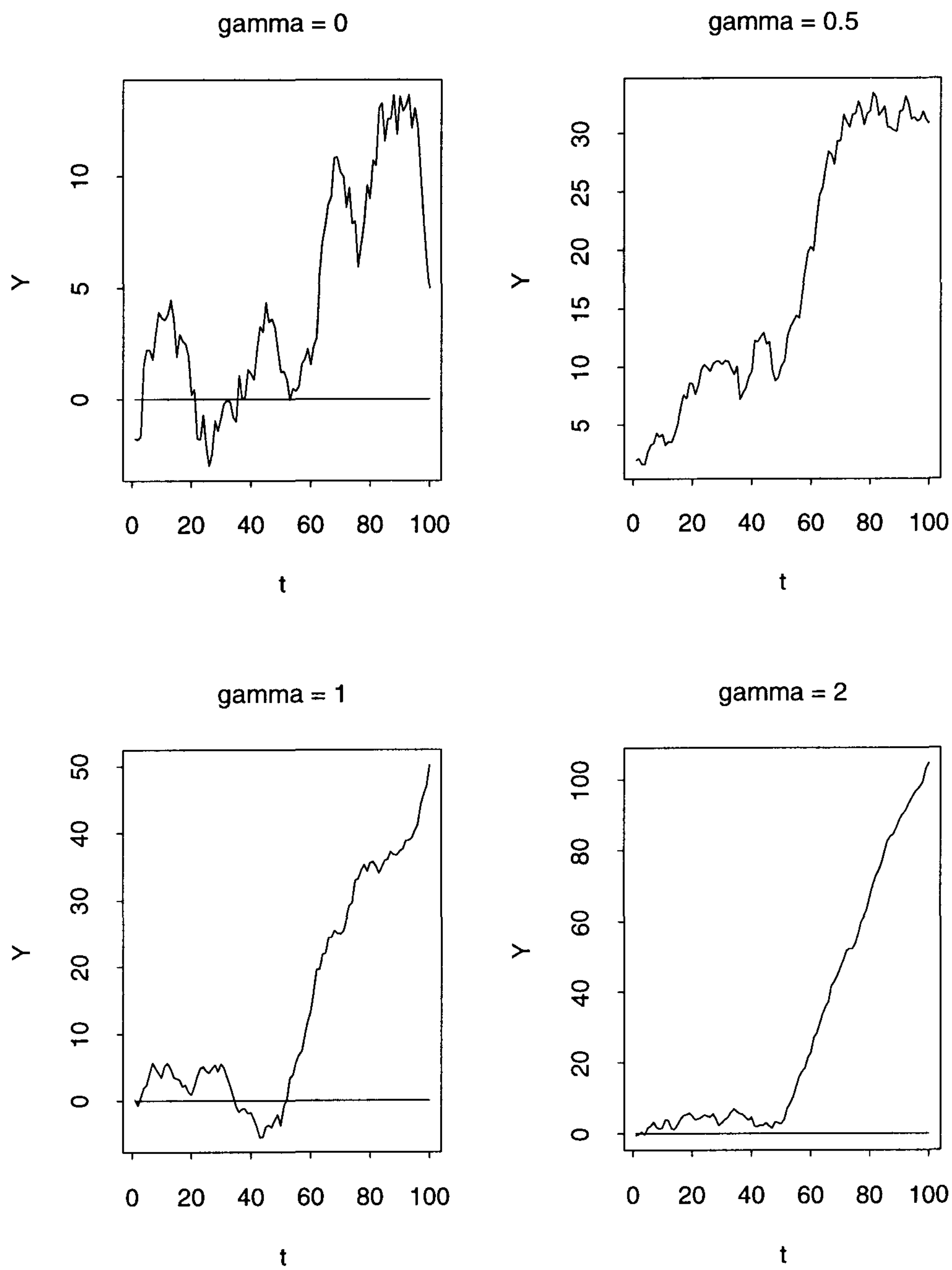

Figure 9: Data from a break-in-slope model (DGP (2), $\rho = 1$, $c = 50$)

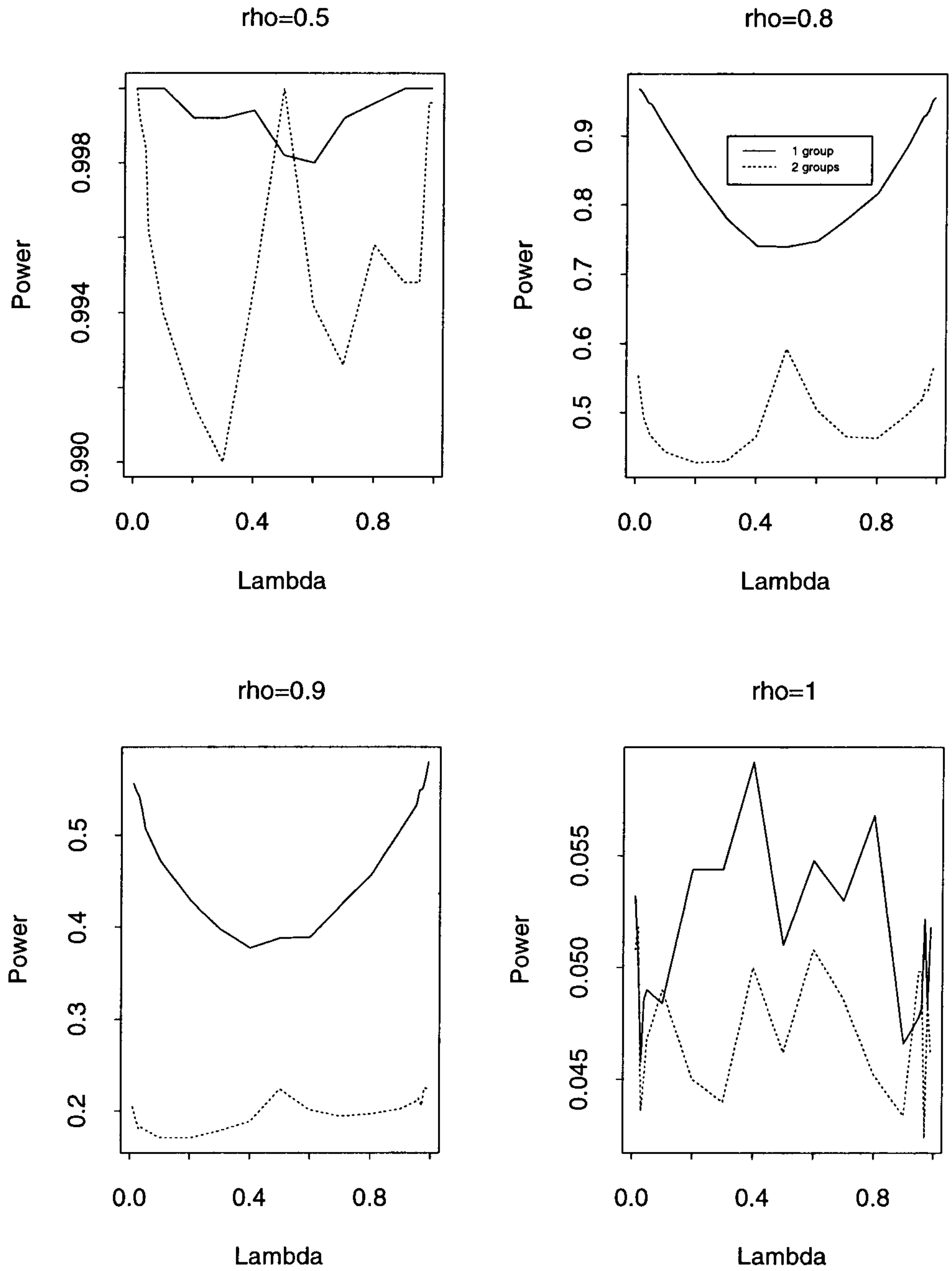

Figure 10: Empirical size and power for DGP (1) using τ_w (1 group) and τ_w^* (2 groups) ($\theta = 2.5$, $\lambda = c/n$)

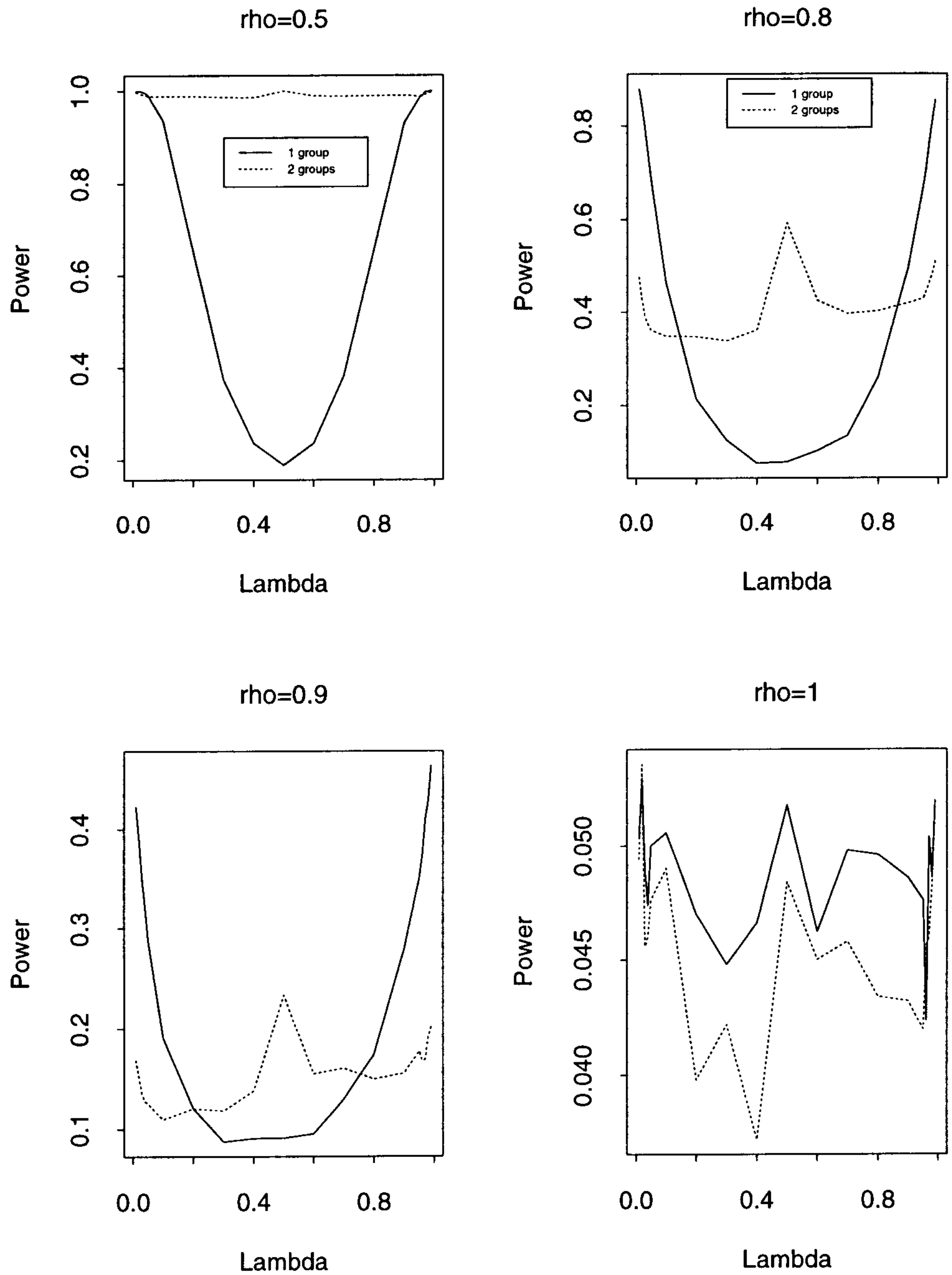

Figure 11: Empirical size and power for DGP (1) using τ_w (1 group) and τ_w^* (2 groups) ($\theta = 5$, $\lambda = c/n$)

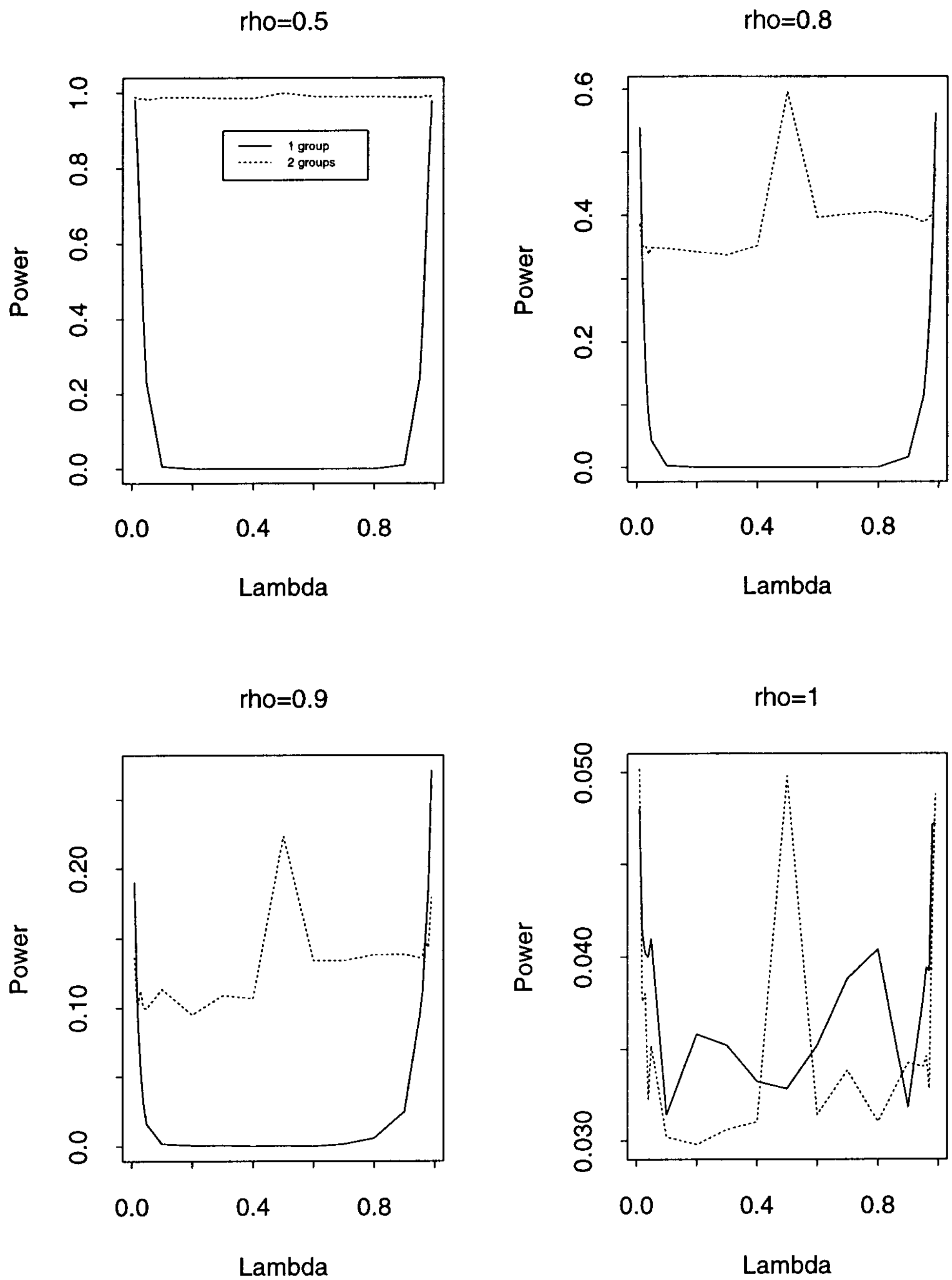

Figure 12: Empirical size and power for DGP (1) using τ_w (1 group) and τ_w^* (2 groups) ($\theta = 10$, $\lambda = c/n$)

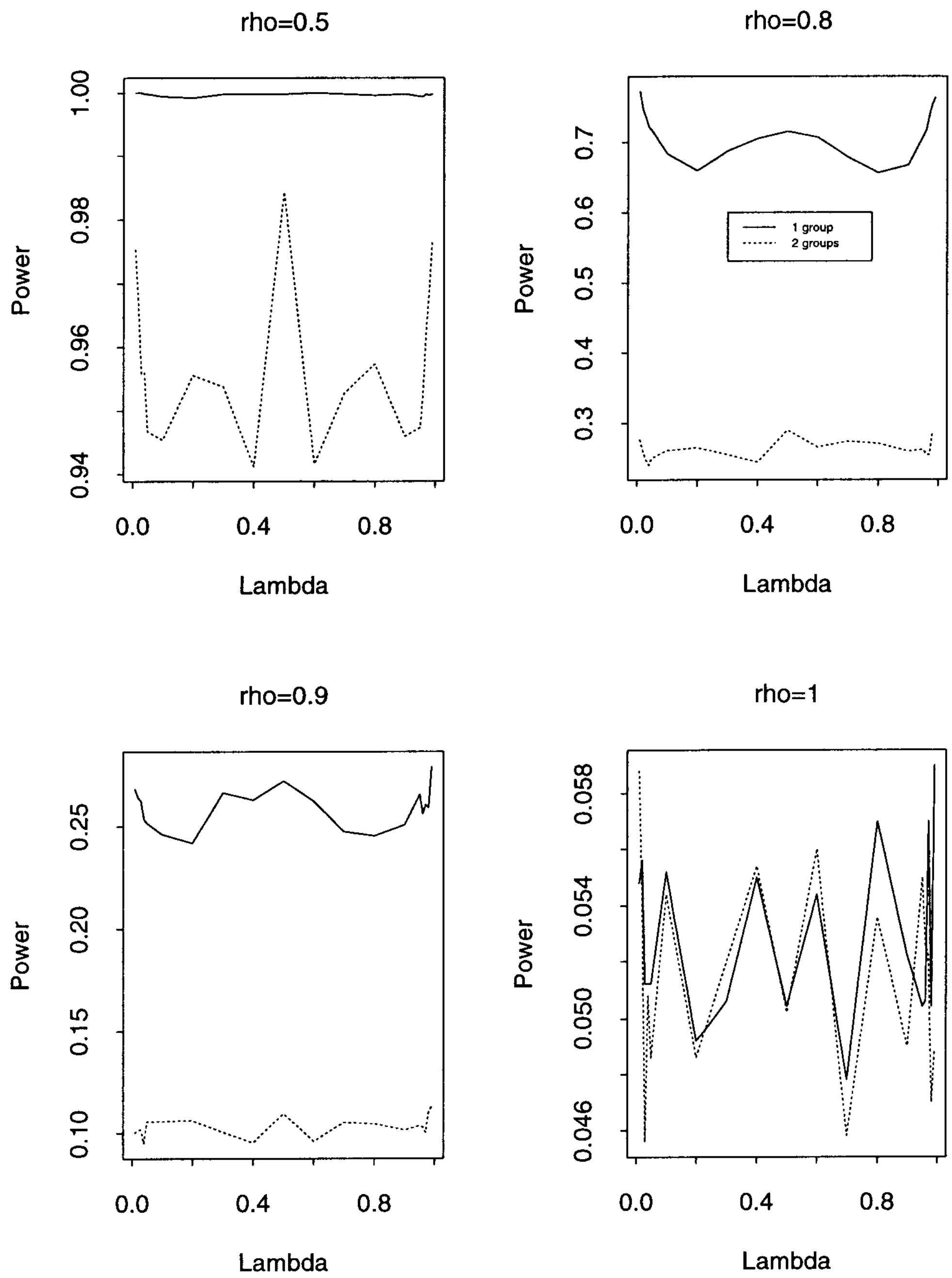

Figure 13: Empirical size and power for DGP (1) using $\tau_{w,\tau}$ (1 group) and $\tau^*_{w,\tau}$ (2 groups) ($\theta = 2.5$, $\lambda = c/n$)

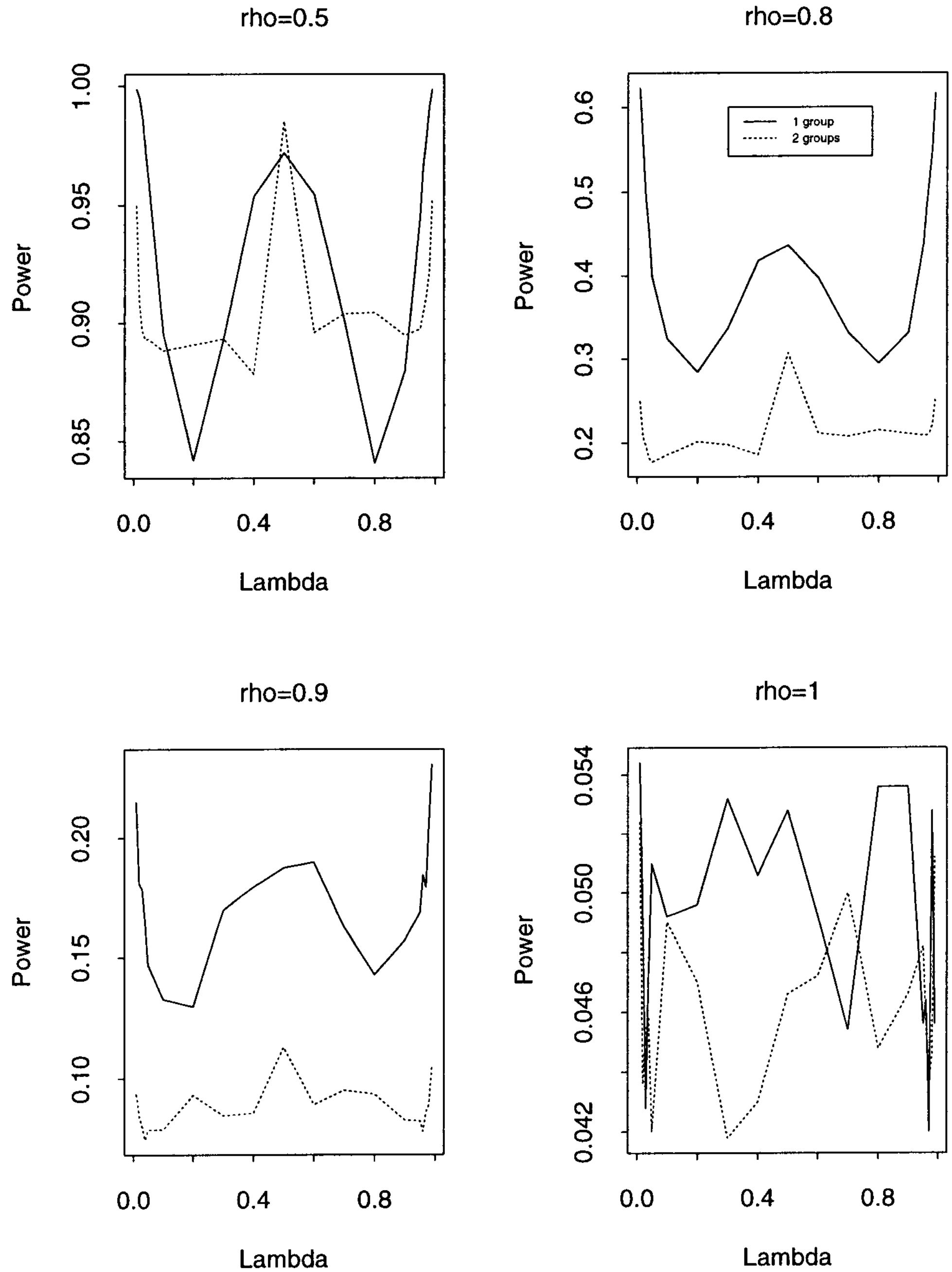

Figure 14: Empirical size and power for DGP (1) using $\tau_{w,\tau}$ (1 group) and $\tau^*_{w,\tau}$ (2 groups) ($\theta = 5$, $\lambda = c/n$)

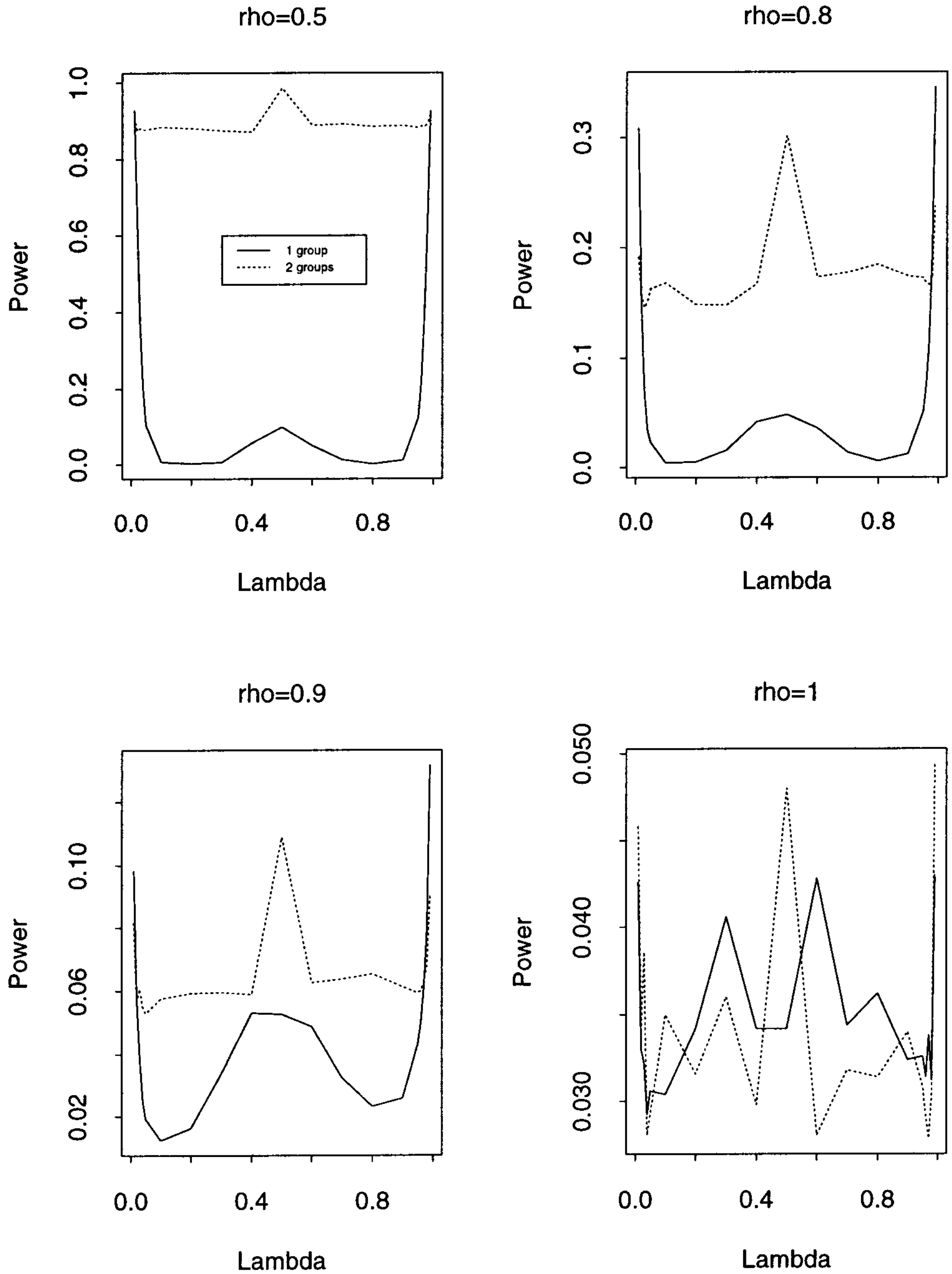

Figure 15: Empirical size and power for DGP (1) using $\tau_{w,\tau}$ (1 group) and $\tau^*_{w,\tau}$ (2 groups) ($\theta = 10$, $\lambda = c/n$)

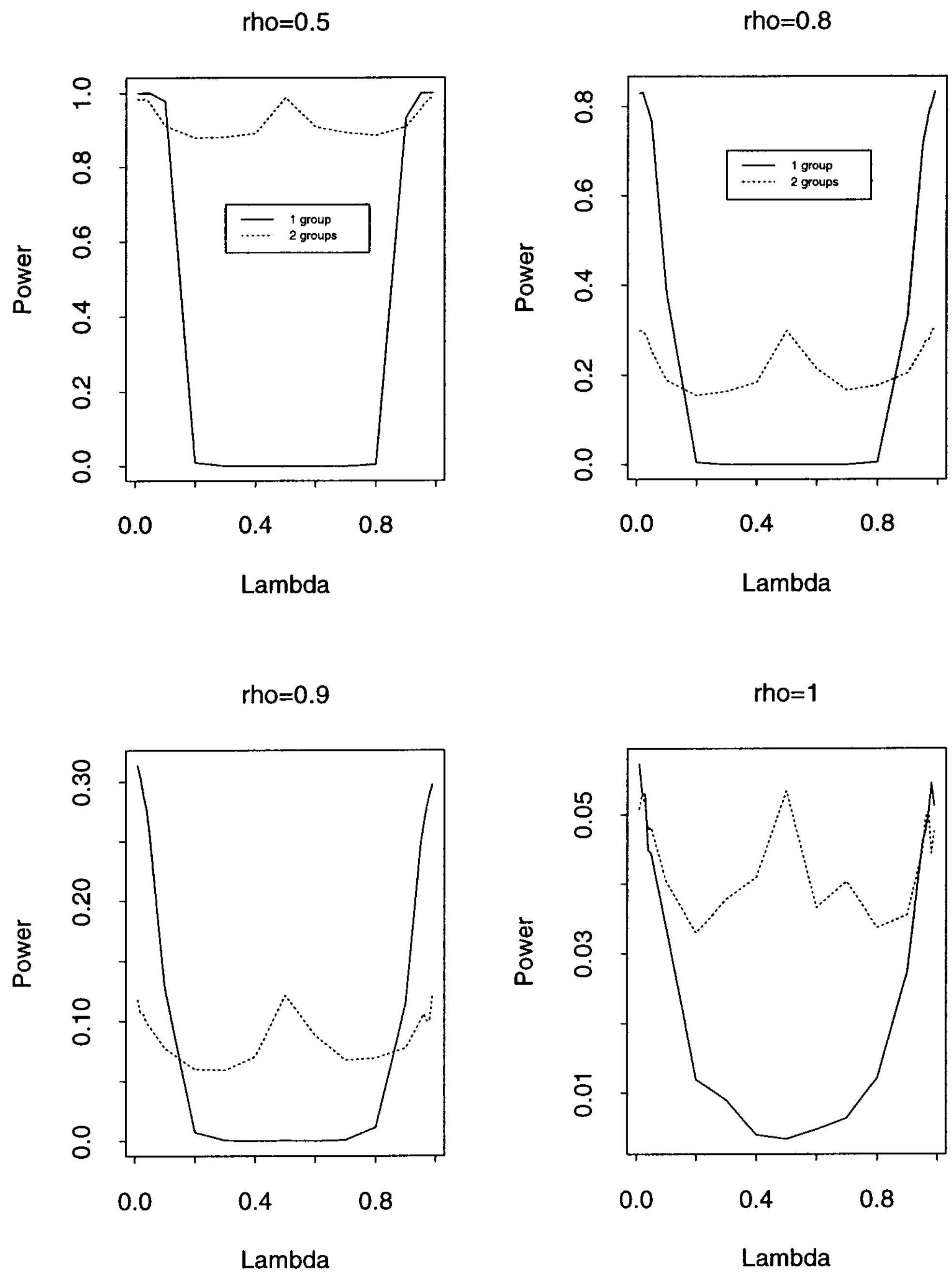

Figure 16: Empirical size and power for DGP (2) using $\tau_{w,\tau}$ (1 group) and $\tau^*_{w,\tau}$ (2 groups) ($\gamma = 0.5$, $\lambda = c/n$)

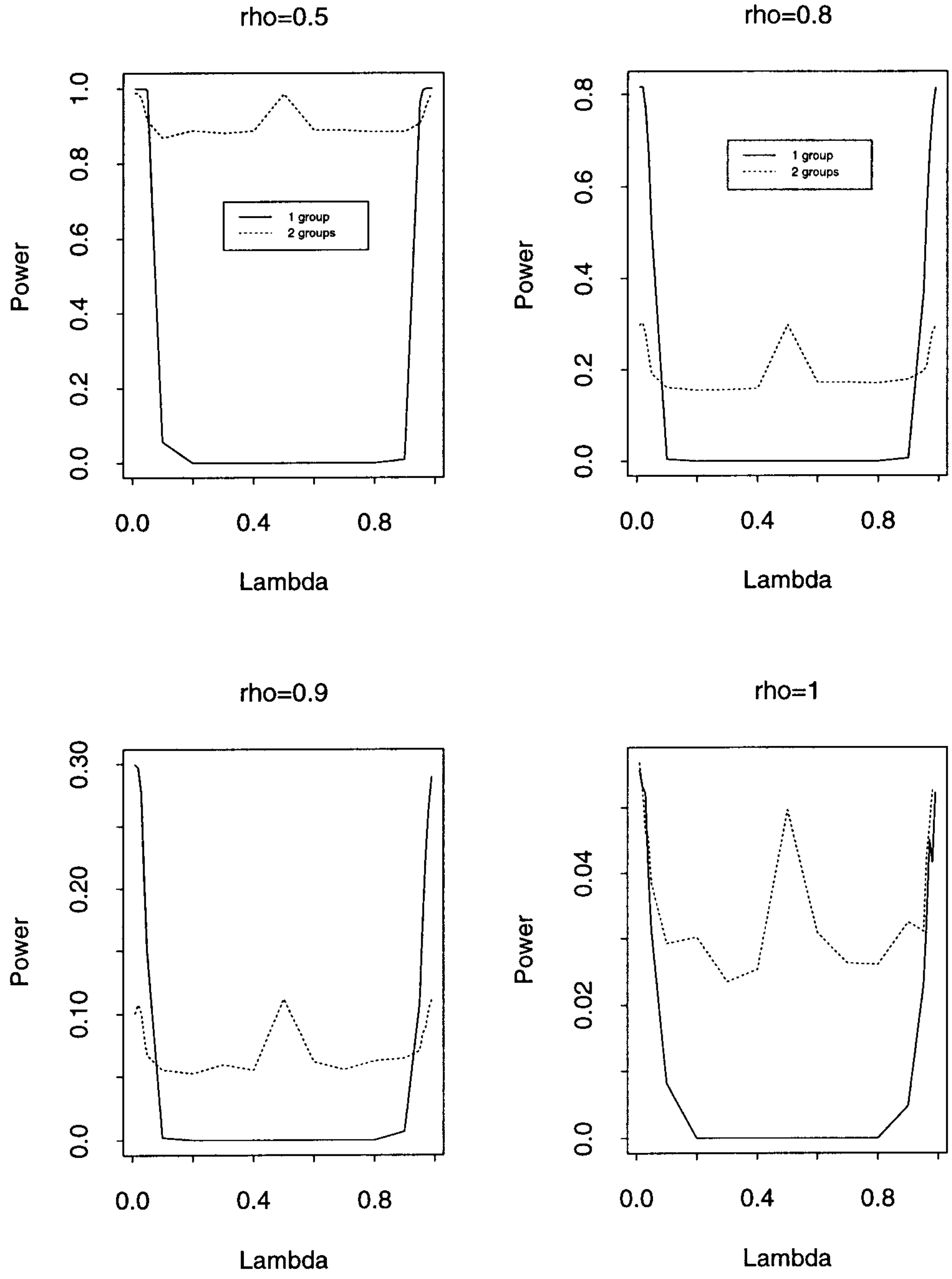

Figure 17: Empirical size and power for DGP (2) using $\tau_{w,\tau}$ (1 group) and $\tau^*_{w,\tau}$ (2 groups) ($\gamma = 1$, $\lambda = c/n$)

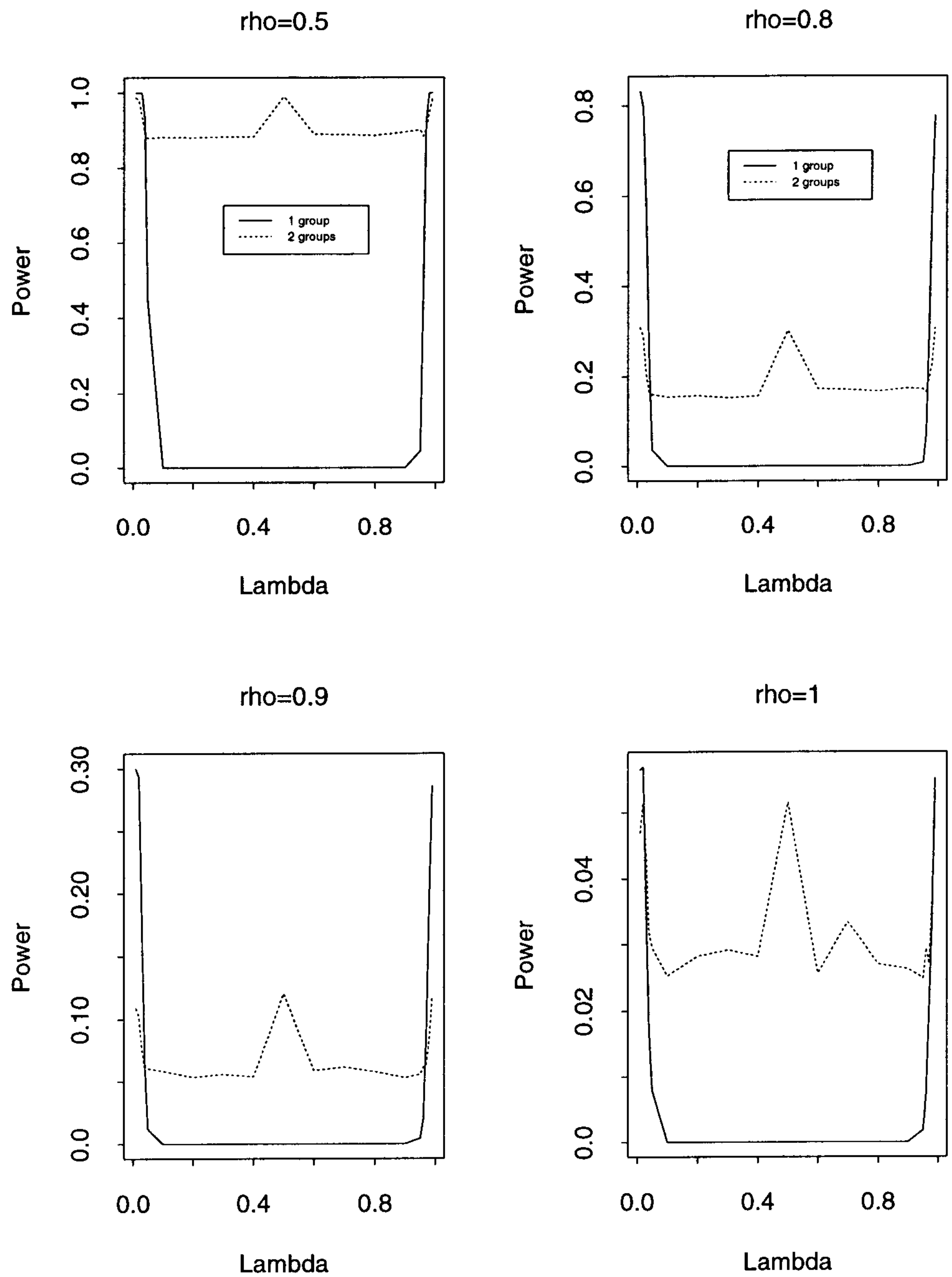

Figure 18: Empirical size and power for DGP (2) using $\tau_{w,\tau}$ (1 group) and $\tau^*_{w,\tau}$ (2 groups) ($\gamma = 2$, $\lambda = c/n$)

Appendix: Limiting distribution of the bisection test statistic

Consider a data generating process

$$Y_t = \theta\sigma I(t > c) + W_t, \quad W_t = \rho W_{t-1} + e_t, \quad t = 1, 2, \cdots, n = 100.$$

We are deriving the limiting distribution of $\tau_w^* \equiv \min(\tau_{w,1}, \tau_{w,2})$ under the null hypothesis of $H_0 : \rho = 1$.

We know that

$$\begin{aligned} \bar{Y} &= \bar{W} + (1-\lambda)m\sigma \\ Y_t - \bar{Y} &= \begin{cases} W_t - \bar{W} - (1-\lambda)m\sigma & \text{if } t \leq c = n\lambda \\ W_t - \bar{W} + \lambda m\sigma & \text{if } t > c = n\lambda. \end{cases} \end{aligned}$$

Under $\rho = 1$,

$$W_t = \sum_{j=1}^{t} e_j.$$

We divide the data into 2 groups with $\frac{n}{2} = 50$ observations each. Without loss of generality we can assume further that

$$51 < c < 100$$

,i.e., the break is in the second group. Then

$$\begin{aligned} W_1 &= e_1 \\ W_2 &= e_1 + e_2 \\ &\vdots \\ W_{50} &= e_1 + e_2 + \cdots + e_{50} \end{aligned}$$

and

$$W_{51} = W_{50} + e_{51}$$

$$
\begin{aligned}
W_{52} &= W_{50} + e_{51} + e_{52} \\
&\vdots \\
W_{100} &= W_{50} + e_{51} + e_{52} + \cdots + e_{100}.
\end{aligned}
$$

Let

$$
\begin{aligned}
\bar{W}_1 &= \frac{1}{50}\sum_{t=1}^{50} W_t \\
&= \frac{1}{50}\sum_{t=1}^{50}\sum_{j=1}^{t} e_j
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{W}_2 &= \frac{1}{50}\sum_{t=51}^{100} W_t \\
&= \frac{1}{50}\sum_{t=51}^{100} (W_{50} + \sum_{j=51}^{t} e_j).
\end{aligned}
$$

Then we can easily show that

$$W_t - \bar{W}_1 = \text{function only of } (e_1, e_2, \cdots, e_{50}) \qquad t = 1, 2, ..., 50$$

and

$$W_t - \bar{W}_2 = \text{function only of } (e_{51}, e_{52}, \cdots, e_{100}) \qquad t = 51, 52, ..., 100.$$

Therefore we can conclude that $(W_1 - \bar{W}_1, W_2 - \bar{W}_1, \cdots, W_{50} - \bar{W}_1)$ and $(W_{51} - \bar{W}_2, W_{52} - \bar{W}_2, \cdots, W_{100} - \bar{W}_2)$ are independent.

Since $51 < c < 100$, for the first group,

$$
\begin{aligned}
Y_t &= W_t \qquad t = 1, 2, ..., 50 \\
\bar{Y}_1 &= \frac{1}{50}\sum_{t=1}^{50} Y_t = \bar{W}_1.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Y_t - \bar{Y}_1 &= W_t - \bar{W}_1 \qquad t = 1, 2, ..., 50 \\
&= \text{function only of } (W_1 - \bar{W}_1, W_2 - \bar{W}_1, \cdots, W_{50} - \bar{W}_1).
\end{aligned}
$$

For the second group,

$$\begin{aligned} Y_t &= \begin{cases} W_t & 51 \le t \le c \\ W_t + m\sigma & c < t \le 100 \end{cases} \\ \bar{Y}_2 &= \frac{1}{50}\sum_{t=51}^{100} Y_t \\ &= \bar{W}_2 + \frac{100-c}{50} m\sigma. \end{aligned}$$

and so

$$\begin{aligned} Y_t - \bar{Y}_2 &= \begin{cases} W_t - \bar{W}_2 - \frac{100-c}{50} m\sigma & 51 \le t \le c \\ W_t - \bar{W}_2 + \frac{c-50}{50} m\sigma & c < t \le 100 \end{cases} \\ &= \text{function only of } (W_{51} - \bar{W}_2, W_{52} - \bar{W}_2, \cdots, W_{100} - \bar{W}_2). \end{aligned}$$

Therefore we can conclude that $(Y_1 - \bar{Y}_1, Y_2 - \bar{Y}_1, \cdots, Y_{50} - \bar{Y}_1)$ and $(Y_{51} - \bar{Y}_2, Y_{52} - \bar{Y}_2, \cdots, Y_{100} - \bar{Y}_2)$ are independent as are

$$\tilde{\rho}_{w,1} = \frac{\sum_{t=2}^{50}(Y_t - \bar{Y}_1)(Y_{t-1} - \bar{Y}_1)}{\sum_{t=2}^{49}(Y_t - \bar{Y}_1)^2 + \frac{1}{50}\sum_{t=1}^{50}(Y_t - \bar{Y}_1)^2}$$

and

$$\tilde{\rho}_{w,2} = \frac{\sum_{t=52}^{100}(Y_t - \bar{Y}_2)(Y_{t-1} - \bar{Y}_2)}{\sum_{t=52}^{99}(Y_t - \bar{Y}_2)^2 + \frac{1}{50}\sum_{t=51}^{100}(Y_t - \bar{Y}_2)^2}.$$

Now let's think about $\tau_{w,1}$ and $\tau_{w,2}$. We also know that $\tau_{w,1}$ and $\tau_{w,2}$ are independent since, by definition,

$$\tau_{w,1} = \frac{\tilde{\rho}_{w,1} - 1}{\sqrt{\tilde{\sigma}_{w,1}^2\{\sum_{t=2}^{49}(Y_t - \bar{Y}_1)^2 + \frac{1}{50}\sum_{t=1}^{50}(Y_t - \bar{Y}_1)^2\}^{-1}}}$$

and

$$\tau_{w,2} = \frac{\tilde{\rho}_{w,2} - 1}{\sqrt{\tilde{\sigma}_{w,2}^2\{\sum_{t=52}^{99}(Y_t - \bar{Y}_2)^2 + \frac{1}{50}\sum_{t=51}^{100}(Y_t - \bar{Y}_2)^2\}^{-1}}}$$

where

$$\tilde{\sigma}_{w,1}^2 = \frac{1}{48}[\sum_{t=2}^{50} w_t\{(Y_t - \bar{Y}_1) - \tilde{\rho}_{w,1}(Y_{t-1} - \bar{Y}_1)\}^2$$

$$
\begin{aligned}
&+\sum_{t=1}^{49}(1-w_{t+1})\{(Y_t-\bar{Y}_1)-\tilde{\rho}_{w,1}(Y_{t+1}-\bar{Y}_1)\}^2] \\
= &\ \frac{1}{48}[\sum_{t=2}^{50}\{(Y_t-\bar{Y}_1)-\tilde{\rho}_{w,1}(Y_{t-1}-\bar{Y}_1)\}^2 \\
&+(1-\tilde{\rho}_{w,1}^2)\{(Y_1-\bar{Y}_1)^2-\frac{1}{50}\sum_{t=1}^{50}(Y_t-\bar{Y}_1)^2\}]
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\sigma}_{w,2}^2 = &\ \frac{1}{48}[\sum_{t=52}^{100} w_t\{(Y_t-\bar{Y}_2)-\tilde{\rho}_{w,2}(Y_{t-1}-\bar{Y}_2)\}^2 \\
&+\sum_{t=51}^{99}(1-w_{t+1})\{(Y_t-\bar{Y}_2)-\tilde{\rho}_{w,2}(Y_{t+1}-\bar{Y}_2)\}^2] \\
= &\ \frac{1}{48}[\sum_{t=52}^{100}\{(Y_t-\bar{Y}_2)-\tilde{\rho}_{w,2}(Y_{t-1}-\bar{Y}_2)\}^2 \\
&+(1-\tilde{\rho}_{w,2}^2)\{(Y_{51}-\bar{Y}_2)^2-\frac{1}{50}\sum_{t=51}^{100}(Y_t-\bar{Y}_2)^2\}].
\end{aligned}
$$

Our example so far has $n = 100$. We can now generalize to n observations and split into 2 groups of $\frac{n}{2}$ observations each.

By Theorem 10.1.8, Fuller, $\tau_{w,1}$ and $\tau_{w,2}$ are independent with common limiting distribution

$$\frac{0.5(T^2-1)-TH-G+2H^2}{\sqrt{G-H^2}}$$

since we showed in Huh and Dickey (1999) that a fixed level shift does not affect the limiting distribution of τ_w. Recall that we showed that the asymptotic distributions of the tests based on the symmetric estimators under $\rho = 1$ are unaffected by a structural break of fixed size, as suggested for the tests based on the OLS estimator by Amsler and Lee (1995).

Therefore

$$
\begin{aligned}
&P(\tau_w^* \le x) \\
= &\ 1-P(\min(\tau_{w,1},\tau_{w,2})>x)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - P(\tau_{w,1} > x, \tau_{w,2} > x) \\
&= 1 - P(\tau_{w,1} > x)P(\tau_{w,2} > x) \\
&= 1 - \{1 - P(\tau_{w,1} \leq x)\}\{1 - P(\tau_{w,2} \leq x)\} \\
&\overset{n\to\infty}{\to} 1 - \{1 - P(X \leq x)\}\{1 - P(X \leq x)\} \\
&= 2P(X \leq x) - \{P(X \leq x)\}^2
\end{aligned}
$$

where

$$X \sim \frac{0.5(T^2 - 1) - TH - G + 2H^2}{\sqrt{G - H^2}}.$$

Here

$$
\begin{aligned}
G &= \int_0^1 W^2(t)dt, \\
H &= \int_0^1 W(t)dt \\
\text{and} \qquad T &= W(1)
\end{aligned}
$$

where $W(t)$ is the standard Wiener process.

References

Amsler, C. and Lee, J. (1995), An LM test for a unit root in the presence of a structural change. Econometric Theory 11, 359-368.

Dickey, D. A. and Fuller, W. A. (1979), Distribution of the estimators for autoregressive time series with a unit root. Journal of the American Statistical Association 74, 427-431.

Fuller, W. A. (1996), *Introduction to Statistical Time Series.* Wiley, New York, NY.

Huh, S. and Dickey, D. A. (1999), Comparison of break induced size distortions among unit root test statistics. Unpublished manuscript, North Carolina State University, Raleigh, NC.

Leybourne, S. J., Mills, T. C. and Newbold, P. (1998), Spurious rejections by Dickey-Fuller tests in the presence of a break under the null. Journal of Econometrics 87, 191-203.

Nelson, C. R. and Plosser, C. I. (1982), Trends and random walks in macroeconomic time series: Some evidence and implications. Journal of Monetary Economics 10, 139-162.

Pantula, S. G., Gonzalez-Farias, G. and Fuller, W. A. (1994), A comparison of unit-root test criteria. Journal of Business & Economic Statistics 12, 449-459.

Perron, P. (1989), The great crash, the oil price shock, and the unit root hypothesis. Econometrica 57, 1361-1401.

Perron, P. (1997), Further evidence on breaking trend functions in macroeconomic variables. Journal of Econometrics 80, 355-385.

Vogelsang, T. J. and Perron, P. (1998), Additional tests for a unit root allowing for a break in the trend function at an unknown time. International Economic Review 39, 1073-1100.

Zivot, E. and Andrews, D. W. K. (1992), Further evidence on the great crash, the oil price shock, and the unit root hypothesis. Journal of Business & Economic Statistics 10, 251-270.