ON THE RECONSTRUCTION OF THE COVARIANCE OF STATIONARY GAUSSIAN PROCESSES OBSERVED THROUGH ZERO MEMORY NONLINEARITIES - PART II

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ABSTRACT

We consider the problem of reconstructing the variance R(0) of a zero mean stationary Gaussian process observed through a zero memory nonlinearity f(x), from the knowledge of f and the first two moments of the output process. The reconstruction is shown to be feasible for certain interval windows, convex nonlinearities and discontinuous unimodal nonlinearities. The paper is the completion of the investigation begun in Cambanis and Masry (1978) where the reconstruction of the *normalized* covariance R(t)/R(0) was considered.

KEY WORDS AND PHRASES: RECONSTRUCTION OF STATIONARY GAUSSIAN COVARIANCES, ZERO MEMORY NONLINEARITIES

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1. INTRODUCTION

Let $X = \{X_t, -\infty < t < \infty\}$ be a real stationary Gaussian processes with zero mean and continuous covariance function $R(t)$. We observe the stationary process $Y = \{f(X_t), -\infty < t < \infty\}$, where f is a real Borel measurable function on the real line. We are concerned with the reconstruction of the covariance function $R(t)$, $-\infty < t < \infty$, from the knowledge of the nonlinearity f, the mean

$$\mu_1 = E[f(X_t)],$$

and the correlation function

$$C(t) = E[f(X_{t+s})f(X_s)], \quad -\infty < t < \infty,$$

of the "output" process Y.

In Cambanis and Masry (1978), which will be referred to as CM(1978) in the following, we have presented procedures by which the *normalized* covariance function $\frac{R(t)}{R(0)}$ is reconstructed from the correlation function $C(t)$ for broad classes of nonlinearities f. The purpose of this brief paper is to complete the investigation of the problem under consideration by providing procedures for the reconstruction of the *variance* $R(0)$ from the knowledge of f and the first two moments

$$\mu_k = E[f^k(X_t)] \quad , \quad k = 1,2 \ ,$$

of the output process Y.

In Section 2 we study some properties of moments of functions of Gaussian random variables, which form the basis for the reconstruction of $R(0)$ given in Section 3. The reconstruction of $R(0)$ is shown (Theorems 1 and 2, Section 3) to be feasible for certain interval windows, convex or concave nonlinearities as well as certain discontinuous unimodal nonlinearities. Section 4 provides a discussion relating the results of this paper with those of CM(1978) and indicating the classes of nonlinearities for which the reconstruction of $R(t)$, $-\infty < t < \infty$, from $C(t)$, $-\infty < t < \infty$, and μ_1 is feasible.

2. SOME MOMENT PROPERTIES OF FUNCTIONS OF GAUSSIAN RANDOM VARIABLES

In this section we study the moment of a function of a Gaussian random variable with zero mean as a function of its variance, i.e.

$$m_k(\sigma) = E[f^k(\xi)] , \quad 0 < \sigma < \infty, \tag{1}$$

where k is a positive integer and ξ is a Gaussian random variable with zero mean, variance σ^2 and probability density function $\phi(x;\sigma)$. f is a real (almost everywhere nonconstant) Borel measurable function on the real line satisfying

$$E[|f(\xi)|^k] < \infty .$$

<u>PROPOSITION 1</u>. *Let* k *be a positive integer,* $0 < \sigma_1 < \sigma_2 < \infty$, *and let* $f(x)$ *satisfy*

(C1) $$f^k(x),\ x^2 f^k(x) \in L_1(\phi(x;\sigma)dx),\ \sigma_1 \leq \sigma \leq \sigma .$$

(i) *If* $f(x)$ *is of bounded variation on each finite interval, determines a* σ-*finite signed measure* df *on* $(-\infty,\infty)$ *and is such that*

(C2) $$\lim_{|x|\to\infty} x\, f^k(x)\, \phi(x;\sigma) = 0,$$

(C3) $$\int_{-\infty}^{\infty} |x\, f^{k-1}(x)|\, \phi(x;\sigma)\, d|f|(x) < \infty ,$$

$\sigma_1 \leq \sigma \leq \sigma_2$,

then $m_k(\sigma)$ *is absolutely continuous on* $[\sigma_1,\sigma_2]$ *and*

$$m_k'(\sigma) = \frac{k}{\sigma} \int_{-\infty}^{\infty} x\, f^{k-1}(x)\, \phi(x;\sigma)\, df(x) \quad \text{a.e. on } [\sigma_1,\sigma_2] .$$

(ii) If f(x) *has a first derivative which is absolutely continuous on each finite interval and* f(x) *satisfies (C2) as well as*

(C4) $$x \frac{df^k(x)}{dx}, \quad \frac{d^2f^k(x)}{dx^2} \in L_1(\phi(x;\sigma)dx),$$

(C5) $$\lim_{|x|\to\infty} \frac{df^k(x)}{dx} \phi(x;\sigma) = 0,$$

$\sigma_1 \le \sigma \le \sigma_2$,

then $m_k(\sigma)$ *is absolutely continuous on* $[\sigma_1,\sigma_2]$ *and*

$$m_k'(\sigma) = \sigma \int_{-\infty}^{\infty} \frac{d^2f^k(x)}{dx^2} \phi(x;\sigma)dx \quad \text{a.e. on } [\sigma_1,\sigma_2] .$$

Proof. We first note from $\phi(x;\sigma) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp(-x^2/2\sigma^2)$ that

$$\frac{\partial}{\partial x} \phi(x,\sigma) = -\frac{x}{\sigma^2} (x;\sigma), \tag{2}$$

and

$$\frac{\partial}{\partial \sigma} \phi(x;\sigma) = \frac{x^2-\sigma^2}{\sigma^3} \phi(x;\sigma) . \tag{3}$$

Since $\partial\phi(x;\sigma)/\partial\sigma$ is continuous, it follows that $\phi(x;\sigma)$ is absolutely continuous in σ on any closed and finite subinterval of $(0,\infty)$ and thus

$$\phi(x;\sigma) = \phi(x;1) + \int_1^{\sigma} \frac{\partial\phi(x;u)}{\partial u} du, \quad \sigma > 0 .$$

It follows that

$$m_k(\sigma) = \int_{-\infty}^{\infty} f^k(x)\, \phi(x;1)dx + \int_{-\infty}^{\infty} f^k(x) \left[\int_1^{\sigma} \frac{\partial\phi(x;u)}{\partial u} du\right]dx .$$

Hence if for each $\sigma_1 \le \sigma \le \sigma_2$ we have

$$\int_{-\infty}^{\infty} \int_1^{\sigma} |f^k(x)| \left|\frac{\partial\phi(x-u)}{\partial u}\right| du\, dx < \infty , \tag{4}$$

then by Fubini's theorem

$$m_k(\sigma) = m_k(1) + \int_1^\sigma [\int_{-\infty}^{\infty} f^k(x) \frac{\partial\phi(x;u)}{\partial u} dx]du,$$

and thus $m_k(\sigma)$ is absolutely continuous on $[\sigma_1,\sigma_2]$ and

$$m_k'(\sigma) = \int_{-\infty}^{\infty} f^k(x) \frac{\partial\phi(x;\sigma)}{\partial\sigma} dx \quad \text{a.e. on } [\sigma_1,\sigma_2].$$

It is clear from (3) and condition (C1) that (4) is indeed satisfied. We thus have by (3) that a.e. on $[\sigma_1,\sigma_2]$

$$m_k'(\sigma) = \int_{-\infty}^{\infty} f^k(x) \frac{x^2-\sigma^2}{\sigma^3} \phi(x;\sigma)dx$$

$$= \frac{1}{\sigma^3} \int_{-\infty}^{\infty} x^2 f^k(x) \phi(x;\sigma)dx - \frac{1}{\sigma} \int_{-\infty}^{\infty} f^k(x)\phi(x;\sigma)dx. \quad (5)$$

(i) For all $-\infty < a < b < \infty$ we have by (2) and integration by parts

$$\frac{1}{\sigma^3} \int_a^b x^2 f^k(x) \phi(x;\sigma)dx = -\frac{1}{\sigma} \int_a^b x f^k(x) \frac{\partial\phi(x;\sigma)}{\partial x} dx$$

$$= -\frac{1}{\sigma} [x f^k(x) \phi(x;\sigma)]_a^b + \frac{k}{\sigma} \int_a^b x f^{k-1}(x) \phi(x;\sigma) df(x)$$

$$+ \frac{1}{\sigma} \int_a^b f^k(x) \phi(x;\sigma)dx . \quad (6)$$

Under the conditions (C2) and (C3), letting $a \downarrow -\infty$, $b \uparrow \infty$ and using the dominated convergence theorem in (6), we obtain

$$\frac{1}{\sigma^3} \int_{-\infty}^{\infty} x^2 f^k(x) \phi(x;\sigma)dx = \frac{k}{\sigma} \int_{-\infty}^{\infty} x f^{k-1}(x) \phi(x;\sigma) df(x)$$

$$+ \frac{1}{\sigma} \int_{-\infty}^{\infty} f^k(x) \phi(x;\sigma)dx$$

and the result follows by (5). Note that if $x f^{k-1}(x) df(x) \geq 0$ on $(-\infty,\infty)$ (or ≤ 0) then conditions (C1) and (C2) imply (C3): Let $a \downarrow -\infty$, $b \uparrow \infty$ in (6) and use the monotone convergence theorem.

(ii) Since $x \frac{d}{dx} f^k(x) = k x f^{k-1}(x) \frac{d}{dx} f(x)$, it follows that condition (C4) implies condition (C3) and thus under (ii) the conditions in (i) are satisfied so that $m_k(\sigma)$ is absolutely continuous on $[\sigma_1,\sigma_2]$ and a.e. on $[\sigma_1,\sigma_2]$

$$m_k'(\sigma) = \frac{k}{\sigma} \int_{-\infty}^{\infty} x\, f^{k-1}(x)\, f'(x)\, \phi(x;\sigma)dx$$

$$= \frac{1}{\sigma} \int_{-\infty}^{\infty} x \frac{df^k(x)}{dx} \phi(x;\sigma)dx$$

$$= -\sigma \int_{-\infty}^{\infty} \frac{df^k(x)}{dx} \frac{\partial\phi(x;\sigma)}{\partial x} dx \qquad (7)$$

where the last equality follows by (2). As in part (i), integrating by parts we have for all $-\infty < a < b < \infty$,

$$\int_a^b \frac{df^k(x)}{dx} \frac{\partial\phi(x;\sigma)}{\partial x} dx = \left[\frac{df^k(x)}{dx} \phi(x;\sigma) \right]_a^b - \int_a^b \frac{d^2f^k(x)}{dx^2} \phi(x;\sigma)dx. \qquad (8)$$

Under the conditions (C4) and (C5), letting $a \downarrow -\infty$, $b \uparrow \infty$ and using the dominated convergence theorem we obtain

$$\int_a^b \frac{df^k(x)}{dx} \frac{\partial\phi(x;\sigma)}{\partial x} dx = - \int_{-\infty}^{\infty} \frac{d^2f^k(x)}{dx^2} \phi(x;\sigma)dx,$$

and the result follows by (7). Note again that if $d^2f^k(x)/dx^2 \geq 0$ a.e. on $(-\infty,\infty)$ (or ≤ 0) then $d^2f^k(x)/dx^2 \in L_1(\phi(x;\sigma)dx)$ follows from the remaining conditions and could thus be deleted from (C4): Let $a \downarrow -\infty$, $b \uparrow \infty$ in (8) and use the monotone convergence theorem. □

REMARKS: It is possible to delete some of the conditions (C1) - (C5) in Proposition 1 under appropriate circumstances. The following cases are of particular interest.

1) As was pointed out in the proof of Proposition 1, if

$$x\, f^{k-1}(x)\, df(x) \geq 0 \text{ on } (-\infty,\infty) \text{ (or } \leq 0)$$

then (C3) follows from (C1) and (C2). Similarly, if

$$\frac{d^2f^k(x)}{dx^2} \geq 0 \text{ a.e. on } (-\infty,\infty) \text{ (or } \leq 0)$$

then the integrability condition $d^2f^k(x)/dx^2 \in L_1(\phi(x;\sigma)dx)$ may be deleted from (C4).

2) The limiting conditions (C2) and (C5) may be deleted under appropriate *uniform* continuity assumption as follows. We first note that if $g(x) \in L_1(dx)$ and $g(x)$ is uniformly continuous on $(-\infty,\infty)$, then $\lim_{|x|\to\infty} g(x) = 0$. Thus, if

$$f(x) \text{ is uniformly continuous on } (-\infty,\infty)$$

then (C2) follows from the integrability condition (C1). Similarly, if

$$f^{k-1}(x)f'(x) \text{ is uniformly continuous on } (-\infty,\infty)$$

then (C5) follows from the first integrability condition in (C4).

3) The limiting condition (C2)(respectively (C5)) may be deleted when $f(x)$ (respectively $f'(x)$ has a polynomial growth $0(|x|^n)$ or exponential growth $0(\exp(\alpha|x|^{\beta}))$, $\alpha > 0$, $0 < \beta < 2$, as $|x| \to \infty$.

4) All the conditions (C1) - (C5) are satisfied in case the nonlinearity $f(x)$ is a polynomial.

The case where f is an interval window function, $f(x) = 1_{(a,b)}(x)$, is of particular practical importance. We have,

<u>PROPOSITION 2</u>. *Let* $f(x) = 1_{(a,b)}(x)$, $-\infty \leq a < b \leq \infty$,

(i) $m_1(\sigma)$ *is strictly monotonic on* $(0,\infty)$ *if and only if the interval* (a,b) *has any one of the following forms*

a) $(-\infty, b)$, $b \neq 0$, $b < \infty$,

b) (a,∞), $a \neq 0$, $-\infty < a$,

c) (a,b), $-\infty < a \leq 0 \leq b < \infty$.

(ii) If $-\infty < a < b < \infty$ *and* $r > 0$, *then* $m_1(\sigma)$ *is strictly monotonic for* $0 < \sigma \leq r$ *if and only if* a, $\delta = b-a$ *satisfy*

$$(a \leq 0,\ \delta \geq -a) \quad \text{or} \quad a \geq a_3(\delta,r) \quad \text{or} \quad a \leq a_4(\delta,r)$$

where

$$a_3(\delta,r) = r\, a_3(\frac{\delta}{r},1)$$

$$a_4(\delta,r) = -a_3(\delta,r) - \delta$$

and $a_3(\delta,1)$ *is the unique solution* $\alpha(\delta) = a_3(\delta,1)$ *of the equation*

$$2\,\ell n(1 + \frac{\delta}{\alpha}) = \delta(\delta + 2\alpha), \quad \alpha > 0, \ \delta > 0,$$

(see Figure 1). We also have that $a_3(\delta,r)$ *is strictly decreasing in* δ *with*

$$a_3(0+,r) = r, \quad a_3'(0+,r) = -\frac{1}{2}, \ a_3(+\infty,r) = 0.$$

Proof. When $f = 1_{(a,b)}$ then

$$m_1(\sigma) = \Phi(\frac{b}{\sigma}) - \Phi(\frac{a}{\sigma})$$

where Φ is the distribution function of the standard normal density $\phi(x) = \phi(x;1)$.

(i) When $-\infty = a < b < \infty$, $b \neq 0$, we have $m_1(\sigma) = \Phi(b/\sigma)$ and

$$m_1'(\sigma) = -\frac{b}{\sigma^2}\ \phi(\frac{b}{\sigma})$$

is strictly positive or negative on $(0,\infty)$ as $b < 0$ or $b > 0$; and similarly when $-\infty < a < b = +\infty$, $a \neq 0$. When $-\infty < a < b < \infty$ we have

$$m_1'(\sigma) = -\frac{b}{\sigma^2}\ \phi(\frac{b}{\sigma}) + \frac{a}{\sigma^2}\ \phi(\frac{a}{\sigma}) = \frac{1}{\sigma^2}\ \phi(\frac{b}{\sigma})(a\, e^{(b^2-a^2)/2\sigma^2} - b).$$

If $a \leq 0 \leq b$ then $m_1'(\sigma) < 0$ on $(0,\infty)$. If $0 < a < b$ then

$$m_1'(\sigma) > 0 \text{ for all } \sigma \text{ such that } 0 < \sigma^2 < \frac{b^2-a^2}{2\,\ell n\,\frac{b}{a}}$$

and

$$m_1'(\sigma) < 0 \text{ for all } \sigma \text{ such that } \frac{b^2-a^2}{2\,\ell n\,\frac{b}{a}} < \sigma^2 ;$$

and similarly if $a < b < 0$. (i) thus follows.

(ii) It is clear from (i) that if $a \leq 0 \leq b$, i.e., $a \leq 0$, $\delta \geq -a$, then $m_1(\sigma)$ is strictly decreasing on $(0,\infty)$. If $0 < a < b$, i.e., $a > 0$, $\delta > 0$, it is again clear from (i) that $m_1(\sigma)$ is strictly monotonic (increasing in fact) for $0 < \sigma \leq r$ if and only if

$$r^2 \leq \frac{b^2-a^2}{2\ \ell n \frac{b}{a}}$$

or equivalently

$$2r^2\ \ell n(1 + \frac{\delta}{a}) \leq \delta(\delta + 2a) \ .$$

It is then clear that when $a > 0$, $\delta > 0$, $m_1(\sigma)$ is strictly monotonic for $0 < \delta \leq r$ if and only if $a \geq a_3(\delta,r)$. Similarly, it is seen that if $a < b < 0$, i.e., $a < 0$, $\delta \leq -a$, then $m_1(\sigma)$ is strictly monotonic for $0 < \sigma \leq r$ if and only if $a \leq a_4(\delta,r)$ where $a_4(\delta,r) = -a_3(\delta,r) - \delta$. Thus, for $a < 0$, $\delta > 0$, $m_1(\sigma)$ is strictly monotonic for $0 < \sigma \leq r$ if and only if $a \leq a_4(\delta,r)$ or $\delta \geq -a$. Finally, the assertions contained in the last sentence of (ii) are established in a straightforward way.

It may be worth noting that the solution of the equation

$$2\ \ell n(1 + \frac{\delta}{a}) = \delta(\delta + 2\alpha), \quad \alpha > 0 \ ,$$

which determines $a_3(\delta,1)$, can be obtained parametrically as a function of x, $1 < x < \infty$, by

$$\alpha(x) = \frac{1}{2}(x-1)\ \delta(x), \qquad \delta^2(x) = \frac{2}{x}\ \ell n\ \frac{x+1}{x-1} \ . \qquad \square$$

3. THE RECONSTRUCTION OF THE VARIANCE R(0)

Throughout this section X is a real stationary Gaussian process with zero mean and continuous covariance function $R(t)$. We consider the problem of determining the variance R(0) when we know the nonlinearity f and either the first moment μ_1 or the second moment μ_2 of the output process $Y = \{f(X_t), -\infty < t < \infty\}$. We have

$$\mu_k = E[f^k(X_t)] = m_k[R^{\frac{1}{2}}(0)], \quad k = 1,2.$$

Since f is known, so is of course the function $m_k(\sigma), \sigma > 0$, and our problem is knowing μ_1 or μ_2 to find conditions on f under which R(0) can be recovered. Propositions 1 and 2 of Section 2 provide a number of such conditions on f.

It is convenient at this point to exclude those f's for which the output moments μ_k provide no information whatever on the input variance. Specifically, we exclude the cases where

$$f^k(x) = a\ 1_{(-\infty,0)}(x) + b\ 1_{(0,\infty)}(x) \quad \text{a.e.} \tag{9}$$

(in which case $\mu_k = \frac{1}{2}(a + b)$ which is independent of R(0)) and for k = 1 the case f(x) = ax + b (in which case $\mu_1 = b$).

We also introduce the following terminology which will facilitate the statement of the main Theorem. A real function f on the real line is called *g-unimodal* (g for generalized) if for some $a \in (-\infty,\infty)$, f is nondecreasing on $(-\infty,a)$ and nonincreasing on (a,∞), or vice versa. The number a is then called a *mode* of f. Note that the mode need not be an extremum of f; and that unimodal functions in the usual sense (i.e., continuous functions which are strictly increasing on $(-\infty,a]$ and strictly decreasing on $[a,\infty)$, or vice versa) are g-unimodal. Note also that f is g-unimodal with mode a if and only if f has bounded variation on each finite interval and

$$(x - a)\ df(x) \geq 0 \ \text{ on } (-\infty,\infty) \quad (\text{or} \leq 0 \ \text{ on } (-\infty,\infty)).$$

A function f is called *c-unimodal* if it has a first derivative f' which is absolutely continuous on each finite interval and $f''(x) \geq 0$ a.e. on $(-\infty,\infty)$, or $f''(x) \leq 0$ a.e. on $(-\infty,\infty)$. Here "c" stands for both continuous and convex or concave. It is important to note that unlike c-unimodal functions, g-unimodal functions may have discontinuities (jumps). We can now state our first result on the reconstruction of R(0).

THEOREM 1. *The variance* R(0) *can be reconstructed from the k-th moment* μ_k, $k = 1,2,$ *of the output process* $Y = \{f(X_t),\ -\infty < t < \infty\}$ *when the nonlinearity* f *satisfies*

(C6) $\quad E|f^k(X_t)| < \infty, \qquad E|X_t^2\, f^k(X_t)| < \infty$

(C7) $\quad \lim_{|x|\to\infty} x\, f^k(x)\, \phi(x;\, R^{\frac{1}{2}}(0)) = 0$

and any one of the following conditions

(i) $f^k(x)$ *is g-unimodal with mode zero and is Lebesgue integrable in some open neighborhood of zero.*

(ii) $f^k(x)$ *is c-unimodal and satisfies*

(C8) $\quad E|X_t\, (f^k)'(X_t)| < \infty$

(C9) $\quad \lim_{|x|\to\infty} \dfrac{d\, f^k(x)}{dx}\, \phi(x;\, R^{\frac{1}{2}}(0)) = 0\,.$

Proof. We first note that if $0 < \sigma_1 < \sigma_2 < \infty$ then there exists a positive real number $N = N(\sigma_1,\sigma_2)$ such that $\phi(x;\sigma_1) < \phi(x;\sigma_2)$ whenever $|x| > N$. Thus if g(x) is a measurable function, then

$$\int_{|x|>N} |g(x)|\, \phi(x;\sigma_1)dx \leq \int_{|x|>N} |g(x)|\, \phi(x;\sigma_2)dx\,.$$

It then follows that if g(x) is Lebesgue integrable on bounded intervals, then

$$g \in L_1(\phi(x;\sigma_2)dx) \Longrightarrow g \in L_1(\phi(x;\sigma_1)dx). \qquad (10)$$

(i) Since f^k is g-unimodal with mode zero and is Lebesgue integrable in some open neighborhood of zero, both $f^k(x)$ and $x^2\ f^k(x)$ are Lebesgue integrable on bounded intervals. It then follows from (10) that condition (C6) implies that condition (C1) is satisfied for *all* $0 < \sigma \leq R^{\frac{1}{2}}(0)$. Also, condition (C7) implies that (C2) is satisfied for *all* $0 < \sigma \leq R^{\frac{1}{2}}(0)$. Finally, since f^k is g-unimodal with mode zero we have $x\ df^k(x) = k\ x\ f^{k-1}(x)\ df(x) \geq 0$ on $(-\infty,\infty)$, or ≤ 0 on $(-\infty,\infty)$, and by Remark (1) following Proposition 1, condition (C3) is implied by (C1) and (C2). Thus conditions (C1) - (C3) are satisfied for all $0 < \sigma \leq R^{\frac{1}{2}}(0)$. Let σ_0 be such that $R^{\frac{1}{2}}(0) \leq \sigma_0 < \infty$ and such that conditions (C1) - (C3) are satisfied for all $0 < \sigma \leq \sigma_0$. Then by Proposition 1(i) we have

$$m_k'(\sigma) = \frac{k}{\sigma}\int_{-\infty}^{\infty} x\ f^{k-1}(x)\ \phi(x;\sigma)\ df(x)$$

$$= \frac{1}{\sigma}\int_{-\infty}^{\infty} x\ \phi(x;\sigma)\ df^k(x) \qquad \text{a.e. on } (0,\sigma_0]\ .$$

Now since f^k is g-unimodal with mode zero, then $x\ df^k(x) \geq 0$ on $(-\infty,\infty)$ (or ≤ 0 on $(-\infty,\infty)$) and since $f^k(x)$ is *not* of the form (9), it follows that

$$m_k'(\sigma) > 0 \text{ (or } < 0) \quad \text{a.e. on } (0,\sigma_0]\ .$$

Thus $m_k(\sigma)$ is strictly monotonic on $(0,\sigma_0]$ and $R(0)$ can be reconstructed from $\mu_k = m_k[R^{\frac{1}{2}}(0)]$.

(ii) is shown similarly so that under the assumptions (C6) - (C9) and the c-unimodality of f^k we have that Proposition 1(ii) is applicable and

$$m_k'(\sigma) = \sigma\int_{-\infty}^{\infty} \frac{d^2\ f^k(x)}{dx^2}\ \phi(x;\sigma)dx \qquad \text{a.e. on } (0,\sigma_0]\ .$$

Now since f^k is c-unimodal then $\frac{d^2 f^k(x)}{dx^2} \geq 0$ (or ≤ 0) on $(-\infty,\infty)$. If

$$\text{Leb}\ \ \{x\colon\ [f^k(x)]'' > 0\} > 0$$

or (11)

$$\text{Leb}\ \ \{x\colon\ [f^k(x)]'' < 0\} > 0$$

then $m_k'(\sigma) > 0$ a.e. on $(0,\sigma_0]$ so that $m_k(\sigma)$ is strictly monotonic on $(0,\sigma_0]$ and thus $R(0)$ can be recovered from $\mu_k = m_k[R^{1/2}(0)]$. Finally, if (11) is *not* satisfied then necessarily $[f^k(x)]'' = 0$ a.e. on $(-\infty,\infty)$ in which case $f(x) = ax + b$ for $k = 1$ and $f^2(x) = b$ for $k = 2$. These cases have been excluded. □

It should be noted that conditions (C6) - (C9) have the somewhat unsatisfactory but quite natural feature that they are expressed in terms of the unknown variance $R(0)$. This can be eliminated by assuming that the conditions hold for *all* $0 < \sigma < \infty$. This is certainly satisfied in practical situations where $f^k(x)$ has a polynomial growth $O(|x|^m)$ or exponential growth $O(\exp(\alpha|x|^\beta))$, $\alpha > 0$, $0 < \beta < 2$ as $|x| < \infty$.

In case the nonlinearity f is an interval window function, $f(x) = 1_{(a,b)}(x)$, we have the following theorem on the reconstruction of $R(0)$, which follows directly from Proposition 2.

<u>THEOREM 2</u>. *Let* $f(x) = 1_{(a,b)}(x)$, $-\infty \le a < b \le \infty$.

(i) The variance $R(0)$ *can be reconstructed from the first moment* μ_1 *of the output process* Y *for the following interval windows*

a) $f(x) = 1_{(-\infty,b)}(x)$ *where* $b \neq 0$.

b) $f(x) = 1_{(a,\infty)}(x)$ *where* $a \neq 0$.

c) $f(x) = 1_{(a,b)}(x)$ or $f(x) = 1 - 1_{(a,b)}(x)$ *where* $-\infty<a\le 0\le b<\infty$.

(ii) If an upper bound r^2 *on the variance* $R(0)$ *is known,* $R(0) \le r^2$, *then* $R(0)$ *can be reconstructed from the first moment* μ_1 *of the output process whenever the interval window* $f(x) = 1_{(a,b)}(x)$, $-\infty < a < b < \infty$, *is such that the pair* $(a, \delta = b-a)$ *satisfies one of the following conditions*

a) $a \le a_4(\delta,r)$

b) $a \le 0$ and $\delta \ge -a$

c) $a \ge a_3(\delta,r)$

(See Figure 1).

When the first order distribution of the output process Y is known, rather than the first two moments, we have the following result where $\mathcal{B}(f)$ denotes the σ-field of Borel sets generated by f.

COROLLARY. *(i) The variance* R(0) *can be reconstructed from the first order distribution of the output process* Y *when* f *is such that* $\mathcal{B}(f)$ *contains an interval* (a,b) *of the form*

a) $(-\infty,b)$, $b \neq 0$.

b) (a,∞) , $a \neq 0$.

c) (a,b), *where* $-\infty < a \leq 0 \leq b < \infty$ *and* $a \neq b$.

(ii) If an upper bound r^2 *on the variance* R(0) *is known,* $R(0) \leq r^2$, *then* R(0) *can be reconstructed from the first order distribution of the output process* Y *when* f *is such that* $\mathcal{B}(f)$ *contains an interval* (a,b) *such that the pair* $(a,\ \delta = b-a)$ *satisfies one of the following conditions,*

a) $a \leq a_4\ (\delta,r)$

b) $a \leq 0$ and $\delta \geq -a$

c) $a \geq a_3\ (\delta,r)$.

Proof. The result follows by Theorem 2 and the following observation. Let $(a,b) \in \mathcal{B}(f)$. Then there is a Borel set B such that $(a,b) = f^{-1}(B)$ and thus

$$P[f(X_t) \in B] = P[X_t \in f^{-1}(B)] = P[a < X_t < b] = E[1_{(a,b)}(X_t)]. \qquad \square$$

We conclude this section by noting that aside from the interval windows of Theorem 2, the classes of nonlinearities f for which R(0) can be recovered, from the first two moments of the output process, are:

a) f is g-unimodal with mode zero.

b) f^2 is g-unimodal with mode zero.

c) f is c-unimodal.

d) f^2 is c-unimodal.

We note that continuous convex functions which are symmetric around some point are included in (c), whereas discontinuous symmetric "unimodal" functions are included in (a). Classes (b) and (d) contain certain odd nonlinearities, possibly discontinuous. For instance, all odd quantizers belong to class (b).

4. THE RECONSTRUCTION OF THE COVARIANCE R(t).

Throughout this section X is a real stationary Gaussian process with zero mean and continuous covaraince function R(t). Combining the results of Section 3 which provide the reconstruction of the variance R(0), with the results of CM(1978) which provide the reconstruction of the normalized convariance function R(t)/R(0), we obtain the reconstruction of the covariance function R(t) of the input process from the knowledge of the nonlinearity f and the first moment and correlation function of the output process. This is accomplished for the following classes of nonlinearities. Here, for simplicity, we omit the integrability and the asymptotic conditions on the nonlinearity f (such as (C6) and (C7)); these can be easily traced through the indicated theorems of this paper and the earlier one CM(1978).

1. Monotonic nonlinearities. The relevant theorems are Theorem 2 of CM(1978) and Theorem 1 of this paper. When f(x) is monotonic (possibly discontinuous) such that $f^2(x)$ is either g-unimodal with mode zero or c-unimodal, then *every* covariance function R(t), $-\infty < t < \infty$, can be reconstructed from the correlation function C(t) of the output process $Y = \{f(X_t), -\infty < t < \infty\}$. Included here are smooth limiters, half-wave νth-law devices with $\nu > 0$, full-wave odd νth-law devices with $\nu > 0$ (for definitions see Thomas (1969)), odd quantizers and companders (see Taub and Schilling (1971)).

2. Even nonlinearities. The relevant theorems here are Theorem 5 of CM(1978) and Theorem 1 of this paper. When f(x) is even, g-unimodal with mode zero and satisfies

$$E[(X_t^2 - 1)f(X_t)] \neq 0 \tag{12}$$

then covariance functions R(t), satisfying the smoothness condition at their zero crossings stated in Theorem 5 of CM(1978), can be reconstructed from the correlation function C(t) of the output process. Included here are the full-wave even νth-law devices with $\nu > 0$ and the symmetric windows $f(x) = 1_{(-a,a)}(x)$ and $f(x) = 1 - 1_{(-a,a)}(x)$ for $0 < a < \infty$. Condition (12) says that in the series expansion of f in terms of the even Hermite polynomials $f(x) = \sum_{n=0}^{\infty} a_{2n} H_{2n}(x;R(0))$ the term H_2 should be present. This condition can be weakened to "the term H_{2k} for some $k \geq 1$ should be present" as discussed in the remark preceeding Theorem 6 of CM(1978).

3. General nonlinearities. The relevant theorems here are Theorem 6 of CM(1978) and Theorem 1 of this paper. When f(x) satisfies the condition stated in Theorem 6 of CM(1978) (which is the analogue of (12) for non-even functions) and for k=1 or 2, $f^k(x)$ is either g-unimodal with mode zero or c-unimodal, then covariance functions R(t) satisfying the smoothness conditions at certain negative level crossings stated in Theorem 6 of CM(1978), can be reconstructed from the correlation function C(t) of the output process when k=2, and from the output correlation function and its first moment when k=1.

4. Interval windows. The relevant theorems here are Theorem 1 of CM(1978) and Theorem 2 of this paper. Arbitrary covariance functions R(t) can be reconstructed from the output correlation function C(t) for the following infinite interval window nonlinearities: $1_{(-\infty,b)}$, $b \neq 0$, $-\infty < b < \infty$, and $1_{(a,\infty)}$, $a \neq 0$, $-\infty < a < \infty$. For finite interval window nonlinearities $1_{(a,b)}$ we have the following results.

(i) If an upper bound r^2 on the input power is known, $R(0) = EX_t^2 \leq r^2$, then arbitrary covariance functions R(t) can be reconstructed from the output correlation function provided (a, δ=b-a) belongs to area (A_r) of Figure 1.

(Similar results hold, and the corresponding areas in the (a,δ) half plane are easily determined, when we know that $0 < r^2 < R(0)$ or that $0 < r_1^2 < R(0) < r_2^2 < \infty$).

(ii) If (a,δ) belong to area (B) of Figure 1, $-\frac{3}{4}\delta \leq a \leq -\frac{1}{4}\delta$, and if R satisfies the conditions in Corollary 6 of CM(1978), then R can be reconstructed from the output correlation C.

For $i = 1,2,3,4$ and $r > 0$ the function $a_i(\delta,r)$ is given by

$$a_i(\delta,r) = r\, a_i\left(\frac{\delta}{r}, 1\right)$$

where for $i = 1,2$, $a_i(\delta,1)$ is identical to the function $a_i(\delta)$ defined in Proposition 3 of CM(1978); for $i = 3,4$, $a_i(\delta,1)$ is given in Proposition 2(ii) of this paper. In Figure 1 the functions $a_i(\delta,r)$, $i = 1,2,3,4$ and the area A_r are plotted for $r = 1$.

We finally note that for interval windows $f(x) = 1_{(a,b)}(x)$ for which the pair $(a, \delta=b-a)$ falls in the shaded area of Figure 1, we do not have as yet reconstruction procedures for both $R(0)$ and $R(t)/R(0)$.

5. Using the second order distribution of the output we can construct

(i) any covariance R, if the σ-field $\mathcal{B}(f)$ generated by f contains an interval of the form $(-\infty,b)$, $b \neq 0$, or (a,∞), $a \neq 0$;

(ii) covariances R with $R(0) \leq r^2$, if $\mathcal{B}(f)$ contains an interval (a,b) such that $(a, \delta=b-a)$ is in area (A_r) of Figure 1;

(iii) covariances R satisfying the conditions in Corollary 6 of CM(1978), if $\mathcal{B}(f)$ contains an interval with (a,δ) in area (B) of Figure 1.

6. Finally if $R(t) \geq 0$, or if R has a rational spectral density, or if it is bandlimited, and if $f^k(x)$ is g-unimodal with mode zero or c-unimodal, for $k = 1$ or 2, then R can be reconstructed from the output correlation C and, for $k = 1$, its first moment.

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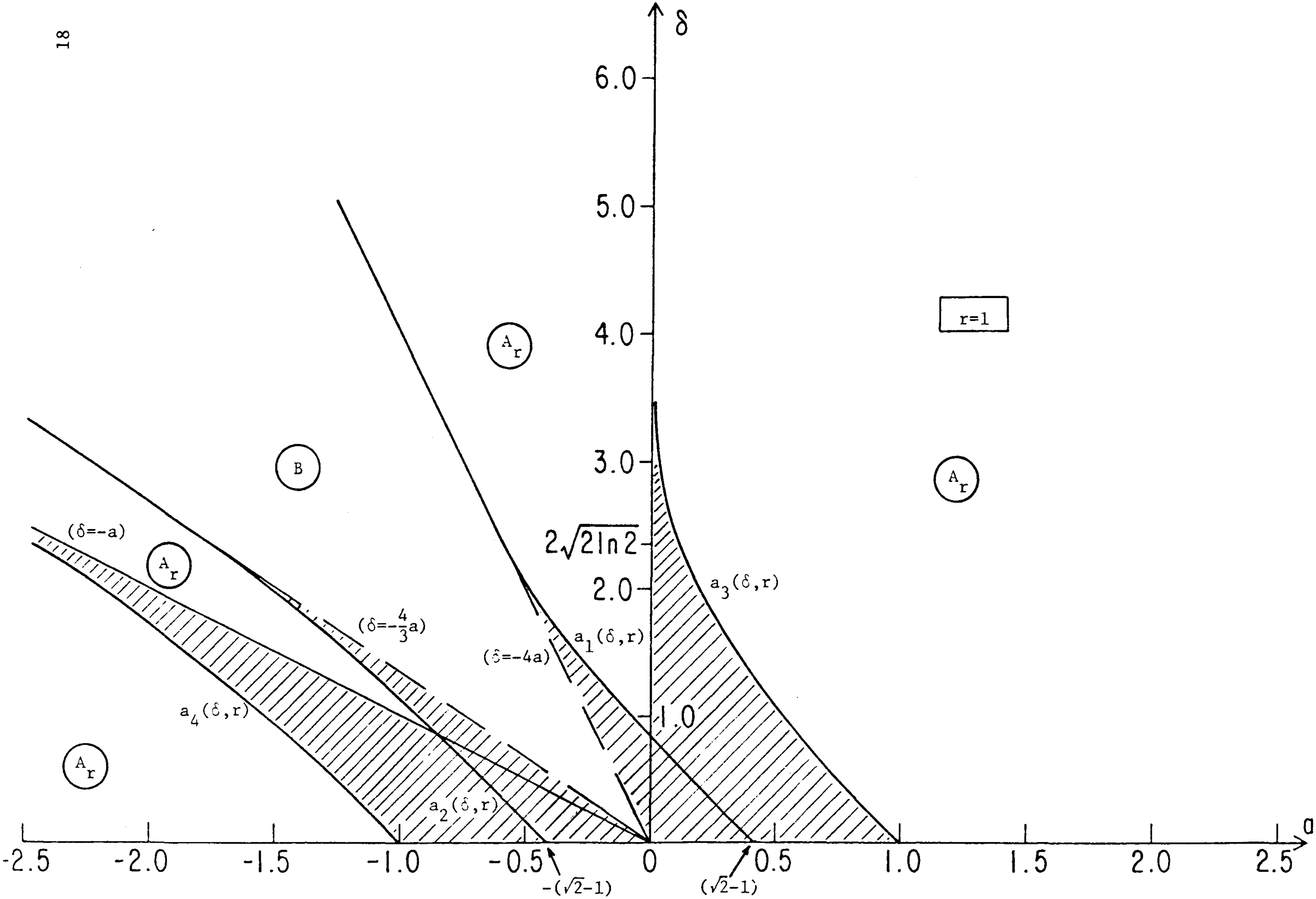

Figure 1: The (a,δ) half plane of Proposition 2.

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We consider the problem of reconstructing the variance R(0) of a zero mean stationary Gaussian process observed through a zero memory nonlinearity f(x), from the knowledge of f and the first two moments of the output process. The reconstruction is shown to be feasible for certain interval windows, convex nonlinearities and discontinuous unimodal nonlinearities. The paper is the completion of the investigation begun in Cambanis and Masry (1978) where the reconstruction of the *normalized* covariance R(t)/R(0) was considered.

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