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ABSTRACT. Conditions on the boundary and parameters that produce ordering in the first passage time distributions of two different diffusion processes are proved making use of comparison theorems for stochastic differential equations. Three applications of interest in stochastic modeling are presented: a sensitivity analysis for diffusion models characterized by means of first passage times, the comparison of different diffusion models where first passage times represent an important feature and the determination of upper and lower bounds for first passage time distributions.

Keywords: diffusion process, first passage time, comparison theorem, Feller process, Ornstein-Uhlenbeck process

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1. INTRODUCTION

The study of the first passage time through boundaries for one-dimensional diffusion processes is important in a variety of problems ranging from biology to finance, statistics, reliability and queuing theory (e.g., Ricciardi and Sato, 1990; Smith, 1992; Ball and Roma, 1993; Di Crescenzo and Nobile, 1995). Since it is seldom possible to obtain analytical results, numerical or simulation approaches are primarily used in these application areas to determine features of the models (cf. Giorno et al., 1989, Giraudo and Sacerdote, 1999 and Giraudo et al. 2001). Alternatively, techniques for obtaining the asymptotic behavior of first passage time (FPT) distributions are available (cf. Nobile et al., 1985), and in several instances at least the few first moments of the FPT are known in analytical form (cf. Sato, 1978; Ricciardi and Sacerdote, 1979). Unfortunately, the complicated form of the resulting moment expressions limit their practical use when comparing features of different stochastic models (cf. Lánský et al., 1995) or when studying the dependence from the involved parameters. However, in many instances, it would be helpful to understand the dependence of the model features on the individual parameters used in the stochastic differential equation (SDE), or to determine which form of a SDE gives rise to more realistic features. The utility of the stochastic ordering for FPTs to deal with these problems was illustrated in two recent application papers (Sacerdote and Smith, 2000a and 2000b). The first one compared two neural diffusion models where the FPT represents the interspike times while in the second a sensitivity analysis for the FPT to different parameters in a neural model with reversal potentials was examined in terms of stochastic ordering. However the FPTs are of interest in many other contexts and the same approach used for the neural models can be generalized to examine these different applications as well.

In this paper we consider three classes of problems and investigate under which instances the notion of stochastic ordering is useful for their study. In particular our objectives will be:

a. a sensitivity analysis for diffusion models characterized by means of FPTs;

b. the comparison of different models where FPTs are the significant feature of interest;

c. the determination of suitable upper and lower bounds for FPTs distributions of diffusions when the closed form expressions are not available.

We explicitly remark that, although the comparison of random variables via stochastic ordering is extensively discussed in the literature (cf. and references cited therein), particularly with reference to reliability problems, the use of the comparison theorems for FPTs seems limited to discrete state instances (e.g. Di Crescenzo and Ricciardi, 1996; Shaked and Shanthikumar, 1988; Assaf et al., 1985; Li and Shaked, 1995 and 1995, Lee and Lynch, 1997) or to very specific instances (cf. Di Crescenzo and Ricciardi, 1996 and 2001).

In this paper we make use of well-known comparison theorems between sample paths of the diffusion processes under consideration to determine conditions for ordering FPTs and we apply our ordering results to examples of problems of types a-c. Since the Skorohod comparison theorem (cf. Skorohod, 1965) various improvements to this theorem have been proposed (cf. Yamada, 1973; O'Brien, 1980; Yamada and Ogura, 1981; Gal'čuk and Davis, 1982; Huang, 1984; Ikeda and Watanabe, 1989; Ouknine, 1990) primarily aiming to relax some of the hypotheses introduced in the original version. The equality of the diffusion terms of the involved processes is a common hypothesis in these theorems. As discussed in Hajek (1983), equalizing diffusion terms makes the method cumbersome when the final objective is the comparison of the processes. However in this paper we wish to compare FPTs and in this case equalizing the diffusion terms with suitable space transformations has not important consequences. Furthermore the martingale approach proposed in Hajek (1983) to dominate a process by an other with larger diffusion gives results in the sense of convex order that has not immediate implications on FPTs comparisons. Therefore here we limit ourselves to report in Section 2 the version due to Yamada (cf. Yamada, 1973) for diffusion processes on the real line and the version due to Nakao (cf. Nakao, 1973) for a SDE with reflecting boundary that will be used in Section 3 to prove theorems about FPTs. We explicitly note that comparison theorems for SDEs have been mainly used to prove existence and uniqueness theorems for solutions of SDE while applications seems to focus on optimal control problems (cf. Huang, 1984; Ikeda and Watanabe, 1989 and references quoted therein). The use of comparison theorems to determine ordering between FPTs seems to be a useful application not yet sufficiently investigated.

In Section 2 we briefly introduce the necessary mathematical background on stochastic ordering. We limit ourselves to results and notations needed in this paper, while referring to Shaked and Shanthikumar, 1994 for a complete description.

Section 3 is devoted to prove sufficient conditions that guarantee suitable ordering between FPTs corresponding to diffusion processes. A corollary enables us to pinpoint how different diffusion processes can give rise to FPTs with the same distribution for particular choices of the parameters values. This result seems of particular interest when statistical estimation is made using FPTs measures.

Finally in Section 4 we present some applications that make use of the results of Section 3. In particular we show how stochastic ordering can be used to understand the role of individual parameters in complex models, to distinguish FPTs features of two different diffusions and to determine bounds for FPTs distributions..

2. Mathematical background

Throughout this paper $(\Omega, \mathcal{A}, P)$ will be the underlying probability space and we assume that it is large enough to hold the processes we will discuss. We consider two time-homogeneous diffusion processes $X_1(t), X_2(t)$ originating at $X_1(0) = X_2(0) = x_0$, crossing a boundary $S > x_0$, and characterized by drift and diffusion coefficient $\mu_i(x)$ and $\sigma_i^2(x), i = 1, 2$. The SDE can be written in the usual way as:

$$dX_i(t) = \mu_i(X_i(t))dt + \sigma_i(X_i(t))dW(t) \quad i = 1, 2 \tag{2.1}$$

where $W(t)$ indicates a standard Wiener process. Let $\Re$ be the diffusion interval if not differently specified. We choose x_0 and S belonging to the intersection of the two diffusion intervals and we indicate with

$$T_{X_i}(S|x_0) = \inf\{t > 0 : X_i(t) \geq S;\ X_i(t_0) = x_0\} \tag{2.2}$$

the first passage time random variable of $X_i(t)$ through S.

For the specified processes, if pathwise uniqueness of solutions holds, Yamada, 1973 proved the following version of Skorohod theorem:

Theorem 1. *(Skorohod-Yamada) Let $\sigma_1(x) = \sigma_2(x)$ and $\mu_1(x) \leq \mu_2(x), x \in \Re$ then, for every $t \in (0, \infty)$, one has*

$$X_1(t) \overset{wp1}{\leq} X_2(t) \tag{2.3}$$

where $\overset{wp1}{\leq}$ indicates that the inequality holds with probability one.

Remark 1. *Yamada considered non-homogeneous diffusion processes but we will limit ourselves to homogeneous ones to avoid more complex* *notations. However our results can be extended to non-homogeneous processes when uniqueness of the solution is guaranteed.*

Nakao (cf. Nakao, 1973) extends (2.3) to Skorohod's SDEs with reflection

$$dX_i(t) = \mu_i(X_i(t))dt + \sigma_i(X_i(t))dW(t) + d\phi_t, \qquad i = 1, 2 \tag{2.4}$$

where

$$\phi_t = \int_0^t I_{\{0\}}(X_i(s))\, d\phi_s \tag{2.5}$$

and $I_{\{0\}}(\cdot)$ is the indicator function of the set $\{0\}$.

Theorem 2. *(Nakao) Consider eq.(2.4) and let $X_1(0) \le X_2(0)$ a.s. If*

(i) $\sigma_1(x) = \sigma_2(x) = \sigma(x)$, $\mu_i(x), i = 1, 2$ are Borel measurable on $\Re$

(ii) $\sigma(x)$ is of bounded variation on any compact interval

(iii) there exists a constant $c > 0$ such that $\sigma(x) \ge c$, $x \in \Re$.

If $\mu_1(x) \le \mu_2(x)$ almost everywhere, then, with probability one $X_1(t) \le X_2(t)$ for $t \ge 0$.

As a further comment we note that these theorems deal with the almost sure ordering notion between two stochastic processes driven by the same Wiener process. However if we consider the two SDEs as driven by two independent Wiener processes, then the theorem implies the existence of a stochastic ordering between the two processes. We recall that:

Definition 1. *Two random variables Y and Z such that*

$$P(Y > u) \leq P(Z > u), \forall u \in \Re \tag{2.6}$$

are ordered in the (usual) stochastic order and we indicate that Y is smaller than Z in this order by writing

$$Y \leq^{st} Z. \tag{2.7}$$

It can be proven (cf. Shaked and Shathikumar, 1994) that almost sure order implies stochastic order, but the reverse implication is not true.

As far as the ordering of FPTs of diffusion processes is concerned, the known results can be summarized by means of the following lemmas that will be used in the next section.

Lemma 1. *Consider the FPTs of any diffusion process X, originating at x_0, through the boundaries S' and S, with $S' \geq S > x_0$. It holds :*

$$T_X(S\,|x_0) \overset{wp1}{\leq} T_X(S'\,|x_0) \tag{2.8}$$

Lemma 2. *Consider the FPTs through S of the diffusion processes Y_1, Y_2 characterized by equal drift and diffusion coefficients but originating with probability one at two points $y_1 < y_2 < S$ respectively. One has:*

$$T_{Y_1}(S\,|y_1) \overset{wp1}{\geq} T_{Y_2}(S\,|y_2). \tag{2.9}$$

Lemma 3. *Let $T_X(S\,|x_0)$ and $T_{X^*}(S\,|x_0)$ be the FPTs of two processes obeying the same SDE with the same initial condition through* *the same boundary S. Suppose the process X is defined on (α, β) while X^* is constrained by a reflecting boundary at $a \in (\alpha, x_0)$. It holds:*

$$T_X(S\,|x_0) \overset{wp1}{\geq} T_{X^*}(S\,|x_0). \tag{2.10}$$

3. Sufficient conditions for FPTs stochastic ordering

In this Section we determine suitable conditions on the coefficients of (2.1) to prove almost sure or stochastic ordering for the corresponding FPTs through assigned boundaries.

In order to pursue this goal we introduce for each process X_i, verifying (2.1) or (2.4), a space transformation

$$y = g_i(x) = \int_{\alpha_i}^{x} \frac{dz}{\sigma_i(z)}, \qquad i = 1, 2 \tag{3.1}$$

with α_i lower bound of the diffusion interval, that changes the process X_i into a process Y_i having unit infinitesimal variance and with drift (cf. Karlin and Taylor, 1981)

$$\mu_{Y_i}(y) = \frac{1}{\sigma_i(x)} \left(-\frac{1}{4} \frac{d\sigma_i^2}{dx} + \mu_i(x) \right) \Bigg|_{x = g_i^{-1}(y)} . \tag{3.2}$$

We can now prove the following

Lemma 4. *Consider the process Y_2, let $g_i(x), i = 1, 2$ be defined from (3.1) with $X_i(t)$ solution of (2.1) on $\Re$, and suppose:*

$$\sigma_1^2(x) \geq \sigma_2^2(x)\,, \qquad x \leq S \tag{3.3}$$

and

$$\mu_{Y_2}(y) \geq \mu_{Y_2}(y + (g_2(x_0) - g_1(x_0))), \quad y \leq g_2(S) - (g_2(x_0) - g_1(x_0)) \tag{3.4}$$

then it holds:

$$T_{Y_2}(g_2(S)\,|g_2(x_0)) \overset{wp1}{\geq} T_{Y_2}(g_1(S)\,|g_1(x_0)). \tag{3.5}$$

Proof. From (3.1), making use of the hypothesis (3.3) one has

$$g_1(x) \leq g_2(x), \qquad x \leq S. \tag{3.6}$$

Consider now the points $g_1(S), g_1(x_0), g_2(S), g_2(x_0)$, due to (3.6) they can be ordered in two different ways:

a. $g_1(x_0) \leq g_2(x_0) \leq g_1(S) \leq g_2(S)$

b. $g_1(x_0) \leq g_1(S) \leq g_2(x_0) \leq g_2(S)$,

depending upon $g_2(x_0) \leq g_1(S)$. We discuss here explicitly case a., since case b. can be studied in a similar way. Consider the process Y_2^*:

$$Y_2^* = Y_2 - (g_2(x_0) - g_1(x_0)), \tag{3.7}$$

its drift is:

$$\mu_{Y_2^*}(y) = \mu_{Y_2}(y + (g_2(x_0) - g_1(x_0))) \tag{3.8}$$

and due to (3.4), it holds:

$$\mu_{Y_2}(y) \geq \mu_{Y_2^*}(y) \qquad y \leq g_2(S) - (g_2(x_0) - g_1(x_0)). \tag{3.9}$$

From the Skorohod-Yamada comparison theorem we have:

$$Y_2 \overset{wp1}{\geq} Y_2^* \tag{3.10}$$

and hence, it holds:

$$T_{Y_2^*}(g_1(S)\,|g_1(x_0))\overset{wp1}{\geq}T_{Y_2}(g_1(S)\,|g_1(x_0)). \tag{3.11}$$

Let us now remark that due to (3.3) and (3.1) we have $\int_{x_0}^{S}\frac{dz}{\sigma_2(z)}\equiv(g_2(S)-g_2(x_0))\geq(g_1(S)-g_1(x_0))\equiv\int_{x_0}^{S}\frac{dz}{\sigma_1(z)}$, hence, making use of Lemma 1:

$$\begin{aligned}T_{Y_2^*}(g_2(S)-(g_2(x_0)-g_1(x_0))\,|g_1(x_0)) \\ = T_{Y_2}(g_2(S)\,|g_2(x_0))\overset{wp1}{\geq}T_{Y_2^*}(g_1(S)\,|g_1(x_0))\end{aligned} \tag{3.12}$$

and the thesis follows. ∎

Let now be (α_i,β_i) be the diffusion intervals of the processes $X_i(t)$, $i=1,2$ verifying eq. (2.4). Note that if $g_1(\alpha_1)=g_2(\alpha_2)$ we can assume $g(\alpha_i)=0, i=1,2$ since this condition can always be fulfilled by means of a shift on the process $Y_i(t)$. It holds the following:

Lemma 5. *Consider the process Y_2, let $g_i(x), i=1,2$ be defined from (3.1) with $X_i(t)$ solution of (2.4) on (α_i,β_i), let $g_1(\alpha_1)=g_2(\alpha_2)=0$. If $Y_2(t)$ verifies the hypotheses of Nakao theorem and conditions (3.3),(3.4) than (3.5) holds.*

Proof. In analogy with Lemma 4 we introduce the process $Y_2^*(t)$,defined by (3.7) on $(-(g_2(x_0)-g_1(x_0)), g_2(\beta_2)-(g_2(x_0)-g_1(x_0)))$. Further consider the process $Y_2^{**}(t)$, obtained from $Y_2^*(t)$ by introducing a reflecting boundary in zero. The drifts of $Y_2^{**}(t)$ and $Y_2^*(t)$ coincides and it holds

$$\mu_{Y_2}(y) \geq \mu_{Y_2^{**}}(y), \qquad y \in (0, g_2(S) - (g_2(x_0) - g_1(x_0))). \tag{3.13}$$

and hence, making use of Nakao theorem we have $Y_2 \geq Y_2^{**}$ and $T_{Y_2^{**}}(g_1(S)|g_1(x_0)) \geq T_{Y_2}(g_1(S)|g_1(x_0))$. From Lemma 3 we have $T_{Y_2^*}(g_1(S)|g_1(x_0)) \geq T_{Y_2^{**}}(g_1(S)|g_1(x_0))$ and hence (3.11) still holds. The proof can then be carried on in perfectly analogy with the proof of Lemma 4. ∎

Remark 2. *If $\mu_{Y_2}(y)$ is differentiable and if*

$$\frac{d\mu_{Y_2}(y)}{dy} \leq 0, \quad y \in [g_2(\alpha_2), g_2(S) - (g_1(S) - g_1(x_0))]. \tag{3.14}$$

then hypothesis (3.4) is verified. Recalling that the drift coefficient is strictly related to the process speed, with higher speed when the drift increases, we can give an heuristic interpretation of our hypothesis (3.4) in the last two Lemmas. Indeed if (3.14) holds the process evolves slower as y increases and the time to cross an interval of fixed amplitude results larger if the process starting point is larger. Hence for $\delta > 0$, one has

$$T_{Y_2}(S|x_0) \leq T_{Y_2}(S+\delta|x_0+\delta). \tag{3.15}$$

Note that Skorohod-Yamada theorem refers to processes with diffusion interval $\Re$ while Nakao theorem can be applied to diffusions on (α, β) but the case of infinite diffusion interval corresponds to $\alpha = -\infty, \beta = \infty$. Hence in the sequel we refer to Skorohod-Yamada-Nakao theorem, which version is in use will be clear from the context.

We can now prove the following

Theorem 3. *Let $g_1(\alpha_1) = g_2(\alpha_2)$ and let the processes $Y_i(t), i = 1,2$ verify hypotheses of Skorohod-Yamada-Nakao theorem. If the drifts (3.2) satisfy the inequalities:*

$$\mu_{Y_1}(y) \geq \mu_{Y_2}(y) \qquad \forall y \in [g_1(\alpha_1), g_2(S)] \tag{3.16}$$

$$\frac{d\mu_{Y_2}(y)}{dy} \leq 0 \qquad \forall y \in [g_1(\alpha_1), g_2(S)] \tag{3.17}$$

and the infinitesimal variances are such that

$$\sigma_1^2(x) \geq \sigma_2^2(x) \tag{3.18}$$

then for any $x_0 \in (\max(\alpha_1, \alpha_2), S)$, $S \in (\max(\alpha_1, \alpha_2), \min(\beta_1, \beta_2))$, $x_0 < S$, one has

$$T_{X_1}(S\,|x_0) \overset{wp1}{\leq} T_{X_2}(S\,|x_0) \tag{3.19}$$

Proof. From the comparison theorem of Skorohod-Yamada-Nakao and the first of our hypotheses about drifts, it holds:

$$Y_1 \overset{wp1}{\geq} Y_2. \tag{3.20}$$

Hence for $\forall t \geq 0$ and $\forall a, b \in [g_1(\alpha_1), g_2(S)]$, $a < b$, we have:

$$T_{Y_1}(b\,|a) \overset{wp1}{\leq} T_{Y_2}(b\,|a). \tag{3.21}$$

Let us choose $a = g_1(x_0), b = g_2(S)$. Observe now that (3.1) is a space transformation, not involving time and is monotone non-decreasing. Hence from (3.21) follows:

$$T_{X_1}(S\,|x_0) = T_{Y_1}(g_1(S)\,|g_1(x_0)) \overset{wp1}{\leq} T_{Y_2}(g_1(S)\,|g_1(x_0)). \tag{3.22}$$

From Lemmas 4 and 5 we obtain:

$$(3.23) \qquad T_{X_1}(S\,|x_0) \overset{wp1}{\leq} T_{Y_2}(g_1(S)\,|g_1(x_0)) \overset{wp1}{\leq} T_{Y_2}(g_2(S)\,|g_2(x_0)) = T_{X_2}(S\,|x_0).$$

∎

Remark 3. *Conditions (3.16) and (3.17) involve properties of the transformed processes and hence have not a direct interpretation on the original process. However they are easy to check by means of simple computations on the assigned processes and are verified by a large class of diffusion processes. For example if $\mu_2(x) = 0$ and $\sigma^2(x)$ is convex and increasing (or decreasing) than from (3.2) follows (3.17). An other example is the process solution of*

$$(3.24) \qquad dX(t) = (aX(t) + b)\,dt + cX(t)\,dW(t)$$

known in the finance modeling as the Black and Scholes model. This process verifies (3.17) if $b > 0$. Furthermore condition (3.18) corresponds to our intuition that when the diffusion coefficient gets larger the process becomes more volatile making easier the boundary crossing.

Corollary 1. *If $g_1(\alpha_1) = g_2(\alpha_2)$ and if the drifts (3.2) satisfy the equality:*

$$(3.25) \qquad \mu_{Y_1}(y) = \mu_{Y_2}(y) \qquad \forall y \in [g_1(\alpha_1), g_2(S)]$$

and the infinitesimal variances are such that (3.18) holds, then for any $y_0 \in [g_1(\alpha_1), g_2(S)]$, $\Sigma \in [g_1(\alpha_1), g_2(S)]$, $y_0 < \Sigma$ one has

$$(3.26) \qquad T_{X_1}(g_1^{-1}(\Sigma)\,|g_1^{-1}(y_0)) = T_{X_2}(g_2^{-1}(\Sigma)\,|g_2^{-1}(y_0)).$$

Proof. The processes $Y_1(t)$ and $Y_2(t)$ have unit infinitesimal variance and equal drift due to (3.25), hence they verify the same SDE and for any $y_0 \in [g_1(\alpha_1), g_2(S)]$, $\Sigma \in [g_1(\alpha_1), g_2(S)]$, $y_0 < \Sigma$ one has $T_{Y_1}(\Sigma\,|y_0) = T_{X_2}(\Sigma\,|y_0)$. Making then use of the inverse transformations $g_1^{-1}(y)$ and $g_2^{-1}(y)$ the thesis (3.26) follows for the FPTs of the original processes. ∎

Note that the only property of transformations (3.1) used in the proof of Theorem 3 is its monotonicity due to hypothesis (3.18). Hence we can generalize this theorem to more general transformations than 3.1:

Theorem 4. *Let $\gamma_1(x)$ and $\gamma_2(x)$ be two monotone non-decreasing transformations changing the processes X_1 and X_2 into two processes Y_1' and Y_2' respectively, characterized by equal infinitesimal variance, verifying the hypotheses of Skorohod-Yamada-Nakao theorem and such that*

$$\begin{aligned} \gamma_1(x_0) &\leq \gamma_2(x_0) \\ \gamma_1(S) &\leq \gamma_2(S) \\ \gamma_1(S) - \gamma_1(x_0) &\leq \gamma_2(S) - \gamma_2(x_0). \end{aligned} \tag{3.27}$$

If $\gamma_1(\alpha_1) = \gamma_2(\alpha_2)$ and if the drifts (3.2) verify the inequalities:

$$\mu_{Y_1'}(y) \geq \mu_{Y_2'}(y) \qquad \forall y \in [\gamma_1(\alpha_1), \gamma_2(S)] \tag{3.28}$$

$$\frac{d\mu_{Y_2'}(y)}{dy} \leq 0 \qquad \forall y \in [\gamma_1(\alpha_1), \gamma_2(S)] \tag{3.29}$$

then for any $x_0 \in (\max(\alpha_1,\alpha_2), S)$, $S \in (\max(\alpha_1,\alpha_2), \min(\beta_1,\beta_2))$, $x_0 < S$ *one has*

$$T_{X_1}(S\,|x_0) \overset{wp1}{\leq} T_{X_2}(S\,|x_0). \tag{3.30}$$

Proof. From the considered transformations, the drifts and infinitesimal variances of the processes X_i are transformed into (cf. Karlin and Taylor, 1981)

$$\mu'_{Y'_i}(y) = \left(\frac{1}{2}\sigma_i^2(x)\frac{d^2\gamma_i(x)}{dx^2} + \mu_i(x)\frac{d\gamma_i(x)}{dx}\right)\Bigg|_{x=\gamma_i^{-1}(y)} \tag{3.31}$$

and

$$\sigma_1^2(y) = \sigma_2^2(y) \tag{3.32}$$

for the processes $Y'_i, i = 1, 2$. Here

$$\sigma^2_{Y'_i}(y) = \sigma^2_{X_i}(x)\left[\frac{d\gamma_i(x)}{dx}\right]^2\Bigg|_{x=\gamma_i^{-1}(y)}. \tag{3.33}$$

One then can proceed as in the proof of Theorem 3. ∎

Corollary 2. *Consider the transformations* $\gamma_i(x), i = 1, 2$ *defined in the Theorem 4. If* $\gamma_1(\alpha_1) = \gamma_2(\alpha_2)$ *and if the drifts (3.2) satisfy the equality:*

$$\mu_{Y'_1}(y) = \mu_{Y'_2}(y) \qquad \forall y \in [\gamma_1(\alpha_1), \gamma_2(S)] \tag{3.34}$$

then for any $x_0 \in (\max(\alpha_1,\alpha_2), S)$, $S \in (\max(\alpha_1,\alpha_2), \min(\beta_1,\beta_2))$, $x_0 < S$ *one has:*

$$T_{X_1}(\gamma_1^{-1}(S)\,|\gamma_1^{-1}(x_0)) = T_{X_2}(\gamma_2^{-1}(S)\,|\gamma_2^{-1}(x_0)). \tag{3.35}$$

Proof. It is an immediate consequence of the equality of FPTs of the transformed processes. ∎

Remark 4. *If $g_2(\alpha_2) < g_1(\alpha_1)$ one can introduce a reflecting boundary to constrain the processes X_2 on a suitable interval in order that for the new transformed process Y_2^*, $g_2(\alpha_2) = g_1(\alpha_1)$. We can then apply the theorem 4 to this new process and finally extend the result to the original process by noting that, as proved in Lemma 3 the introduction of a reflecting boundary can only decrease the FPT. The case $g_2(\alpha_2) > g_1(\alpha_1)$ can be studied in a similar way by introducing a reflecting boundary to constrain the processes X_1 on a suitable interval in order to obtain $g_2(\alpha_2) = g_1(\alpha_1)$. However, in this case the reflecting boundary cannot be removed to obtain an order relationship for the original processes, since in this case the inequality does not hold if we increase the FPT. An analogous procedure can be carried out when the transformations γ_i are used.*

Remark 5. *If stochastic ordering with probability 1 holds between FPTs and the two FPTs means are equal then (cf. Shaked and Shanthikumar, 1994) the two random variables are equal. Note that, if $g_2(\alpha_2) = g_1(\alpha_1)$ the only instance this happens corresponds to Corollaries 1 and 2, but if $g_2(\alpha_2) < g_1(\alpha_1)$ it could be verified in other instances.*

Remark 6. *If in equations (2.1) we introduce independent Wiener processes for each $X_i(t)$, it is easy to verify that Lemmas 1 and 3 and*

Theorems 3 and 4 keep their validity provided the usual stochastic order is substituted for the order with probability one in their theses.

Remark 7. *Conditions of Theorems 3 and 4 are sufficient conditions. FPTs can be ordered under less restrictive conditions than those of the Theorems. Indeed, since ordering of FPTs is connected with the ordering of the processes' sample paths in any strip* $[S-\varepsilon, S]$ *with* $\varepsilon > 0$ *arbitrary, necessary conditions should deal with sample path ordering on this strip. However comparison theorems for process sample paths hold on the entire diffusion interval and it seems difficult to determine general conditions for sample path comparisons on strips.*

4. Applications

Here we make use of ordering among FPTs to pursue our 3 different objectives. First we show how to use of Theorems 2-3 can help to investigate the dependence of FPTs distributions on the process parameters. In a second example, we compare different diffusion processes that could be used to model the same phenomenon (cf. Lánský et al, 1995) and we recognize the different features that determine an order between the respective FPTs. Finally we consider an example where an unknown FPT distribution admits, as lower and upper bounds, two FPTs distributions that are known in closed form.

Sensitivity analysis for a Feller process. Let us consider a Feller process, i.e. a diffusion process on $(0,\infty)$ characterized by linear drift and linear diffusion coefficient

$$\mu(x) = px + q \qquad (4.1)$$
$$\sigma^2(x) = 2rx.$$

This process is of interest in various applications such as population growth, population genetics, neural models and is often studied as an example of a multiplicative noise model. Note that a discussion similar to that concerning this process could be repeated for the Black and Sholes model in finance.

Different boundary conditions can be considered at 0 (cf. Karlin and Taylor, 1981) we use here a reflecting boundary at 0. We explicitly note that the FPT distribution through $S > 0$ is not known analytically but numerical computations can be found for example in Lánský et al, 1995. Expressions for the mean and variance of FPTs have been determined (cf. Giorno et al., 1988), but they are very complex and it is hard to ascertain the dependences on the parameters. In order to determine the dependence of the FPT distribution on the parameters p, q, r we compare the FPTs of two Feller processes with parameters p_i, q_i, r_i, $i = 1, 2$ originated at $x_0 \geq 0$ through a boundary $S > x_0$. A typical study of interest for this model is the study of FPTs dependency upon the noise level r_i since this is generally a quantity characterizing different environments. Intuitively one might expect that a larger noise intensity ($r_1 > r_2$) would imply that X_1 has a faster FPT. However, as it can be shown by using the Theorems of Section 3, this does not need to be true.

In order to deal with this problem we consider the processes $Y_i, i = 1, 2$ obtained from the processes $X_i, i = 1, 2$ via transformations:

$$\begin{array}{l} \gamma_1(x) = x \\ \gamma_2(x) = \frac{r_1}{r_2} x. \end{array} \tag{4.2}$$

The corresponding infinitesimal moments are:

$$\begin{array}{ll} \mu_{Y_1}(y) = p_1 y + q_1 & \sigma^2_{Y_1}(y) = 2 r_1 y \\ \mu_{Y_2}(y) = p_2 y + q_2 \frac{r_1}{r_2} & \sigma^2_{Y_2}(y) = 2 r_1 y. \end{array} \tag{4.3}$$

Hence if $r_1 \geq r_2$ and one of the instances a, b, c considered in Table I are verified than the hypotheses of Theorem 4 are fulfilled and $T_{X_1} \overset{wp1}{\leq} T_{X_2}$ holds. In this Table and in the following one x refers to allowable values for the diffusion interval.

Interchanging the role of the variables X_1 and X_2 one immediately verifies that hypotheses of Theorem 4 are satisfied in the three cases a, b, c considered in Table II when $r_2 \geq r_1$. In these instances we can conclude $T_{X_2} \overset{wp1}{\leq} T_{X_1}$.

The cases considered in the second lines of Tables I and II require the moving of the reflecting boundary from 0 to $q_2 - q_1 \frac{r_2}{r_1}$ and $q_1 - q_2 \frac{r_1}{r_2}$ respectively. Similarly the cases considered in the third lines correspond to an absorbing boundary S that respects the constraints on x.

Hence, as the Tables suggest, an increase of the noise intensity does not guarantee a faster FPT. In Fig.1 a numerical example illustrates an instance where ordering does not occur, even in the weaker usual stochastic order, despite the increase of the noise. Note that the considered case does not fit the sufficient conditions of the Tables. Indeed here we choose $r_1 = 1, r_2 = 0.2, p_1 = -0.1, p_2 = -0.2, q_1 = q_2 = 1, S = 2$ which satisfy the condition in the second line of Table I but we fix a reflecting boundary at 0 and $x_0 = 0.05$. Hence the diffusion interval is $(0, 2)$ and the condition on the third column in now not met since $0 < q_2 - q_1 \frac{r_2}{r_1} = 0.98$.

In Fig.2 for $S = 2$ and $x_0 = 1$ we present three Feller processes that are stochastically ordered in the usual sense with respect to a Feller process characterized by $p_1 = -1$ and $q_1 = r_1 = 1$. All these processes are characterized by a smaller value of $r_2 = 0.8$. The FPTs distribution of the process X_1 is illustrated as the second curve (dashed line) from the top in Fig.2. Each of the two lower curves is ordered with regard to T_{X_1} distribution, exemplifying two instances of Table I namely the first and third lines. Hence they are also ordered with probability one but this cannot be illustrated via their cumulative distribution functions. These cases could be considered examples of instances when the intuition about the role of a decrement of the noise intensity can be confirmed. The upper curve considered in Fig. 2 is ordered in the opposite direction, with regard to T_{X_1} in the usual sense, but in this instance the intuition suggests a reverse order. However we cannot establish the order between the random variables making use of Table II. Indeed the parameters values satisfy conditions of the third line of the Table but the boundary and the initial value are not in the prescribed diffusion interval. This example proves that ordering relationships can hold for a wider range of parameter values than that determined via the Tables.

Furthermore in looking at the Tables it is interesting to note the dependence of the FPTs order on the ratios $\frac{q_i}{r_i}$.

An analogous analysis can be performed with respect to the other parameters p and q, determining suitable instances where an order can be recognized. However we do not focus here on a detailed analysis which could easily be performed following the above mentioned method instead we direct our attention to the use of the Corollary for determining instances where two Feller processes are characterized by the same FPT distribution.

Indeed if $p_1 = p_2$ and $\frac{q_1}{r_1} = \frac{q_2}{r_2}$ from Corollary 2 we have:

$$T_{X_1}\left(\frac{r_1 S}{r_2}\,\middle|\,\frac{r_1 x_0}{r_2}\right) = T_{X_2}(S\,|x_0). \tag{4.4}$$

This result has implication for the statistical estimation of parameters namely it proves the nonidentifiability of the model via FPTs observations.

Comparison of Ornstein-Uhlenbeck and Feller FPTs. Let us now compare FPTs of a Feller process (4.1) with the analogous FPT of an Ornstein-Uhlenbeck process, having infinitesimal moments:

$$\mu_2(x) = \beta x; \qquad \sigma_2^2(x) = \sigma^2. \tag{4.5}$$

where $\beta < 0$ and $\sigma^2 > 0$. These two models are considered in the literature as alternative models in a neurobiological context. The use of the

Feller model is often suggested as an improvement of the Ornstein-Uhlenbeck one since it has a lower bound for its diffusion interval which seems more realistic when modeling the membrane potential (cf. Lánský et al, 1995, Sacerdote and Smith, 2000). This situation may arise in other application areas as well, where the existence of a natural lower bound for the diffusion interval could suggest a shift from the Ornstein-Uhlenbeck to the Feller model. We explicitly underline how both the models correspond to a quadratic potential well but the Ornstein-Uhlenbeck one is characterized by additive noise while the Feller one by multiplicative noise. Hence it appears useful to try to determine differences and similarities between the two models by means of ordering relationships and also shedding some light on the different roles of multiplicative versus additive noise.

Let the two processes X_1(4.1), and X_2 (4.5) both originate at x_0 and consider a boundary S. Here the two diffusion intervals do not coincide since the Ornstein-Uhlenbeck is defined on $(-\infty, +\infty)$. However one might wish to compare the FPTs of the two processes and this can be done if x_0 and S are positive, even though some new difficulty arises when we make use of Theorems 3 and 4.

To deal with this comparison, we first make use of Theorem 3 and we establish order relationships between the FPTs of the two processes for certain choices of parameter values.

Let

$$g_1(x) = \sigma \sqrt[2]{\frac{2x}{r}} \tag{4.6}$$

$$g_2(x) = x. \tag{4.7}$$

be two transformations on the processes X_1 and X_2 respectively.

The processes X_1 and X_2 are transformed via (4.6) and (4.7) into the processes Y_1 and Y_2 characterized by infinitesimal variance

$$\sigma^2_{Y_1}(y) = \sigma^2_{Y_2}(y) = \sigma^2 \tag{4.8}$$

and drifts

$$\mu_{Y_1}(y) = \frac{p}{2}y + \frac{\sigma^2}{y}\left(\frac{q}{r} - \frac{1}{2}\right) \tag{4.9}$$

$$\mu_{Y_2}(y) = \beta y. \tag{4.10}$$

The hypotheses (3.27) are verified for transformations (4.6) and (4.7) if $x_0 \geq \frac{\sigma^2}{2r}$. The hypotheses of Theorem 3 are verified in each of the following instances if a reflecting boundary at 0 is introduced for the Ornstein-Uhlenbeck process and $\beta < 0$

$$\begin{array}{llll} a. \; \frac{q}{r} \geq \frac{1}{2}; & \frac{p}{2} \geq \beta & if \;\; x \geq 0 \\ b. \; \frac{q}{r} \geq \frac{1}{2}; & \beta > \frac{p}{2} & if \;\; x < \frac{r}{2}\sqrt[2]{\frac{\frac{q}{r} - \frac{1}{2}}{\beta - \frac{p}{2}}} \\ c. \; \frac{q}{r} \leq \frac{1}{2}; & \beta < \frac{p}{2} & if \;\; x > \frac{r}{2}\sqrt[2]{\frac{\frac{1}{2} - \frac{q}{r}}{\frac{p}{2} - \beta}}. \end{array} \tag{4.11}$$

Here the diffusion intervals respect the constraints on x implied by each of the three conditions. In these cases, indicating X_2' the Ornstein-Uhlenbeck process with reflecting boundary at the origin we conclude:

$$T_{X_1}(S\,|x_0) \overset{wp1}{\leq} T_{X_2'}(S\,|x_0) \tag{4.12}$$

From Lemma 3 we can now remove the reflecting boundary and obtain, when (4.11) are verified:

$$T_{X_1}(S\,|x_0) \overset{wp1}{\leq} T_{X_2}(S\,|x_0). \tag{4.13}$$

In Fig. 3 two examples illustrate (4.13) when the usual stochastic order is considered. The first and the second curves from the top correspond to a case where (4.13) holds since condition b in (4.11) is verified. Indeed for the Feller process (curve at the top) we have:

$$p = -3 \; q = 1 \; r = 1 \; S = 0.5 \; x_0 = 0.25 \tag{4.14}$$

while for the Ornstein-Uhlenbeck process (second curve from the bottom) we have:

$$\beta = -1 \; \sigma^2 = 0.5 \; S = 0.5 \; x_0 = 0.25. \tag{4.15}$$

The third and forth curves from the top illustrate a case where (4.13) holds since condition a in (4.11) is satisfied.

Indeed in this case we have:

$$p = -1 \; q = 1 \; r = 1 \; S = 2 \; x_0 = 1 \tag{4.16}$$

for the Feller process (third curve from the top) and

$$\beta = -1 \; \sigma^2 = 2 \; S = 2 \; x_0 = 1 \tag{4.17}$$

for the Ornstein-Uhlenbeck process (lower curve). These results seem to confirm an intuitive idea which attributes faster FPTs to the Feller process, whose diffusion interval is smaller than the analogous one for the Ornstein-Uhlenbeck process.

In order to check the generality of this intuition we interchange now the roles of X_1 and X_2 considering the Ornstein-Uhlenbeck process as process X_1 and the Feller process as process X_2 in Theorem 3 and we check if there exist instances where the order is reversed. Now (3.27) holds for

$$S \leq \frac{\sigma^2}{2r}. \tag{4.18}$$

The hypothesis (3.28) of Theorem 3 is verified in each of the following instances if a reflecting boundary at 0 is introduced for the Ornstein-Uhlenbeck process:

$$\begin{aligned} &a.\ \frac{q}{r} \leq \frac{1}{2}; \quad \frac{p}{2} \leq \beta \quad if\ \ x \geq 0 \\ &b.\ \frac{q}{r} \geq \frac{1}{2}; \quad \beta \geq \frac{p}{2} \quad if\ \ x \geq \frac{r}{2}\sqrt[2]{\frac{\frac{q}{r}-\frac{1}{2}}{\beta-\frac{p}{2}}} \\ &c.\ \frac{q}{r} \leq \frac{1}{2}; \quad \beta \leq \frac{p}{2} \quad if\ \ x \leq \frac{r}{2}\sqrt[2]{\frac{\frac{1}{2}-\frac{q}{r}}{\frac{p}{2}-\beta}} \end{aligned} \tag{4.19}$$

and the diffusion interval respect the constraints on x implied by each of the three conditions while (3.29) is verified for the transformed process Y, if

$$\frac{p}{2} - \frac{\sigma^2}{y^2}\left(\frac{q}{r} - \frac{1}{2}\right) \leq 0. \tag{4.20}$$

The range of y is now on the transformed interval. In these cases if X_2' is the Ornstein-Uhlenbeck process with reflecting boundary at the origin we conclude:

$$T_{X_1}(S\,|x_0) \geq T_{X_2'}(S\,|x_0) \tag{4.21}$$

Unfortunately in this case we cannot use Lemma 3 to remove the reflecting boundary and hence we have not a counterexample for our intuition about the role of the multiplicative noise with respect to the additive one. However in some instances the order relationship holds for the process X_2, at least in the usual stochastic order and, even if we cannot prove it by the use of our theorems, we can determine such a counterexample with the aid of numerical computations . An example of this type is illustrated in Fig. 4 where we compare FPTs distribution of a Feller process with parameters:

$$p = -1 \; q = 0.2 \; r = 0.5 \; S = 2 \; x_0 = 1 \tag{4.22}$$

with the FPTs distribution of an Ornstein-Uhlenbeck process on $(-\infty, \infty)$ with parameters:

$$\beta = -0.5 \; \sigma^2 = 2 \; S = 2 \; x_0 = 1. \tag{4.23}$$

These choices for the parameters of the processes correspond to case a of (4.19), and the order relation (4.21) holds for the Ornstein-Uhlenbeck process constrained by a reflecting boundary at 0. However, as shown in Fig. 4, the order relation is true even for the process without a reflecting boundary.

Lower and upper bounds for FPTs. Closed form expressions for the FPT distribution are known only in rare instances which are not interesting for a direct use in the applications. However, these solutions can be used to obtain upper and lower bounds for the expression of FPTs distribution when the stochastic order between the involved FPTs holds. In order to illustrate an example of bounds obtained by means of ordering relationships let us consider the diffusion process with drift

$$\mu(x) = \beta x + \mu \tag{4.24}$$

and with diffusion coefficient σ^2. This process can be viewed as a generalization of the Ornstein-Uhlenbeck process (4.5) since it can be obtained from it via a shift transformation. The FPT probability density function $f_S(t)$ through the boundary $S = \frac{\mu}{\beta}$ of a process (4.24) originated in x_0 at time $t_0 = 0$ can be proved to be (cf. Ricciardi, 1977):

$$f_S(t) = \frac{2(\mu - \beta x_0)\sqrt{\beta}}{\sqrt{\pi\sigma^2(e^{2\beta t}-1)}(1-e^{-2\beta t})} \exp\left\{-\frac{(\mu - \beta x_0)^2}{\beta\sigma^2(e^{2\beta t}-1)}\right\}. \tag{4.25}$$

We now consider the FPT of a Feller process (4.1), characterized by $p = -1, q = 1.5$ and $r = 0.5$, originated in $x_0 = 0.1$ through the boundary $S = 2$. In this instance, the FPT distribution cannot be determined in closed form. In Fig. 5 we plot the FPT distribution of this process, obtained via the numerical solution of the integral equation proposed in Giorno et al., 1989 and the FPTs distributions corresponding to the two processes (4.24) originated in $x_0 = 0.1$, with $\beta = -1, \mu = 2, \sigma^2 = 1$, $S = 2$ and $\beta = -\frac{1}{4}, \mu = 2$, $\sigma^2 = 1$, $S = 8$, respectively. Fig. 5 shows the existence of an order relationship between the three FPTs distributions.

Since the FPTs distributions of the two Ornstein-Uhlenbeck processes with $\beta = -1$ and $\beta = -\frac{1}{4}$ are known in closed form with the boundaries $S = 2$ and $S = 8$ respectively, the order relationships between these FPTs and the Feller one allow to determine an upper and a lower bound for the unknown Feller FPT distribution.

We conclude by remarking how this new approach to the study of diffusion FPTs by means of order relationships could be useful in other applications and one may wish to wide the range of investigations to FPTs of non-homogeneous diffusion or of jump-diffusion processes.

References

[1] Assaf, F.D., Shaked, M. and Shanthikumar, J., G. *1st passage times with PFR densities.*J. Appl. Prob. 1985; **22**, 185-196.

[2] Ball, C.A. and Roma, A. *A jump diffusion model for european monetary system*, Journal of International Money and Finance 1993; **12**, 475-492

[3] Di Crescenzo, A. and Nobile, A.G. *Diffusion approximation to a queueing system with time-dependent arrival and service rates*, Queueing Systems 1995; **19**, 41-62

[4] Di Crescenzo, A. and Ricciardi L.M. *Comparing failure times via diffusion models and likelihood ratio ordering.* IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences 1996; Vol. E79-A, no.9, 1429-1432.

[5] Di Crescenzo, A. and Ricciardi, L.M. *Comparing first-passage-times for semi-Markov skip-free processes.* Stat. Prob. Lett. 1996; **30**, 247-256.

[6] Di Crescenzo, A. and Ricciardi, L.M. *On a discrimination problem for a class of stochastic processes with ordered first-passage times* Appl. Stoch. Models Bus. Ind. 2001; **17** 17 no. 2, 205–219.

[7] Gal'čuk, L.I. and Davis, M.H.A. *A note on a comparison theorem for equations with different diffusions.* Stochastics 1982; **6**, 147-149.

[8] Giorno, V., Lánský, P., Nobile, A.G. and Ricciardi, L.M. *Diffusion approximation and first-passage-time problem for a model neuron. III. A birth-and-death approach*, Biol. Cybern. 1988; **58**, 387-404.

[9] Giorno, V., Nobile, A.G., Ricciardi, L.M. and Sato, S., *On the evaluation of first-passage-time probability densities via non-singular integral equations*, Adv. Appl. Prob. 1989; **21**, 20-36

[10] Giraudo, M.T. and Sacerdote, L. *An improved technique for the simulation of first passage times for diffusion processes.* Commun. Statist.-Simula. *1999;* **28** (4), 1135-1163

[11] Giraudo, M.T., Sacerdote, L. and Zucca, C. *A MonteCarlo method for the simulation of first passage times of diffusion processes.* Meth. Comp. Appl. Prob. 2001, **3**, 215-231

[12] Hajek, B. *Mean stochastic comparison of diffusions.* Z. Wahrscheinlichkeitstheorie verw. Gebiete 1985, **68**, 315-329.

[13] Huang, Z. Y. *A comparison theorem for solutions of stochastic differential equations and its applications.* Proc. A.M.S. 1984; **91**, n. 4, 611-617.

[14] Ikeda, N. and Watanabe, S. *Stochastic differential equations and diffusion processes.* North Holland Math. Lib., 1989;

[15] Karlin, S. and Taylor, H. M. *A second course in stochastic processe*s Academic Press, 1981.

[16] Lánský, P., Sacerdote, L. and Tomassetti, F. *On the comparison of Feller and Ornstein-Uhlenbeck models for neural activity.* Biol. Cybern. 1995; **73**, 457-465.

[17] Lee, S. and Lynch, J. *Total positivity of Markov chains and the failure rate character of some first passage times.* Adv. Appl. Prob. 1997; **29**, 713-732.

[18] Li, H.J. and Shaked, M. *On the first passage times for Markov-processes with monotone convex transition kernels.* Stoch. Proc. Appl. 1995; **58**, (2) 205-216.

[19] Li, H.J. and Shaked, M. *Ageing first-passage times of Markov processes: a matrix approach.* J. Appl. Prob. 1997; **34**, (1) 1-13.

[20] Nakao, S. *Comparison theorems for solutions of one-dimensional stochastic differential equations.* Proceedings of the second Japan_USSR Symposium on Probability Theory. pp. 310-315 Lecture Notes in Math. Vol. 330, Springer, 1973 .

[21] Nobile, A.G., Ricciardi, L.M. and Sacerdote, L. *Exponential trends of first-passage-time densities for a class of diffusion processes with steady-state distribution*, J. Appl. Prob. 1985; **22**, 611-618 .

[22] O'Brien, G.L. *A new comparison theorem for solutions of stochastic differential equations.* Stochastics 1980; **3**, 245-249.

[23] Ouknine, Y. *Comparison et non-confluence des solutions d'équations differentielles stochastiques unidimensionnelles.* Probab. Math. Statist. 1990; **11**, n. 1, 37-46

[24] Ricciardi, L.M. Diffusion Processes and Related Topics in Biology. Lecture Notes in Biomathematics **14**,Springer Verlag Berlin 1977.

[25] Ricciardi, L.M. and Sacerdote, L. *The Ornstein-Uhlenbeck process as a model for neuronal activity*, Biol. Cybern. 1979; **3 5**, 1-9

[26] Ricciardi, L.M.. and Sato, S., Diffusion processes and first-passage-time problems, Lectures in Applied Mathematics and Informatics, L.M. Ricciardi, Ed. Manchester University Press, Manchester, 1990

[27] Sacerdote, L. and Smith, C.E. *A qualitative comparison of some diffusion models for neural activity via stochastic ordering.* Biol. Cybern. 2000; **83**, 543-551.

[28] Sacerdote, L. and Smith, C.E. *New parameter relationships determined via stochastic ordering for spike activity in a reversal potential neural model* Biosystems 2000, **58,** 59-65

[29] Sato, S., *On the moments of the firing interval of the diffusion approximated model neuron*, Math. Biosci. 1978; **39**, 53-70

[30] Shaked, M. and Shanthikumar, J., G. *On the 1st passage times of pure jump-processes.* J.Appl. Prob. 1988; **20** (2), 427-446.

[31] Shaked, M. and Shanthikumar, J., G. Stochastic orders and their applications. Academic Press, Inc. Boston 1994.

[32] Smith, C.E. *A note on neuronal firing and input variability* J.Theor. Biol. 1992; **154**, 271-275.

[33] Skorohod, A.V. Studies in the theory of random processes. Addison-Wesley Pub. Comp., inc.1965.

[34] Yamada, T. *On a comparison theorem for solutions of stochastic differential equations and its applications.* J. Math. Kyoto Univ. 1973; **13-3**, 497-512.

[35] Yamada, T. and Ogura, Y. *On the strong comparison theorems for solutions of stochastic differential equations.* Z. Wahrsch. Verw. Gebiete 1981; **56**, 3-19.

[36]