Systems of Frequency Curves Generated by Transformations of Logistic Variables

by

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1. Introduction

The systems of distributions (for a random variable Y) defined by

$$S_L: \quad Z = \gamma + \delta \ln Y \qquad (0 < Y) \tag{1.1}$$

$$S_B: \quad Z = \gamma + \delta \ln\{Y/(1-Y)\} \qquad (0 < Y < 1) \tag{1.2}$$

$$S_U: \quad Z = \gamma + \delta \sinh^{-1} Y \tag{1.3}$$

with $\delta > 0$, and Z a unit normal variable, have been described by Johnson (1949); and with Z a standard Laplace (double exponential) variable, (S_L', S_B', S_U') by Johnson (1954). In the present paper we study analogous systems, generated by ascribing to Z a standard logistic distribution with probability density function (pdf)

$$f_Z(z) = e^z(1 + e^z)^{-2} \tag{1.4}$$

or, equivalently, with cumulative distribution function (cdf)

$$F_Z(z) = (1 + e^{-z})^{-1} . \tag{1.5}$$

The percentile function (inverse of cdf) is

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$$z = \ell n\{F_Z(z)/(1 - F_Z(z))\}. \tag{1.6}$$

We will denote these new systems of distributions by L_L, L_B, L_U according as transformations (1.1), (1.2), (1.3), respectively, are used. In view of the closeness of the shapes of the logistic and normal distributions, it is to be expected that the new systems will exhibit some similarity in shape to S_L, S_B, and S_U.

In fact since (Johnson and Kotz (1970), p.6)

$$|\{1 + \exp(-\pi x/\sqrt{3})\}^{-1} - \Phi(\tfrac{16x}{15})| < 0.01 \quad \text{for all } x \tag{1.7a}$$

where $\Phi(u) = (\sqrt{2\pi})^{-1} \int_{-\infty}^{u} e^{-\frac{1}{2}t^2}\, dt$ the difference between the cdf of $S_{L,B,U}$ with parameters γ, δ and $L_{L,B,U}$ (with the same values of ξ and λ) with parameters $\psi\gamma$, $\psi\delta$, where

$$\psi = \frac{\pi}{\sqrt{3}} \cdot \frac{15}{16} \doteqdot 1.7 \tag{1.7b}$$

cannot exceed 0.01. Although this gives good agreement in the central part of the distribution, there can be gross disparities in percentile points in the tails.

If an S curve and an L curve are fitted using the same first four moments, we do not find that (i) the ratios (L/S) for γ values and for δ values will each be about 1.7, and (ii) the values of ξ and λ will be the same. Section 5.4 (Table 4) contains an example wherein this is clearly illustrated. There can be very considerable disparity between the percentiles in the tails of the distributions even though (1.7a) is valid.

The very simple formula (1.6) for z in terms of $F_Z(z)$ is an obvious practical advantage of the new systems. Percentile points of fitted distributions can be obtained very simply.

2. The Log-Logistic (L_L) Distribution

This distribution has been studied by Shah and Dave (1963). The pdf is

$$f_Y(y) = \delta\, e^{-\gamma}\, y^{\delta-1}\, (1 + e^{-\gamma}\, y^{\delta})^{-2} \quad (y \geq 0;\ \delta > 0) \tag{2.1}$$

from which it can be seen that it is a special case of Burr's (1942) Type XII system of distributions. Dubey (1966) called it the Weibull-exponential distribution, and fitted it to some business failure data presented by Lomax (1954).

The pdf is unimodal. The mode is at $y = 0$ for $\delta \leq 1$ (giving a reversed J-shaped curve); it is at $y = e^{-\gamma}(\delta-1)/(\delta+1)$ if $\delta > 1$. The cdf is

$$F_Y(y) = 1 - (1 + e^{-\gamma}\, y^{\delta})^{-1} \tag{2.2}$$

and the inverse cdf is

$$y = e^{\Omega}\, [F_Y(y)/(1-F_Y(y))]^{1/\delta} \tag{2.3}$$

where $\Omega = \gamma/\delta$.

From (1.1) and (1.4), the r-th moment of Y about zero is

$$\mu'_r = \int_{-\infty}^{\infty} e^{r(z-\gamma)/\delta}\, e^z\, (1 + e^z)^{-2}\, dz\ .$$

Putting $u = e^z\,(1 + e^z)^{-1}$ so that $du/dz = e^z(1 + e^z)^{-2}$ and $e^z = u/(1-u)$

$$\mu'_r = e^{-r\Omega} \int_0^1 u^{r/\delta}\, (1-u)^{-r/\delta}\, du = e^{-r\Omega}\, B(1+r\delta^{-1},\ 1-r\delta^{-1})$$

$$= e^{-r\Omega}\, r\theta \operatorname{cosec} r\theta \tag{2.4}$$

with $\theta = \pi/\delta$, provided $r < \delta$. (If $r \geq \delta$, μ'_r is infinite.)

In particular

$$E[Y] = e^{-\Omega}\, \theta \operatorname{cosec} \theta \tag{2.5}$$

and

$$\mathrm{Var}(Y) = e^{-2\Omega}\,\theta\,\mathrm{cosec}^2\theta\,(\tan\theta - \theta)\ . \tag{2.6}$$

The r-th central moment, μ_r, is a multiple (depending on θ, but not on Ω) of $\exp(-r\Omega)$, and so the moment-ratios $\mu_r/\mu_2^{r/2}$ - in particular $\sqrt{\beta_1} = \mu_3/\mu_2^{3/2}$ and $\beta_2 = \mu_4/\mu_2^2$ - do not depend on Ω, but only on θ. The $(\sqrt{\beta_1},\beta_2)$ points lie on a line which passes through the 'logistic point' (0,4.2) (corresponding to $\theta \to 0$ $(\delta \to \infty)$). Figure 1 shows that L_L ('log-logistic') line, and also the S_L (lognormal) line.

The $(\sqrt{\beta_1},\beta_2)$ points for $L_B(L_U)$ distributions lie 'above' ('below') the L_L line in Figure 1. This is analogous to the situation for S_B, S_U and the lognormal line.

Table 1 gives values of β_2 for selected values of $\sqrt{\beta_1}$, for points on the L_L line.

3. The L_B System

If transformation (1.2) is valid, with Z standard logistic, then the pdf of Y is

$$f_Y(y) = \delta\, e^{\gamma}\, y^{\delta-1}\,(1-y)^{\delta-1}\,\{(1-y)^{\delta} + e^{\gamma}\, y^{\delta}\}^{-2} \qquad (0 < y < 1)\ . \tag{3.1}$$

If $\delta = 1$, we get a 'power function' pdf

$$f_Y(y) = e^{\gamma}(1 - y + e^{\gamma}\, y)^{-2} \qquad (0 < y < 1)\ . \tag{3.2}$$

If, in addition, $\gamma = 0$ we have a standard uniform distribution.

In contrast to the situations when Z has a normal or Laplace distribution, L_B curves cannot be multimodal. There is a single mode (if $\delta > 1$) or antimode (if $\delta < 1$) at the unique value of y between 0 and 1 satisfying the equation

$$e^{\gamma} = \left(\frac{1-y}{y}\right)^{\delta} \cdot \frac{\delta-1+2y}{\delta+1-2y} \quad . \tag{3.3}$$

The right hand side of (3.3) is equal to 1 when $y = \frac{1}{2}$, whatever the value of δ. The derivative of the logarithm of the right hand side is

$$\delta(1-\delta^2)\ y^{-1}(1-y)^{-1}\ \{\delta^2 - (1-2y)^2\} \ . \tag{3.4}$$

If $\delta > 1$, this is negative,and so the mode is at $y \gtrless \frac{1}{2}$ according as $\gamma \gtrless 0$.

For $\delta < 1$, the curve is U-shaped, and the antimode is between $\frac{1}{2}(1-\delta)$ and $\frac{1}{2}(1+\delta)$. Since (3.4) is positive for $\frac{1}{2}(1-\delta) < y < \frac{1}{2}(1+\delta)$ it follows that the antimode is at $\frac{1}{2}(1-\delta) < y < \frac{1}{2}$ for $\gamma < 0$ and $\frac{1}{2} < y < \frac{1}{2}(1+\delta)$ for $\gamma > 0$.

TABLE 1: The Log-logistic Line

$\sqrt{\beta_1}$	β_2	δ	$\sqrt{\beta_1}$	β_2	δ	$\sqrt{\beta_1}$	β_2	δ
0.05	4.21	174.2	0.55	4.96	16.26	1.1	7.47	8.73
0.10	4.22	87.14	0.60	5.11	14.98	1.2	8.16	8.13
0.15	4.26	58.16	0.65	5.27	13.90	1.3	8.94	7.63
0.20	4.30	43.69	0.70	5.45	12.98	1.4	9.81	7.21
0.25	4.35	35.02	0.75	5.65	12.18	1.5	10.79	6.86
0.30	4.42	29.26	0.80	5.86	11.49	1.6	11.88	6.55
0.35	4.50	25.15	0.85	6.08	10.88	1.7	13.11	6.28
0.40	4.60	22.08	0.90	6.32	10.35	1.8	14.47	6.04
0.45	4.71	19.70	0.95	6.58	9.87	1.9	16.00	5.84
0.50	4.83	17.80	1.00	6.86	9.45	2.0	17.71	5.65

(For $\sqrt{\beta_1} = 0$, $\beta_2 = 4.2$ and $\delta = \infty$)

The $(\sqrt{\beta_1}, \beta_2)$ line corresponding to $\delta = 1$, which is the 'lower' boundary of the region with U-shaped curves, is shown in Figure 1. Apart from (3.2), there are no J - or reverse-J shaped curves in the L_B system.

Following an analysis similar to that leading to (2.4) we obtain

$$\mu_r' = \int_{-\infty}^{\infty} \{1 + e^{-(z-\gamma)/\delta}\}^{-r} e^z(1 + e^z)^{-2} dz$$

$$= \int_0^1 \{1 + e^{\Omega} (1 - u)^{1/\delta} u^{-1/\delta}\}^{-r} du \qquad (3.5)$$

with $\Omega = \gamma/\delta$, as in Section 2.

Generally, the integral in (3.5) must be evaluated by quadrature. (Of course, for some special cases - such as $\delta = 1$ - explicit solutions can be obtained.) Expansion of the integrand as a power series in $(1-u^{-1})^{1/\delta}$ cannot be valid over the whole range of integration. Dichotomy of the interval (0,1) according as $(1-u^{-1})^{1/\delta} \gtrless e^{\Omega}$ (i.e. $u \lesseqgtr (1+e^{-\Omega})^{-1}$) leads to valid expansions but the resulting incomplete beta functions must themselves (in general) be evaluated by quadrature.

4. The L_U System

If the transformation (1.3) is valid, the pdf of Y is

$$f_Y(y) = \frac{\delta e^{\gamma}}{(y^2+1)^{\frac{1}{2}}} \cdot \frac{(y+\sqrt{(y^2+1)})^{\delta}}{\{1+e^{\gamma}(y+\sqrt{(y^2+1)})^{\delta}\}^2} \qquad (4.1)$$

This curve is unimodal, with mode at the unique value of y satisfying the equation

$$\delta\{1 - e^{\gamma}(y + \sqrt{(y^2+1)})\} = y(y^2+1)^{-\frac{1}{2}} \qquad (4.2)$$

Note that, since $\delta > 0$, the left hand side of (4.2) is a monotonic decreasing function of y; the right hand side is monotonic increasing.

The r-th moment of Y about zero is

$$\mu_r' = 2^{-r}\int_{-\infty}^{\infty}\{e^{(z-\gamma)/\delta} - e^{-(z-\gamma)/\delta}\}^r e^z(1+e^z)^{-2}\,dz$$

$$= 2^{-r}\sum_{j=0}^{r}(-1)^j\binom{r}{j}e^{-(r-2j)\Omega}\int_{-\infty}^{\infty}e^{(r-2j)z/\delta}e^z(1+e^z)^{-2}\,dz$$

$$= 2^{-r}\sum_{j=0}^{r}(-1)^j\binom{r}{j}e^{-(r-2j)\Omega}(r-2j)\theta\operatorname{cosec}(r-2j)\theta \qquad (4.3)$$

provided $r < \delta$. If $r \geq \delta$, μ_r' is infinite. When $r = 2j$, '0 cosec 0' is interpreted as $\lim_{\theta\to 0}\theta\operatorname{cosec}\theta = 1$. (As in Section 2, $\Omega = \gamma/\delta$ and $\theta = \pi/\delta$.)

Since $(-\alpha)\operatorname{cosec}(-\alpha) = \alpha\operatorname{cosec}\alpha$, (4.3) can be simplified.

For r even:

$$\mu_r' = 2^{-r}(-1)^{\frac{1}{2}r}\binom{r}{\frac{1}{2}r} + 2^{-(r-1)}\sum_{j=0}^{\frac{1}{2}r-1}(-1)^j\binom{r}{j}(r-2j)\theta\operatorname{cosec}(r-2j)\theta\cosh(r-2j)\Omega \qquad (4.4)$$

For r odd:

$$\mu_r' = -2^{-(r-1)}\sum_{j=0}^{\frac{1}{2}(r-1)}(-1)^j\binom{r}{j}(r-2j)\theta\operatorname{cosec}(r-2j)\theta\sinh(r-2j)\Omega \qquad (4.5)$$

In particular

$$\begin{aligned}
\mu_1' &= -\theta\operatorname{cosec}\theta\sinh\Omega\\
\mu_2' &= \theta\operatorname{cosec}2\theta\cosh 2\Omega - \frac{1}{2}\\
\mu_3' &= -\frac{3}{4}\theta(\operatorname{cosec}3\theta\sinh 3\Omega - \operatorname{cosec}\theta\sinh\Omega)\\
\mu_4' &= \frac{1}{2}\theta(\operatorname{cosec}4\theta\cosh 4\Omega - 2\operatorname{cosec}2\theta\cosh 2\Omega) + \frac{3}{8}
\end{aligned} \qquad (4.6)$$

so that

$$\begin{aligned}
E[Y] &= -\theta\operatorname{cosec}\theta\sinh\Omega\\
\operatorname{Var}(Y) &= \tfrac{1}{2}\{(\theta\operatorname{cosec}\theta)^2 - 1\} + \tfrac{1}{2}\theta\operatorname{cosec}^2\theta(\tan\theta - \theta)\cosh 2\Omega
\end{aligned} \qquad (4.7)$$

5. Fitting the Distributions

5.1. General. The methods of fitting described by Johnson (1949) for the S systems are also applicable to the L systems, with the advantage that the cdf of the logistic is simpler than that of the normal distribution.

Introducing the location and scale parameters ξ, λ respectively, by the transformation $Y = (X-\xi)/\lambda$ we obtain a three parameter family for L_L:

$$Z = \gamma + \delta \, \ell n \, (X-\xi) \qquad (X > \xi) \tag{5.1}$$

and four parameter families

$$Z = \gamma + \delta \, \ell n \, \left(\frac{X-\xi}{\xi+\lambda-X}\right) \qquad (\xi < X < \xi + \lambda) \tag{5.2}$$

$$Z = \gamma + \delta \, \ell inh^{-1} \, \left(\frac{X-\xi}{\lambda}\right) \tag{5.3}$$

for L_B, L_U respectively.

We consider fitting by the methods of moments, percentile points and maximum likelihood.

Moments. If four parameters are to be fitted, the first four moments of the distribution of X are equated to those of the fitted curve. It is convenient to use them in the form of mean (μ_1'), the variance (μ_2) and the moment ratios $\sqrt{\beta_1}$ ($= \mu_3/\mu_2^{3/2}$) and β_2 ($= \mu_4/\mu_2^2$). Since $\sqrt{\beta_1}$ and β_2 are determined by γ, δ and (for L_B and L_U, as for S_B and S_U) conversely, the first step is to determine γ, δ from the specified values of $\sqrt{\beta_1}$ and β_2. This could be done using special tables (Table 1,4), or of course a computer program. Once these values are found the values of ξ and λ are determined from the equations

$$\begin{aligned} E[X|\gamma,\ \delta] &= \xi + \lambda E[Y|\gamma,\ \delta] \\ \operatorname{var}(X|\gamma,\ \delta) &= \lambda^2 \operatorname{var}(Y|\gamma,\ \delta) \end{aligned} \tag{5.4}$$

wrt μ

$$= \int_{-\infty}^{\infty} \left[F_n(e^{\sigma x+\theta} + \mu) - F_0(x) \right] dF_n(e^{\sigma x+\theta} + \mu)$$

$$e^{\sigma x+\theta} + \mu = e^{\sigma y+\theta} \qquad \sigma e^{\sigma x+\theta} dx =$$

$$x = \frac{\ln\left[e^{\sigma y+\theta} - \mu\right] - \theta}{\sigma}$$

$$dx = \left[\frac{\sigma e^{\sigma y+\theta}}{e^{\sigma y+\theta} - \mu}\right] dy$$

$$\int_{\mu}^{\infty} g(\mu,x)\,dx$$

$$\int_{\mu+h}^{\infty} g(\mu+h,x)\,dx - \int_{\mu}^{\infty} g(\mu,x)\,dx$$

$$\int_{\mu+h}^{\mu} g(\mu+h,x)\,dx + \int_{\mu}^{\infty} \left[g(\mu+h,x) - g(\mu,x)\right] dx$$

$$\downarrow$$

$$g(\mu+h,x)\,[\mu - (\mu+h)] \qquad \int_{\mu}^{\infty} g'(\mu,x)\,dx$$

$$\downarrow$$

$$-g(\mu,\mu)$$

$$\int_{\mu} 2[\qquad] [-f_0(\qquad)\left(\frac{1}{\sigma} - \frac{1}{(y-\mu)}\right)\left(\frac{1}{y-\mu}\right)$$

$$+[\qquad]^2$$

eq. 1

$$\frac{1}{\sigma} \int_{-\infty} \left[F_n\left(e^{\alpha x+\theta} + \mu\right) - F_0(\,)\right] f_0(x)\, e^{-(\sigma x+\theta)}\,dx = 0$$

Some Applications of Minimum Distance Estimation

$Y_i = \gamma + \delta \ln X_i \sim$ logistic

$Y = \ln X \sim$ logistic unknown scale or location

$\ln$

$F_n(\sigma x + \theta)$

$$Y = \ln \frac{(X-\mu)}{\sigma} - \theta$$

$$\int_{-\infty}^{\infty} \left[\frac{1}{n} \sum I\left(\ln \frac{(X_i-\mu)}{\sigma} - \theta \le x \right) - F_0(x) \right]^2 dx$$

$$X_i \le e^{(\sigma x + \theta)} + \mu$$

$$\int_{-\infty}^{\infty} \left[F_n\left(e^{\sigma x + \theta} + \mu \right) - F_0(x) \right]^2 dx$$

$$y = e^{\sigma x + \theta} + \mu$$

$$dy = \sigma e^{\sigma x + \theta} dx = \sigma(y-\mu) dx$$

$$\frac{1}{\sigma} \int_{\mu}^{\infty} \left[F_n(y) - F_0\left(\frac{\ln(y-\mu)}{\sigma} - \theta \right) \right]^2 \left(\frac{1}{y-\mu} \right) dy$$

wrt θ

$$\frac{1}{\sigma}\int_{\mu}^{\infty}\left[F_n \qquad\qquad\right]\frac{1}{\sigma}f_0(\qquad)\frac{1}{y-\mu}\,dy + \cdots = 0$$

$$\int_{-\infty}^{\infty}\left[F_n(e^{\sigma x+\theta}+\mu)-F_0(x)\right]f_0(x)\,dx = 0$$

wrt σ

$\ln(y-\mu) = \sigma x + \theta$

$$\frac{1}{\sigma}\int_{\mu}^{\infty}\left[\qquad\right]\left[-f_0(\qquad)\right]\left[-\frac{\ln(y-\mu)+\theta}{\sigma^2}\right]\left(\frac{1}{y-\mu}\right)dy$$

$$= \int_{-\infty}^{\infty}\left[\qquad\right]x\,f_0(x)\,dx$$

<u>Logistic</u>

① Use min-D and Anderson-Darling

② Use m.l. and Anderson-Darling

① gives up a little eff. in est. σ. However, instead of two computational steps m.l. + GOF, we just do one.

② Robustness of $\hat{\sigma}$:

IC of M.L. $\sim \psi_0(x/\sigma)(x/\sigma)$

$$\psi_0(x) = -\frac{f_0'}{f_0}(x) \sim F_0(x) - \tfrac{1}{2}$$

So IC [illegible] $\sim x$

IC of Min-D $\sim \int_{-\infty}^{x} y f_0(y)\,dy$ is bounded

Percentile Points. If four parameters are to be fitted, then (using (1.6)), four equations of form

$$Z_{P_j} = \ln\left(\frac{P_j}{1-P_j}\right) = \tilde{\gamma} + \tilde{\delta}\, f\left(\frac{\hat{x}_{P_j} - \tilde{\xi}}{\tilde{\lambda}}\right) \quad (j = 1,2,3,4) \tag{5.5}$$

where $\hat{X}_{P_j}$ is the estimated 100 P_j% point of the distribution of X, have to be solved for $\tilde{\gamma}$, $\tilde{\delta}$, $\tilde{\xi}$ and $\tilde{\lambda}$.

Solutions of especially simple form are obtained by taking symmetric percentiles with $P_3 = 1-P_2$; $P_4 = 1-P_1$.

It is interesting to note that with this choice of P's, provided the distribution of Z is symmetrical about zero then the solutions of (5.5) for $\tilde{\xi}$ and $\tilde{\lambda}$ must be linked by the same equation

$$f\left(\frac{\hat{x}_{P_1}-\tilde{\xi}}{\tilde{\lambda}}\right) + f\left(\frac{\hat{x}_{1-P_1}-\tilde{\xi}}{\tilde{\lambda}}\right) = f\left(\frac{\hat{x}_{P_2}-\tilde{\xi}}{\tilde{\lambda}}\right) + f\left(\frac{\hat{x}_{1-P_2}-\tilde{\xi}}{\tilde{\lambda}}\right) \tag{5.6}$$

whatever the actual distribution of Z.

In fact, from

$$\tilde{\gamma} + \tilde{\delta}\, f\left(\frac{\hat{x}_{P_j}-\tilde{\xi}}{\tilde{\lambda}}\right) = -Z_{1-P_j} = -\tilde{\gamma} - \tilde{\delta}\, f\left(\frac{\hat{x}_{1-P_j}-\tilde{\xi}}{\tilde{\lambda}}\right) \quad (j = 1,2)$$

it follows that the common value of each side of (5.6) is $-2\tilde{\Omega}$.

Another relation is

$$Z_{1-P_1}\left\{f\left(\frac{x_{1-P_2}-\tilde{\xi}}{\tilde{\lambda}}\right) - f\left(\frac{\hat{x}_{P_2}-\tilde{\xi}}{\tilde{\lambda}}\right)\right\} = Z_{1-P_2}\left\{f\left(\frac{\hat{x}_{1-P_1}-\tilde{\xi}}{\tilde{\lambda}}\right) - f\left(\frac{\hat{x}_{P_1}-\tilde{\xi}}{\tilde{\lambda}}\right)\right\}. \tag{5.7}$$

This relation does depend on the actual distribution of Z. Equations (5.6) and (5.7) suffice to determine $\tilde{\xi}$ and $\tilde{\lambda}$.

In certain cases (e.g. for L_B - see below) equation (5.6) can be solved explicitly giving $\tilde{\lambda}$ as a function of $\tilde{\xi}$.

If ξ and λ are known, two percentile points suffice to determine γ and δ. Knowledge of ξ and/or λ is more usual when a bounded curve is being fitted. Fitting of L_B in these cases is discussed in Section 5.3.

Maximum Likelihood. If ξ and λ are known, the transformed values of $f((x-\xi)/\lambda)$ can be used, and methods appropriate to fitting logistic distributions (e.g. Johnson and Kotz (1970, Chapter 23)) can be applied. If ξ and λ are not known, a series of pairs of values of these parameters can be used and the corresponding maximized likelihoods calculated, the maximum likelihood estimates being found by trial and error - or, of course, a computer program which directly maximizes the likelihood function can be used. Except in special cases, maximum likelihood calculations are rather lengthy, but not impracticably so.

5.2. Fitting the L_L System. If ξ is known, the transformed values

$$Y_i = \log(X_i - \xi)$$

can be used, and the parameters γ and δ fitted by methods appropriate to fitting a logistic distribution.

When ξ is not known there are three parameter values to be fitted (ξ, γ, and δ). With the method of *percentile points*, it is natural to use the sample median $\hat{X}_{0.5}$ and the upper and lower sample 100 P% points, $\hat{X}_{1-P}$ and $\hat{X}_P$ respectively. Noting (1.6), this leads to the equations

$$\begin{aligned} 0 &= \hat{\gamma} + \hat{\delta}\,\ell n(\hat{X}_{0.5} - \tilde{\xi}) \\ -\ell n\{P/(1-P)\} &= \tilde{\gamma} + \tilde{\delta}\,\ell n(\hat{X}_{1-P} - \tilde{\xi}) \qquad (5.8) \\ \ell n\{P/(1-P)\} &= \tilde{\gamma} + \tilde{\delta}\,\ell n(\hat{X}_P - \tilde{\xi}) \end{aligned}$$

which have the unique solutions

$$\tilde{\xi} = \frac{\hat{X}_{0.5}^2 - \hat{X}_{1-P}\hat{X}_P}{2\hat{X}_{0.5} - \hat{X}_{1-P} - \hat{X}_P}$$

$$\tilde{\delta} = \frac{-2\ \ell n\{P/(1-P)\}}{\ell n\{(\hat{X}_{1-P} - \tilde{\xi})(\hat{X}_P - \tilde{\xi})\}} \qquad (5.9)$$

$$\tilde{\gamma} = -\tilde{\delta}\ \ell n(\hat{X}_{0.5} - \tilde{\xi})$$

provided $\tilde{\xi} < \hat{X}_P$.

Of course, several values of P may be used. The resulting estimates $\tilde{\xi}$, $\tilde{\gamma}$ and $\tilde{\delta}$ should be reasonably consistent with each other if an L_L distribution is suitable. If only one set is to be used, P = 0.10 (i.e. the sample median, and upper and lower sample deciles) would seem to be a good choice.

We now come to the method of *moments*. The skewness parameter $\sqrt{\beta_1} = \mu_3/\mu_2^{3/2}$ is

$$\sqrt{\beta_1(\theta)} = \frac{3\theta \text{ cosec } 3\theta - 6\theta^2 \text{ cosec } \theta \text{ cosec } 2\theta + 3\theta^3 \text{ cosec}^3\theta}{\{\theta \text{ cosec}^2\theta\ (\tan\theta - \theta)\}^{3/2}}$$

$$= \frac{3}{\sqrt{\theta}} \cdot \frac{\text{cosec } 3\theta - \theta \text{ cosec}^2\theta \text{ sec } \theta + \theta^2 \text{ cosec}^3\theta}{\text{cosec}^3\theta\ (\tan\theta - \theta)^{3/2}} \qquad (5.10)$$

Equating (5.3) to $\sqrt{b_1} = m_3/m_2^{3/2}$, the sample skewness ratio, and solving for θ gives the moment estimator, $\tilde{\theta}$, of θ. The function (5.3) is an increasing function of θ, and tends to infinity as $\theta \to \infty$, so provided $\sqrt{b_1} > 0$, there will be a unique value for $\tilde{\theta}$. The estimator for δ is

$$\tilde{\delta} = \pi/\tilde{\theta}\ .$$

<u>Example 1</u>. As can be seen from Figure 1, the L_L line is in the Pearson Type IV region. Suppose we want to use an L_L curve to approximate a Type IV with $\sqrt{\beta_1} = 0.4$, $\beta_2 = 4.6$. From Table 1 (with extra decimal places), $\delta = 22.0803$

yields $\sqrt{\beta_1} = 0.4$, $\beta_2 = 4.5991$ with $E[Y|\gamma=0, \delta) = 1.0034$, $\delta(Y|\gamma=0, \delta) = 0.0828$ whence $\gamma = -55.0$, $\xi = -12.1184$. Standardized percentiles of the fitted distribution are compared with those of the Type IV distribution (obtained by interpolation in the tables of Bouver and Bargmann (1974)) in Table 2.

Table 2. Comparison of Standardized Type IV and L_L Percentiles ($\beta_1 = 0.4$, $\beta_2 = 4.6$)

%	0.1	0.25	0.5	1	2.5	5	10	25	50
Type IV	-3.315	-2.887	-2.574	.2.262	-1.854	-1.532	-1.186	-0.641	-0.046
L_L	-3.285	-2.910	-2.615	-2.310	-1.887	-1.548	-1.185	-.627	-.041

%	75	90	95	97.5	99	99.5	99.75	99.9
Type IV	0.587	1.237	1.686	2.127	2.717	3.182	3.674	4.346
L_L	0.575	1.223	1.682	2.139	2.753	3.231	3.723	4.395

5.3. Fitting the L_B System.

Moments. If all four parameters $(\xi,\lambda,\gamma,\delta)$ have to be estimated, it is necessary to have special tables (i) to determine γ, δ from $\sqrt{\beta_1}$ and β_2 and (ii) giving values of $E[Y|\gamma, \delta)$, and $S.D(Y|\gamma, \delta)$ for these values of γ and δ. Such tables are in preparation but as yet they are not sufficiently extensive for practical use.

Percentile Points. For L_B curves, introducing the notation $\tilde{\xi}_P = \hat{X}_P - \tilde{\xi}$, equations (5.6) and (5.7) become

$$\frac{\tilde{\xi}_{P_1}\ \tilde{\xi}_{1-P_1}}{(\tilde{\lambda}-\tilde{\xi}_{P_1})(\tilde{\lambda}-\tilde{\xi}_{1-P_1})} = \frac{\tilde{\xi}_{P_2}\ \tilde{\xi}_{1-P_2}}{(\tilde{\lambda}-\tilde{\xi}_{P_2})(\tilde{\lambda}-\tilde{\xi}_{1-P_2})} \tag{5.11}$$

and

$$z_{1-P_1}\ \ell n \left| \frac{\tilde{\xi}_{P_2}(\tilde{\lambda}-\tilde{\xi}_{1-P_2})}{(\tilde{\lambda}-\tilde{\xi}_{P_2})\ {}_{1-P_2}} \right| = z_{1-P_2}\ \ell n \left| \frac{\tilde{\xi}_{P_1}(\tilde{\lambda}-\tilde{\xi}_{1-P_1})}{(\tilde{\lambda}-\tilde{\xi}_{P_1})\tilde{\xi}_{1-P_1}} \right| \tag{5.12}$$

respectively. From (5.11)

$$\tilde{\lambda} = \frac{\tilde{\xi}_{P_1}\tilde{\xi}_{1-P_1}(\tilde{\xi}_{P_2}+\tilde{\xi}_{1-P_2}) - \tilde{\xi}_{P_2}\tilde{\xi}_{1-P_2}(\tilde{\xi}_{P_1}+\tilde{\xi}_{1-P_1})}{\tilde{\xi}_{P_1}\tilde{\xi}_{1-P_1} - \tilde{\xi}_{P_2}\tilde{\xi}_{1-P_2}} \tag{5.13}$$

These equations may be solved by a process of trial and error, illustrated in Example 2(a) below.

Techniques of solving similar equations for fitting S_B curves, devised by Bukač (1972), Mage (1978) and Shapiro (1978), may also be used for L_B. Since P_1 and P_2 may be chosen arbitrarily, it is possible to arrange that Z_{1-P_1}/Z_{1-P_2} is an integer. We do this in Example 2(a), though it is not necessary to do so.

Devices for simplifying estimation of the four parameters for S_B (Bukač (1972), Mage (1978), Slifker and Shapiro (1979)) can equally be applied to fitting L_B. We do not pursue these matters here.

In Example 2(b) and 2(c) we fit the same data as in 2(a), when both ξ and λ are assumed known, and when ξ (but not λ) is assumed known.

If the range of variation (and so both ξ and λ) be known, then only two sample percentiles are needed to estimate γ and δ. It is convenient to choose them symmetrically and use the equations

$$\begin{aligned} -Z_{1-P} &= \tilde{\gamma} + \tilde{\delta} \ln \left(\frac{\xi_P}{\lambda-\xi_P}\right) \\ Z_{1-P} &= \tilde{\gamma} + \tilde{\delta} \ln \left(\frac{\xi_{1-P}}{\lambda-\xi_{1-P}}\right) \end{aligned} \tag{5.14}$$

where $\xi_P = \hat{X}_P - \xi$, whence

$$\tilde{\delta} = 2\, Z_{1-P} \left[\ln \frac{\xi_{1-P}(\lambda-\xi_P)}{\xi_P(\lambda-\xi_{1-P})}\right]^{-1} \tag{5.15}$$

and

$$\tilde{\gamma} = -\frac{1}{2}\tilde{\delta} \ln \frac{\xi_P\, \xi_{1-P}}{(\lambda-\xi_P)(\lambda-\xi_{1-P})} \quad . \tag{5.16}$$

If only one end point - say the lower (ξ) - is known, then three sample percentiles are needed, from which to estimate λ, γ and δ. It is convenient to take the sample median $\hat{X}_{0.5}$ in addition to $\hat{X}_P$ and $\hat{X}_{1-P}$. From the resulting equations we obtain

$$\left(\frac{\xi_{0.5}}{\tilde{\lambda}-\xi_{0.5}}\right)^2 = \frac{\xi_P\ \xi_{1-P}}{(\tilde{\lambda}-\xi_P)(\tilde{\lambda}-\xi_{1-P})}$$

whence

$$\tilde{\lambda} = \frac{\xi_{0.5}\ \{\xi_{0.5}\ (\xi_P + \xi_{1-P}) - 2\ \xi_P\ \xi_{1-P}\}}{\xi_{0.5}^2 - \xi_P\ \xi_{1-P}} \tag{5.17}$$

$\tilde{\gamma}$ and $\tilde{\delta}$ are then calculated from (5.15) and (5.16).

Example 2. (a) We will use the data in Table 6-4 of Hahn and Shapiro (1967). For fitting four parameters we use, in addition to the values $\hat{X}_{.09} = 0.84$, $\hat{X}_{.91} = 1.42$ employed by Hahn and Shapiro for fitting S_B, the values $\hat{X}_{.3162} = 0.97$, $\hat{X}_{.6838} = 1.18$. (These values are chosen to make
$Z_{.91} = \ell n\, \frac{0.91}{0.09} = 2.3136 = 3\ \ell n\, \frac{0.6838}{0.3162} = 3\ Z_{.6838}$)

The trial and error calculations are set out briefly below.

ξ	$\tilde{\lambda}(\xi)$	(1) $g(0.09)$	(2) $g(0.3162)$	(1)/(2)
0.5	6.32	-1.097	-0.424	2.59
0.75	0.95	-3.130	-1.009	3.10
0.72	1.12	-2.631	-0.886	2.97
0.73	1.06	-2.779	-0.924	3.01

$$g(P) = \ell n \left| \frac{\xi_P(\tilde{\lambda}-\xi_{1-P})}{\xi_{1-P}(\tilde{\lambda}-\xi_P)} \right|$$

We want $g(0.09)/g(0.3162) = 3$.

Hence we obtain $\tilde{\xi} = 0.73$, $\tilde{\lambda} = 1.06$. These lead to estimates $\tilde{\gamma} = 1.276$, $\tilde{\delta} = 1.665$. Percentage distribution is shown in column (3) of Table 3.

(b) If we take $\xi = 0.5$, $\lambda = 1.5$ and use $\hat{X}_{0.09}$ and $\hat{X}_{0.91}$, we obtain

$$\tilde{\gamma} = 1.049, \quad \tilde{\delta} = 2.740 .$$

Since we are using the same values for ξ and λ as Hahn and Shapiro used in fitting an S_B curve to the same data (and so the same $\hat{X}_{0.09}$ and $\hat{X}_{0.91}$), the values are each in the same ratio (2.3136/1.3408 = 1.726 - note that $\Phi(1.3408) = 0.91$) to the corresponding values for their fitted S_B.

Percentage distributions of the fitted L_B are shown in column (5) of Table 3. Column (4) contains values obtained by Hahn and Shapiro for their fitted S_B.

(c) If $\hat{X}_{0.5} = 1.07$ is used in addition to $\hat{X}_{0.09}$ and $\hat{X}_{0.91}$ we obtain from (5.17)

$$\tilde{\lambda} = 4.36$$

which, of course, agrees with the value obtained by Hahn and Shapiro when fitting an S_B. The values for γ and δ,

$$\tilde{\gamma} = 7.614 \quad \text{and} \quad \tilde{\delta} = 4.019$$

are again in the same ratio (1.726) to Hahn and Shapiro's S_B values.

Percentage distributions of the fitted L_B are shown in column (7) of Table 3. Column (6) contains values obtained by Hahn and Shapiro for their fitted S_B.

Comparing the three L_B fits (columns (3), (5) and (7)) it is noteworthy how similar shaped curves can be obtained with quite different values for γ and δ, by appropriate adjustment of ξ and λ (and conversely).

Table 3 - Comparison of Fitted S_B and L_B Curves

(1) Production Time (min)	(2) Actual %	(3) L_B	(4) S_B	(5) L_B	(6) S_B	(7) L_B
≤ 0.695	0.9	-	0.9	1.5	0.4	0.9
0.695-0.795	3.7	3.7	4.7	4.2	4.5	4.2
0.795-0.895	12.6	14.0	10.3	8.8	12.8	11.0
0.895-0.995	18.4	18.8	15.2	14.6	18.7	18.4
0.995-1.095	18.8	18.6	17.4	18.4	19.1	20.6
1.095-1.195	15.8	15.3	16.9	18.1	15.7	16.8
1.195-1.295	12.2	11.3	13.9	14.3	11.3	11.2
1.295-1.395	7.6	7.8	10.2	9.4	7.4	6.8
1.395-1.495	5.0	5.1	6.1	5.5	4.5	4.0
1.495-1.595	2.8	3.1	3.1	2.9	2.6	2.4
1.595-1.695	1.1	1.7	1.0	1.4	1.4	1.4
1.695-1.795	0.9	0.6	0.3	0.6	0.8	0.9
≥ 1.795	0.2	-	0.0	0.2	0.8	1.5
	ξ	0.73*	0.5		0.5	
	λ	1.06*	1.5		4.36*	
	γ	1.276*	0.608*	1.049*	4.411*	7.614*
	δ	1.665*	1.587*	2.740*	2.328*	4.019*

*estimated

5.4. Fitting L_U Distributions

Moments: Table 4 gives values of δ and $\Omega(= \gamma/\delta)$, corresponding to $\sqrt{\beta_1} = 0.00(0.05)1(\cdot 1)2.0$ combined (for each $\sqrt{\beta_1}$) with twenty values of β_2 increasing by intervals of 0.2, starting from a value just 'below' the L_L line. The simultaneous nonlinear equations

$$\begin{aligned} \sqrt{\beta_1(\gamma,\delta)} &= \sqrt{\beta_1} \\ \beta_2(\gamma,\delta) &= \beta_2 \end{aligned} \tag{5.18}$$

were solved for γ and δ using Brown's (1967) algorithm. The search for values of Ω and δ continued until $|\sqrt{\beta_1(\gamma,\delta)} - \sqrt{\beta_1}| \le 10^{-6}$ and $|\beta_2(\gamma,\delta) - \beta_2| \le 10^{-6}$. The use of the table is illustrated by the following example.

Example 3. We will compare a standardized Pearson Type IV curve with $\sqrt{\beta_1} = 0.9$, $\beta_2 = 8.6$ against an L_U curve having the same first four moments.

From Table 4 we obtain $\delta = 6.0151$ and $\Omega = -0.5250$, where $\gamma = 6.0151 \times (-0.5250) = -3.1580$. Using (4.7) we obtain

$$E[Y] = 0.5753; \quad S.D.[Y] = 0.3712 .$$

Using (5.4) (with 'observed' mean zero, and standard deviation 1) we obtain $\lambda = (0.3712)^{-1} = 2.6940$, $\xi = 0 - (2.6940 \times 0.5753) = -1.5498$. So the fitted L_U curve is defined by the statement that

$$Z = -3.1580 + 6.0151 \sinh^{-1} \{(X + 1.5498/2.6940\}$$

Table 4: To Facilitate Fitting L_U Curves

$\sqrt{\beta_1} = 0.00$

β_2	δ	Ω	μ	σ
4.40	16.1153	0.0000	0.0000	0.1136
4.60	11.7442	0.0000	0.0000	0.1571
4.80	9.8648	0.0000	0.0000	0.1884
5.00	8.7752	0.0000	0.0000	0.2131
5.20	8.0510	0.0000	0.0000	0.2337
5.40	7.5295	0.0000	0.0000	0.2512
5.60	7.1336	0.0000	0.0000	0.2665
5.80	6.8216	0.0000	0.0000	0.2800
6.00	6.5687	0.0000	0.0000	0.2920
6.20	6.3590	0.0000	0.0000	0.3028
6.40	6.1822	0.0000	0.0000	0.3126
6.60	6.0308	0.0000	0.0000	0.3215
6.80	5.8998	0.0000	0.0000	0.3297
7.00	5.7850	0.0000	0.0000	0.3373
7.20	5.6837	0.0000	0.0000	0.3442
7.40	5.5935	0.0000	0.0000	0.3507
7.60	5.5127	0.0000	0.0000	0.3567
7.80	5.4398	0.0000	0.0000	0.3623
8.00	5.3738	0.0000	0.0000	0.3676
8.20	5.3137	0.0000	0.0000	0.3725

$\sqrt{\beta_1} = 0.05$

β_2	δ	Ω	μ	σ
4.40	16.2968	-0.0909	0.0916	0.1127
4.60	11.8071	-0.0638	0.0646	0.1565
4.80	9.8988	-0.0520	0.0529	0.1879
5.00	8.7974	-0.0451	0.0461	0.2128
5.20	8.0668	-0.0404	0.0414	0.2334
5.40	7.5415	-0.0369	0.0380	0.2510
5.60	7.1432	-0.0343	0.0354	0.2663
5.80	6.8294	-0.0322	0.0333	0.2798
6.00	6.5752	-0.0304	0.0316	0.2918
6.20	6.3647	-0.0290	0.0302	0.3026
6.40	6.1870	-0.0277	0.0289	0.3124
6.60	6.0351	-0.0266	0.0279	0.3214
6.80	5.9036	-0.0257	0.0270	0.3296
7.00	5.7884	-0.0249	0.0262	0.3372
7.20	5.6868	-0.0241	0.0254	0.3441
7.40	5.5963	-0.0235	0.0248	0.3506
7.60	5.5153	-0.0229	0.0242	0.3566
7.80	5.4422	-0.0223	0.0236	0.3623
8.00	5.3760	-0.0218	0.0232	0.3675
8.20	5.3157	-0.0214	0.0227	0.3725

$\sqrt{\beta_1} = 0.10$

β_2	δ	Ω	μ	σ
4.40	16.8772	-0.1905	0.1927	0.1103
4.60	12.0026	-0.1306	0.1325	0.1549
4.80	10.0035	-0.1057	0.1076	0.1867
5.00	8.8648	-0.0912	0.0933	0.2118
5.20	8.1148	-0.0815	0.0837	0.2325
5.40	7.5779	-0.0745	0.0767	0.2502
5.60	7.1720	-0.0690	0.0713	0.2656
5.80	6.8530	-0.0647	0.0671	0.2792
6.00	6.5950	-0.0612	0.0636	0.2913
6.20	6.3815	-0.0582	0.0607	0.3021
6.40	6.2017	-0.0557	0.0582	0.3120
6.60	6.0481	-0.0535	0.0560	0.3210
6.80	5.9150	-0.0516	0.0542	0.3292
7.00	5.7987	-0.0499	0.0525	0.3368
7.20	5.6961	-0.0485	0.0510	0.3438
7.40	5.6048	-0.0471	0.0497	0.3503
7.60	5.5230	-0.0459	0.0485	0.3564
7.80	5.4494	-0.0448	0.0474	0.3620
8.00	5.3826	-0.0438	0.0464	0.3673
8.20	5.3219	-0.0429	0.0455	0.3722

$\sqrt{\beta_1} = 0.15$

β_2	δ	Ω	μ	σ
4.40	18.0083	-0.3121	0.3188	0.1065
4.60	12.3522	-0.2040	0.2076	0.1523
4.80	10.1867	-0.1628	0.1661	0.1846
5.00	8.9813	-0.1396	0.1430	0.2100
5.20	8.1972	-0.1243	0.1277	0.2311
5.40	7.6401	-0.1132	0.1167	0.2489
5.60	7.2211	-0.1048	0.1084	0.2645
5.80	6.8931	-0.0981	0.1017	0.2782
6.00	6.6285	-0.0926	0.0963	0.2904
6.20	6.4102	-0.0881	0.0918	0.3013
6.40	6.2266	-0.0842	0.0880	0.3113
6.60	6.0698	-0.0809	0.0847	0.3203
6.80	5.9344	-0.0779	0.0818	0.3286
7.00	5.8161	-0.0754	0.0793	0.3362
7.20	5.7118	-0.0731	0.0770	0.3433
7.40	5.6191	-0.0711	0.0750	0.3498
7.60	5.5361	-0.0692	0.0732	0.3559
7.80	5.4614	-0.0676	0.0715	0.3616
8.00	5.3938	-0.0661	0.0700	0.3669
8.20	5.3322	-0.0647	0.0686	0.3719

$\sqrt{\beta_1} = 0.20$

β_2	δ	Ω	μ	σ
4.40	20.0817	-0.4873	0.5089	0.1019
4.60	12.9064	-0.2896	0.2966	0.1486
4.80	10.4634	-0.2259	0.2313	0.1817
5.00	9.1539	-0.1917	0.1967	0.2077
5.20	8.3178	-0.1696	0.1745	0.2290
5.40	7.7305	-0.1539	0.1589	0.2472
5.60	7.2920	-0.1421	0.1471	0.2629
5.80	6.9509	-0.1327	0.1378	0.2768
6.00	6.6767	-0.1252	0.1302	0.2891
6.20	6.4512	-0.1189	0.1240	0.3002
6.40	6.2621	-0.1135	0.1187	0.3102
6.60	6.1009	-0.1089	0.1141	0.3194
6.80	5.9620	-0.1049	0.1101	0.3277
7.00	5.8408	-0.1014	0.1067	0.3354
7.20	5.7341	-0.0983	0.1036	0.3425
7.40	5.6393	-0.0955	0.1008	0.3491
7.60	5.5546	-0.0930	0.0983	0.3552
7.80	5.4784	-0.0907	0.0960	0.3610
8.00	5.4095	-0.0887	0.0940	0.3663
8.20	5.3468	-0.0868	0.0921	0.3713

$\sqrt{\beta_1} = 0.25$

β_2	δ	Ω	μ	σ
4.40	24.2825	-0.8445	0.9511	0.1035
4.60	13.7515	-0.3975	0.4116	0.1444
4.80	10.8595	-0.2985	0.3073	0.1782
5.00	9.3936	-0.2494	0.2567	0.2047
5.20	8.4828	-0.2188	0.2257	0.2265
5.40	7.8528	-0.1975	0.2042	0.2450
5.60	7.3875	-0.1816	0.1882	0.2609
5.80	7.0280	-0.1692	0.1758	0.2750
6.00	6.7408	-0.1592	0.1658	0.2875
6.20	6.5055	-0.1509	0.1575	0.2987
6.40	6.3090	-0.1439	0.1506	0.3089
6.60	6.1420	-0.1380	0.1446	0.3181
6.80	5.9983	-0.1328	0.1395	0.3266
7.00	5.8732	-0.1282	0.1349	0.3344
7.20	5.7633	-0.1242	0.1309	0.3416
7.40	5.6659	-0.1206	0.1273	0.3482
7.60	5.5789	-0.1174	0.1241	0.3544
7.80	5.5008	-0.1145	0.1212	0.3602
8.00	5.4301	-0.1118	0.1185	0.3656
8.20	5.3659	-0.1094	0.1161	0.3707

$\sqrt{\beta_1} = 0.30$

β_2	δ	Ω	μ	σ
4.60	15.0745	-0.5508	0.5833	0.1407
4.80	11.4198	-0.3870	0.4018	0.1742
5.00	9.7192	-0.3157	0.3266	0.2012
5.20	8.7013	-0.2736	0.2831	0.2234
5.40	8.0126	-0.2451	0.2540	0.2423
5.60	7.5109	-0.2242	0.2329	0.2586
5.80	7.1269	-0.2081	0.2166	0.2729
6.00	6.8225	-0.1953	0.2037	0.2856
6.20	6.5745	-0.1847	0.1931	0.2970
6.40	6.3683	-0.1759	0.1842	0.3073
6.60	6.1938	-0.1683	0.1766	0.3167
6.80	6.0440	-0.1618	0.1701	0.3252
7.00	5.9140	-0.1561	0.1643	0.3331
7.20	5.8000	-0.1510	0.1593	0.3404
7.40	5.6992	-0.1466	0.1548	0.3472
7.60	5.6093	-0.1425	0.1508	0.3534
7.80	5.5286	-0.1389	0.1471	0.3593
8.00	5.4558	-0.1356	0.1438	0.3647
8.20	5.3898	-0.1326	0.1408	0.3699
8.40	5.3295	-0.1299	0.1381	0.3747

$\sqrt{\beta_1} = 0.35$

β_2	δ	Ω	μ	σ
4.60	17.3274	-0.8266	0.9291	0.1439
4.80	12.2247	-0.5035	0.5309	0.1705
5.00	10.1592	-0.3954	0.4123	0.1975
5.20	8.9876	-0.3366	0.3500	0.2200
5.40	8.2178	-0.2984	0.3103	0.2393
5.60	7.6672	-0.2711	0.2823	0.2559
5.80	7.2512	-0.2505	0.2612	0.2704
6.00	6.9244	-0.2341	0.2446	0.2834
6.20	6.6602	-0.2208	0.2311	0.2950
6.40	6.4416	-0.2098	0.2199	0.3055
6.60	6.2575	-0.2004	0.2105	0.3150
6.80	6.1001	-0.1923	0.2023	0.3237
7.00	5.9638	-0.1853	0.1953	0.3317
7.20	5.8448	-0.1791	0.1890	0.3391
7.40	5.7396	-0.1736	0.1835	0.3459
7.60	5.6461	-0.1687	0.1786	0.3523
7.80	5.5624	-0.1643	0.1741	0.3582
8.00	5.4869	-0.1603	0.1701	0.3637
8.20	5.4186	-0.1566	0.1664	0.3689
8.40	5.3563	-0.1533	0.1631	0.3738

$\sqrt{\beta_1}$ = 0.40

β_2	δ	Ω	μ	σ
4.80	13.4343	-0.6780	0.7379	0.1699
5.00	10.7623	-0.4972	0.5254	0.1941
5.20	9.3632	-0.4118	0.4316	0.2166
5.40	8.4802	-0.3597	0.3761	0.2360
5.60	7.8638	-0.3238	0.3385	0.2529
5.80	7.4056	-0.2972	0.3109	0.2677
6.00	7.0499	-0.2765	0.2896	0.2809
6.20	6.7648	-0.2599	0.2726	0.2927
6.40	6.5307	-0.2462	0.2585	0.3034
6.60	6.3345	-0.2346	0.2468	0.3131
6.80	6.1676	-0.2248	0.2368	0.3219
7.00	6.0238	-0.2162	0.2281	0.3301
7.20	5.8984	-0.2087	0.2205	0.3376
7.40	5.7880	-0.2021	0.2138	0.3445
7.60	5.6901	-0.1961	0.2078	0.3510
7.80	5.6026	-0.1908	0.2024	0.3570
8.00	5.5239	-0.1860	0.1976	0.3626
8.20	5.4527	-0.1816	0.1931	0.3679
8.40	5.3881	-0.1776	0.1891	0.3728
8.60	5.3290	-0.1740	0.1854	0.3775

$\sqrt{\beta_1}$ = 0.45

β_2	δ	Ω	μ	σ
4.80	15.4222	-1.0249	1.2225	0.1874
5.00	11.6174	-0.6403	0.6934	0.1932
5.20	9.8638	-0.5068	0.5378	0.2137
5.40	8.8176	-0.4330	0.4563	0.2329
5.60	8.1110	-0.3848	0.4044	0.2499
5.80	7.5968	-0.3501	0.3677	0.2649
6.00	7.2035	-0.3238	0.3401	0.2783
6.20	6.8918	-0.3029	0.3185	0.2903
6.40	6.6380	-0.2859	0.3009	0.3012
6.60	6.4269	-0.2717	0.2863	0.3110
6.80	6.2482	-0.2596	0.2739	0.3200
7.00	6.0950	-0.2492	0.2633	0.3283
7.20	5.9619	-0.2402	0.2541	0.3359
7.40	5.8452	-0.2322	0.2459	0.3430
7.60	5.7419	-0.2251	0.2387	0.3496
7.80	5.6499	-0.2187	0.2323	0.3557
8.00	5.5673	-0.2130	0.2264	0.3614
8.20	5.4928	-0.2078	0.2212	0.3667
8.40	5.4252	-0.2031	0.2164	0.3717
8.60	5.3635	-0.1988	0.2120	0.3764

$\sqrt{\beta_1}$ = 0.50

β_2	δ	Ω	μ	σ
5.00	12.9053	-0.8811	1.0095	0.2023
5.20	10.5493	-0.6366	0.6906	0.2132
5.40	9.2579	-0.5250	0.5602	0.2306
5.60	8.4243	-0.4577	0.4850	0.2472
5.80	7.8343	-0.4116	0.4348	0.2622
6.00	7.3916	-0.3775	0.3984	0.2757
6.20	7.0457	-0.3510	0.3704	0.2879
6.40	6.7669	-0.3298	0.3481	0.2989
6.60	6.5370	-0.3122	0.3299	0.3089
6.80	6.3438	-0.2975	0.3146	0.3180
7.00	6.1789	-0.2849	0.3016	0.3265
7.20	6.0365	-0.2740	0.2903	0.3342
7.40	5.9121	-0.2644	0.2805	0.3414
7.60	5.8025	-0.2559	0.2718	0.3480
7.80	5.7050	-0.2484	0.2641	0.3543
8.00	5.6178	-0.2416	0.2571	0.3601
8.20	5.5393	-0.2355	0.2509	0.3655
8.40	5.4681	-0.2299	0.2452	0.3706
8.60	5.4034	-0.2248	0.2400	0.3754
8.80	5.3443	-0.2201	0.2352	0.3799

$\sqrt{\beta_1}$ = 0.55

β_2	δ	Ω	μ	σ
5.00	15.0669	-1.6001	2.3931	0.3150
5.20	11.5340	-0.8409	0.9553	0.2211
5.40	9.8481	-0.6492	0.7078	0.2309
5.60	8.8274	-0.5493	0.5897	0.2456
5.80	8.1321	-0.4854	0.5174	0.2600
6.00	7.6233	-0.4402	0.4676	0.2734
6.20	7.2325	-0.4061	0.4307	0.2855
6.40	6.9219	-0.3792	0.4020	0.2966
6.60	6.6683	-0.3574	0.3789	0.3067
6.80	6.4570	-0.3393	0.3598	0.3160
7.00	6.2779	-0.3239	0.3438	0.3246
7.20	6.1240	-0.3107	0.3300	0.3324
7.40	5.9902	-0.2992	0.3181	0.3397
7.60	5.8728	-0.2891	0.3076	0.3465
7.80	5.7689	-0.2801	0.2983	0.3528
8.00	5.6761	-0.2721	0.2900	0.3587
8.20	5.5928	-0.2648	0.2826	0.3642
8.40	5.5176	-0.2583	0.2758	0.3694
8.60	5.4493	-0.2523	0.2697	0.3742
8.80	5.3870	-0.2469	0.2641	0.3788

$\sqrt{\beta_1}$ = 0.60

β_2	δ	Ω	μ	σ
5.20	13.0664	-1.3000	1.7148	0.2789
5.40	10.6714	-0.8392	0.9550	0.2396
5.60	9.3592	-0.6725	0.7381	0.2471
5.80	8.5111	-0.5784	0.6252	0.2593
6.00	7.9114	-0.5160	0.5536	0.2719
6.20	7.4611	-0.4708	0.5031	0.2837
6.40	7.1090	-0.4362	0.4651	0.2947
6.60	6.8252	-0.4086	0.4353	0.3048
6.80	6.5911	-0.3861	0.4111	0.3141
7.00	6.3943	-0.3672	0.3910	0.3228
7.20	6.2264	-0.3511	0.3740	0.3307
7.40	6.0813	-0.3372	0.3595	0.3381
7.60	5.9545	-0.3251	0.3467	0.3450
7.80	5.8427	-0.3144	0.3355	0.3513
8.00	5.7433	-0.3049	0.3256	0.3573
8.20	5.6543	-0.2963	0.3168	0.3629
8.40	5.5742	-0.2886	0.3087	0.3682
8.60	5.5017	-0.2816	0.3015	0.3731
8.80	5.4357	-0.2753	0.2949	0.3778
9.00	5.3753	-0.2694	0.2888	0.3822

$\sqrt{\beta_1}$ = 0.65

β_2	δ	Ω	μ	σ
5.40	11.8992	-1.2277	1.5784	0.2891
5.60	10.0875	-0.8593	0.9848	0.2578
5.80	9.0064	-0.7040	0.7793	0.2624
6.00	8.2758	-0.6121	0.6670	0.2723
6.20	7.7438	-0.5496	0.5939	0.2830
6.40	7.3367	-0.5037	0.5416	0.2935
6.60	7.0138	-0.4681	0.5020	0.3034
6.80	6.7508	-0.4396	0.4707	0.3126
7.00	6.5318	-0.4161	0.4451	0.3212
7.20	6.3464	-0.3963	0.4238	0.3292
7.40	6.1873	-0.3794	0.4058	0.3366
7.60	6.0492	-0.3647	0.3902	0.3435
7.80	5.9279	-0.3519	0.3766	0.3500
8.00	5.8206	-0.3406	0.3646	0.3560
8.20	5.7249	-0.3304	0.3540	0.3617
8.40	5.6390	-0.3213	0.3444	0.3670
8.60	5.5615	-0.3131	0.3358	0.3720
8.80	5.4911	-0.3056	0.3290	0.3768
9.00	5.4269	-0.2988	0.3209	0.3812
9.20	5.3681	-0.2926	0.3144	0.3854

$\sqrt{\beta_1}$ = 0.70

β_2	δ	Ω	μ	σ
5.60	11.1440	-1.2297	1.5848	0.3102
5.80	9.6753	-0.8950	1.0375	0.2765
6.00	8.7476	-0.7431	0.8311	0.2776
6.20	8.0996	-0.6506	0.7153	0.2850
6.40	7.6174	-0.5867	0.6389	0.2939
6.60	7.2427	-0.5392	0.5838	0.3030
6.80	6.9420	-0.5022	0.5419	0.3118
7.00	6.6949	-0.4723	0.5086	0.3202
7.20	6.4878	-0.4476	0.4813	0.3281
7.40	6.3114	-0.4268	0.4586	0.3355
7.60	6.1592	-0.4089	0.4392	0.3424
7.80	6.0265	-0.3934	0.4225	0.3489
8.00	5.9096	-0.3798	0.4079	0.3549
8.20	5.8059	-0.3677	0.3950	0.3606
8.40	5.7131	-0.3569	0.3836	0.3660
8.60	5.6296	-0.3472	0.3733	0.3711
8.80	5.5541	-0.3384	0.3640	0.3758
9.00	5.4854	-0.3304	0.3556	0.3804
9.20	5.4226	-0.3231	0.3479	0.3846
9.40	5.3650	-0.3164	0.3408	0.3887

$\sqrt{\beta_1}$ = 0.75

β_2	δ	Ω	μ	σ
5.80	10.6310	-1.2778	1.6793	0.3397
6.00	9.3805	-0.9454	1.1133	0.2966
6.20	8.5581	-0.7901	0.8948	0.2931
6.40	7.9694	-0.6943	0.7713	0.2978
6.60	7.5240	-0.6277	0.6896	0.3048
6.80	7.1737	-0.5778	0.6305	0.3125
7.00	6.8900	-0.5388	0.5854	0.3203
7.20	6.6551	-0.5073	0.5495	0.3278
7.40	6.4571	-0.4811	0.5202	0.3349
7.60	6.2876	-0.4590	0.4957	0.3417
7.80	6.1409	-0.4400	0.4748	0.3481
8.00	6.0124	-0.4235	0.4567	0.3542
8.20	5.8989	-0.4089	0.4410	0.3599
8.40	5.7979	-0.3960	0.4270	0.3652
8.60	5.7073	-0.3845	0.4147	0.3703
8.80	5.6257	-0.3741	0.4035	0.3751
9.00	5.5517	-0.3647	0.3935	0.3797
9.20	5.4842	-0.3561	0.3844	0.3839
9.40	5.4225	-0.3483	0.3761	0.3880
9.60	5.3659	-0.3411	0.3684	0.3919

$\sqrt{\beta_1} = 0.80$

β_2	δ	Ω	μ	σ
6.00	10.2742	-1.3685	1.8665	0.3813
6.20	9.1702	-1.0114	1.2165	0.3196
6.40	8.4219	-0.8461	0.9731	0.3100
6.60	7.8763	-0.7443	0.8369	0.3113
6.80	7.4581	-0.6733	0.7472	0.3162
7.00	7.1260	-0.6202	0.6826	0.3224
7.20	6.8551	-0.5786	0.6333	0.3289
7.40	6.6295	-0.5449	0.5943	0.3355
7.60	6.4384	-0.5169	0.5623	0.3419
7.80	6.2741	-0.4933	0.5356	0.3481
8.00	6.1314	-0.4729	0.5129	0.3540
8.20	6.0061	-0.4552	0.4933	0.3595
8.40	5.8951	-0.4396	0.4761	0.3649
8.60	5.7961	-0.4258	0.4610	0.3699
8.80	5.7072	-0.4134	0.4475	0.3747
9.00	5.6269	-0.4023	0.4355	0.3792
9.20	5.5540	-0.3922	0.4246	0.3835
9.40	5.4875	-0.3830	0.4147	0.3876
9.60	5.4265	-0.3745	0.4056	0.3915
9.80	5.3704	-0.3668	0.3973	0.3952

$\sqrt{\beta_1} = 0.85$

β_2	δ	Ω	μ	σ
6.20	10.0236	-1.5141	2.1985	0.4462
6.40	9.0243	-1.0969	1.3577	0.3479
6.60	8.3289	-0.9132	1.0707	0.3293
6.80	7.8142	-0.8017	0.9149	0.3263
7.00	7.4158	-0.7246	0.8138	0.3285
7.20	7.0971	-0.6670	0.7416	0.3329
7.40	6.8355	-0.6221	0.6869	0.3381
7.60	6.6166	-0.5857	0.6436	0.3437
7.80	6.4305	-0.5555	0.6084	0.3493
8.00	6.2700	-0.5300	0.5790	0.3547
8.20	6.1302	-0.5080	0.5541	0.3601
8.40	6.0071	-0.4889	0.5326	0.3652
8.60	5.8979	-0.4721	0.5138	0.3701
8.80	5.8003	-0.4572	0.4973	0.3748
9.00	5.7125	-0.4439	0.4826	0.3793
9.20	5.6330	-0.4315	0.4694	0.3835
9.40	5.5608	-0.4210	0.4575	0.3876
9.60	5.4948	-0.4111	0.4467	0.3915
9.80	5.4343	-0.4020	0.4368	0.3952
10.00	5.3786	-0.3936	0.4278	0.3987

$\sqrt{\beta_1} = 0.90$

β_2	δ	Ω	μ	σ
6.40	9.8674	-1.7795	2.9282	0.5809
6.60	8.9309	-1.2101	1.5599	0.3865
6.80	8.2718	-0.9950	1.1960	0.3529
7.00	7.7789	-0.8689	1.0096	0.3437
7.20	7.3944	-0.7830	0.8920	0.3424
7.40	7.0849	-0.7195	0.8094	0.3445
7.60	6.8298	-0.6701	0.7475	0.3482
7.80	6.6155	-0.6303	0.6988	0.3525
8.00	6.4327	-0.5974	0.6594	0.3572
8.20	6.2747	-0.5696	0.6267	0.3619
8.40	6.1367	-0.5457	0.5991	0.3666
8.60	6.0151	-0.5250	0.5753	0.3712
8.80	5.9069	-0.5068	0.5546	0.3756
9.00	5.8101	-0.4906	0.5363	0.3800
9.20	5.7229	-0.4762	0.5201	0.3841
9.40	5.6439	-0.4632	0.5056	0.3881
9.60	5.5720	-0.4514	0.4926	0.3919
9.80	5.5063	-0.4407	0.4807	0.3955
10.00	5.4459	-0.4309	0.4700	0.3990
10.20	5.3903	-0.4218	0.4601	0.4024

$\sqrt{\beta_1} = 0.95$

β_2	δ	Ω	μ	σ
6.60	9.7701	-2.5098	6.2171	1.1941
6.80	8.8814	-1.3688	1.8770	0.4461
7.00	8.2464	-1.0976	1.3644	0.3840
7.20	7.7676	-0.9490	1.1285	0.3651
7.40	7.3920	-0.8506	0.9863	0.3590
7.60	7.0883	-0.7790	0.8890	0.3580
7.80	6.8371	-0.7238	0.8172	0.3596
8.00	6.6255	-0.6797	0.7615	0.3625
8.20	6.4446	-0.6434	0.7167	0.3660
8.40	6.2879	-0.6128	0.6798	0.3698
8.60	6.1508	-0.5866	0.6487	0.3738
8.80	6.0297	-0.5640	0.6221	0.3778
9.00	5.9220	-0.5441	0.5990	0.3817
9.20	5.8254	-0.5264	0.5787	0.3856
9.40	5.7384	-0.5107	0.5608	0.3894
9.60	5.6594	-0.4965	0.5447	0.3930
9.80	5.5875	-0.4837	0.5303	0.3965
10.00	5.5217	-0.4720	0.5172	0.3999
10.20	5.4613	-0.4614	0.5053	0.4032
10.40	5.4055	-0.4516	0.4945	0.4063

$\sqrt{\beta_1}$ = 1.00

β_2	δ	Ω	μ	σ
7.00	8.8717	-1.6214	2.4828	0.5618
7.20	8.2499	-1.2330	1.6086	0.4297
7.40	7.7787	-1.0475	1.2845	0.3934
7.60	7.4075	-0.9306	1.1037	0.3796
7.80	7.1065	-0.8477	0.9847	0.3744
8.00	6.8569	-0.7849	0.8991	0.3731
8.20	6.6463	-0.7351	0.8338	0.3740
8.40	6.4658	-0.6944	0.7819	0.3761
8.60	6.3093	-0.6603	0.7396	0.3788
8.80	6.1722	-0.6313	0.7041	0.3819
9.00	6.0510	-0.6063	0.6740	0.3852
9.20	5.9431	-0.5843	0.6479	0.3885
9.40	5.8462	-0.5650	0.6252	0.3919
9.60	5.7588	-0.5477	0.6051	0.3952
9.80	5.6795	-0.5322	0.5872	0.3985
10.00	5.6073	-0.5182	0.5711	0.4017
10.20	5.5411	-0.5055	0.5566	0.4048
10.40	5.4803	-0.4938	0.5434	0.4078
10.60	5.4242	-0.4831	0.5314	0.4107
10.80	5.3722	-0.4733	0.5203	0.4135

$\sqrt{\beta_1}$ = 1.10

β_2	δ	Ω	μ	σ
7.60	8.3382	-1.7704	2.9200	0.6917
7.80	7.8646	-1.3511	1.8503	0.5020
8.00	7.4903	-1.1532	1.4690	0.4472
8.20	7.1861	-1.0281	1.2587	0.4233
8.40	6.9333	-0.9390	1.1212	0.4115
8.60	6.7194	-0.8712	1.0225	0.4056
8.80	6.5360	-0.8173	0.9474	0.4030
9.00	6.3767	-0.7731	0.8879	0.4022
9.20	6.2370	-0.7360	0.8393	0.4026
9.40	6.1134	-0.7043	0.7987	0.4038
9.60	6.0032	-0.6769	0.7642	0.4054
9.80	5.9043	-0.6528	0.7344	0.4074
10.00	5.8149	-0.6315	0.7083	0.4095
10.20	5.7337	-0.6125	0.6853	0.4118
10.40	5.6597	-0.5954	0.6648	0.4141
10.60	5.5919	-0.5799	0.6464	0.4164
10.80	5.5296	-0.5658	0.6298	0.4188
11.00	5.4720	-0.5529	0.6147	0.4211
11.20	5.4187	-0.5411	0.6010	0.4234
11.40	5.3692	-0.5301	0.5883	0.4257

$\sqrt{\beta_1}$ = 1.20

β_2	δ	Ω	μ	σ
8.20	8.0376	-2.4435	5.8615	1.3851
8.40	7.6442	-1.5981	2.4388	0.6482
8.60	7.3250	-1.3295	1.8123	0.5333
8.80	7.0601	-1.1732	1.5108	0.4861
9.00	6.8364	-1.0658	1.3256	0.4616
9.20	6.6446	-0.9857	1.1973	0.4477
9.40	6.4783	-0.9227	1.1021	0.4394
9.60	6.3323	-0.8715	1.0277	0.4344
9.80	6.2033	-0.8288	0.9678	0.4315
10.00	6.0883	-0.7924	0.9182	0.4300
10.20	5.9851	-0.7610	0.8763	0.4295
10.40	5.8919	-0.7335	0.8404	0.4296
10.60	5.8073	-0.7093	0.8091	0.4302
10.80	5.7301	-0.6876	0.7816	0.4311
11.00	5.6595	-0.6681	0.7572	0.4323
11.20	5.5945	-0.6505	0.7354	0.4336
11.40	5.5345	-0.6345	0.7158	0.4350
11.60	5.4790	-0.6198	0.6979	0.4365
11.80	5.4274	-0.6064	0.6817	0.4381
12.00	5.3794	-0.5940	0.6668	0.4397

$\sqrt{\beta_1}$ = 1.30

β_2	δ	Ω	μ	σ
9.00	7.5308	-2.3441	5.3169	1.3512
9.20	7.2437	-1.6612	2.6191	0.7306
9.40	7.0022	-1.4063	1.9838	0.6010
9.60	6.7958	-1.2527	1.6658	0.5433
9.80	6.6173	-1.1453	1.4671	0.5115
10.00	6.4612	-1.0640	1.3282	0.4920
10.20	6.3234	-0.9996	1.2243	0.4793
10.40	6.2008	-0.9468	1.1429	0.4708
10.60	6.0909	-0.9024	1.0771	0.4650
10.80	5.9918	-0.8644	1.0224	0.4611
11.00	5.9019	-0.8315	0.9761	0.4584
11.20	5.8201	-0.8026	0.9363	0.4567
11.40	5.7452	-0.7769	0.9017	0.4557
11.60	5.6763	-0.7539	0.8712	0.4552
11.80	5.6129	-0.7332	0.8440	0.4551
12.00	5.5541	-0.7144	0.8198	0.4552
12.20	5.4996	-0.6973	0.7978	0.4557
12.40	5.4489	-0.6816	0.7780	0.4562
12.60	5.4015	-0.6671	0.7598	0.4570
12.80	5.3572	-0.6538	0.7432	0.4578

$\sqrt{\beta_1}$ = 1.40

β_2	δ	Ω	μ	σ
10.00	6.9974	-1.9166	3.4399	0.9690
10.20	6.8005	-1.5807	2.4112	0.7288
10.40	6.6291	-1.3962	1.9691	0.6344
10.60	6.4784	-1.2717	1.7094	0.5838
10.80	6.3447	-1.1791	1.5339	0.5526
11.00	6.2252	-1.1065	1.4054	0.5319
11.20	6.1177	-1.0473	1.3062	0.5175
11.40	6.0205	-0.9979	1.2268	0.5071
11.60	5.9321	-0.9557	1.1614	0.4994
11.80	5.8513	-0.9191	1.1065	0.4937
12.00	5.7772	-0.8870	1.0594	0.4895
12.20	5.7090	-0.8585	1.0186	0.4863
12.40	5.6459	-0.8331	0.9828	0.4839
12.60	5.5874	-0.8102	0.9511	0.4821
12.80	5.5331	-0.7894	0.9228	0.4809
13.00	5.4824	-0.7704	0.8972	0.4800
13.20	5.4350	-0.7530	0.8741	0.4794
13.40	5.3907	-0.7370	0.8531	0.4791
13.60	5.3490	-0.7223	0.8338	0.4790
13.80	5.3098	-0.7085	0.8161	0.4790

$\sqrt{\beta_1}$ = 1.50

β_2	δ	Ω	μ	σ
10.80	6.8450	-3.4154	15.7446	4.3688
11.00	6.6758	-1.9659	3.6334	1.0731
11.20	6.5265	-1.6529	2.6152	0.8178
11.40	6.3937	-1.4740	2.1544	0.7100
11.60	6.2746	-1.3508	1.8782	0.6497
11.80	6.1673	-1.2582	1.6895	0.6114
12.00	6.0700	-1.1848	1.5504	0.5851
12.20	5.9813	-1.1246	1.4425	0.5661
12.40	5.9001	-1.0740	1.3559	0.5520
12.60	5.8256	-1.0306	1.2843	0.5412
12.80	5.7568	-0.9928	1.2240	0.5328
13.00	5.6931	-0.9595	1.1723	0.5262
13.20	5.6341	-0.9299	1.1274	0.5210
13.40	5.5791	-0.9033	1.0879	0.5168
13.60	5.5277	-0.8793	1.0528	0.5134
13.80	5.4797	-0.8575	1.0215	0.5106
14.00	5.4347	-0.8375	0.9933	0.5084
14.20	5.3923	-0.8192	0.9677	0.5066
14.40	5.3525	-0.8022	0.9444	0.5052
14.60	5.3149	-0.7866	0.9231	0.504[illegible]

$\sqrt{\beta_1}$ = 1.60

β_2	δ	Ω	μ	σ
12.00	6.4694	-2.3521	5.4165	1.6232
12.20	6.3485	-1.8711	3.3040	1.0392
12.40	6.2392	-1.6431	2.5973	0.8545
12.60	6.1400	-1.4956	2.2144	0.7598
12.80	6.0495	-1.3881	1.9661	0.7016
13.00	5.9665	-1.3045	1.7887	0.6622
13.20	5.8903	-1.2366	1.6540	0.6339
13.40	5.8199	-1.1799	1.5474	0.6128
13.60	5.7547	-1.1315	1.4604	0.5964
13.80	5.6941	-1.0896	1.3877	0.5836
14.00	5.6377	-1.0528	1.3259	0.5732
14.20	5.5850	-1.0201	1.2725	0.5648
14.40	5.5357	-0.9908	1.2258	0.5579
14.60	5.4894	-0.9643	1.1845	0.5521
14.80	5.4459	-0.9403	1.1477	0.5473
15.00	5.4049	-0.9183	1.1147	0.5433
15.20	5.3662	-0.8982	1.0848	0.5398
15.40	5.3296	-0.8796	1.0576	0.5369
15.60	5.2950	-0.8623	1.0328	0.5345
15.80	5.2622	-0.8463	1.0100	0.5324

$\sqrt{\beta_1}$ = 1.70

β_2	δ	Ω	μ	σ
13.20	6.2311	-2.5592	6.7043	2.0823
13.40	6.1374	-2.0037	3.8047	1.2296
13.60	6.0513	-1.7582	2.9453	0.9880
13.80	5.9721	-1.6020	2.4942	0.8666
14.00	5.8990	-1.4888	2.2059	0.7923
14.20	5.8313	-1.4010	2.0018	0.7419
14.40	5.7683	-1.3297	1.8478	0.7054
14.60	5.7097	-1.2702	1.7263	0.6779
14.80	5.6549	-1.2194	1.6274	0.6564
15.00	5.6036	-1.1754	1.5450	0.6392
15.20	5.5555	-1.1366	1.4750	0.6253
15.40	5.5102	-1.1021	1.4146	0.6137
15.60	5.4676	-1.0712	1.3619	0.6041
15.80	5.4274	-1.0433	1.3153	0.5960
16.00	5.3893	-1.0178	1.2738	0.5891
16.20	5.3533	-0.9945	1.2366	0.5831
16.40	5.3191	-0.9732	1.2029	0.5780
16.60	5.2866	-0.9534	1.1723	0.5735
16.80	5.2557	-0.9351	1.1444	0.5696
17.00	5.2263	-0.9180	1.1187	0.5662

$\sqrt{\beta_1}$ = 1.80

β_2	δ	Ω	μ	σ
14.60	5.9954	-2.5092	6.3957	2.0763
14.80	5.9241	-2.0480	3.9966	1.3398
15.00	5.8578	-1.8214	3.1585	1.0916
15.20	5.7961	-1.6725	2.6989	0.9601
15.40	5.7384	-1.5626	2.3988	0.8770
15.60	5.6845	-1.4763	2.1835	0.8195
15.80	5.6338	-1.4057	2.0196	0.7771
16.00	5.5863	-1.3463	1.8896	0.7446
16.20	5.5414	-1.2954	1.7833	0.7189
16.40	5.4991	-1.2509	1.6944	0.6981
16.60	5.4591	-1.2117	1.6186	0.6809
16.80	5.4212	-1.1766	1.5531	0.6666
17.00	5.3853	-1.1451	1.4957	0.6544
17.20	5.3511	-1.1165	1.4449	0.6441
17.40	5.3187	-1.0904	1.3997	0.6351
17.60	5.2878	-1.0665	1.3590	0.6273
17.80	5.2583	-1.0444	1.3221	0.6205
18.00	5.2302	-1.0240	1.2886	0.6145
18.20	5.2034	-1.0050	1.2579	0.6092
18.40	5.1777	-0.9873	1.2298	0.6045

$\sqrt{\beta_1}$ = 1.90

β_2	δ	Ω	μ	σ
16.20	5.7796	-2.3853	5.6597	1.9215
16.40	5.7255	-2.0464	4.0034	1.3948
16.60	5.6751	-1.8545	3.2811	1.1715
16.80	5.6274	-1.7207	2.8505	1.0420
17.00	5.5825	-1.6190	2.5580	0.9564
17.20	5.5400	-1.5377	2.3430	0.8951
17.40	5.4998	-1.4702	2.1765	0.8489
17.60	5.4617	-1.4128	2.0429	0.8128
17.80	5.4255	-1.3632	1.9326	0.7838
18.00	5.3910	-1.3195	1.8396	0.7599
18.20	5.3583	-1.2808	1.7599	0.7401
18.40	5.3271	-1.2460	1.6907	0.7232
18.60	5.2973	-1.2146	1.6299	0.7088
18.80	5.2688	-1.1859	1.5759	0.6963
19.00	5.2416	-1.1597	1.5275	0.6854
19.20	5.2156	-1.1356	1.4840	0.6758
19.40	5.1906	-1.1132	1.4444	0.6674
19.60	5.1666	-1.0925	1.4084	0.6598
19.80	5.1437	-1.0732	1.3753	0.6531
20.00	5.1216	-1.0551	1.3449	0.6471

$\sqrt{\beta_1}$ = 2.00

β_2	δ	Ω	μ	σ
17.80	5.6333	-2.8775	9.3328	3.2305
18.00	5.5903	-2.2981	5.1969	1.8380
18.20	5.5494	-2.0440	4.0066	1.4467
18.40	5.5107	-1.8815	3.3860	1.2471
18.60	5.4739	-1.7629	2.9905	1.1228
18.80	5.4388	-1.6702	2.7107	1.0367
19.00	5.4054	-1.5944	2.4997	0.9731
19.20	5.3736	-1.5308	2.3333	0.9240
19.40	5.3432	-1.4760	2.1979	0.8849
19.60	5.3142	-1.4282	2.0850	0.8530
19.80	5.2864	-1.3858	1.9891	0.8265
20.00	5.2597	-1.3480	1.9063	0.8040
20.20	5.2342	-1.3138	1.8340	0.7848
20.40	5.2097	-1.2828	1.7701	0.7681
20.60	5.1862	-1.2544	1.7131	0.7536
20.80	5.1636	-1.2283	1.6620	0.7408
21.00	5.1418	-1.2042	1.6157	0.7295
21.20	5.1209	-1.1818	1.5736	0.7193
21.40	5.1007	-1.1609	1.5351	0.7103
21.60	5.0812	-1.1415	1.4998	0.7021

has a standard logistic distribution. Percentile points of the Type IV curve (from Bouver and Bargmann (1974)) and this L_U curve are compared in Table 5, which also contains values for

(i) the S_U curve with the same first four moments, defined by

$$Z^* = -0.4048 + 1.455 \sinh^{-1} \{(X + 0.3842)/1.0765\}$$

having a standard normal distribution, and

(ii) the S_U curve defined by $Z^{**} = Z/1.7$ (see (1.7 a,b)).

We note that (a) the L_U curve agrees more closely with the Type IV than does the first (moment-fitted) S_U in the tails and (b) although the probability integral of the second S_U curve must agree with the L_U to within 0.01, there is considerable divergence between the percentiles near 0 or 100.

Table 5. Comparison of Type IV, L_U and S_U Standardized Percentiles

%	0.1	0.25	0.5	1	2.5	5	10	25	50
Type IV	-3.345	-2.841	-2.492	-2.164	-1.748	-1.437	-1.114	-0.619	-0.078
L_U	-3.339	-2.865	-2.526	-2.220	-1.777	-1.454	-1.118	-0.609	-0.070
S_U(i)	-3.707	-3.086	-2.656	-2.257	-1.778	-1.417	-1.073	-0.585	-0.081
S_U(ii)	-2.507	-2.281	-2.100	-1.908	-1.628	-1.387	-1.109	-0.632	-0.070

%	75	90	95	07.5	99	99.5	99.75	99.9
Type IV	0.524	1.192	1.690	2.211	2.959	3.586	4.279	5.316
L_U	0.520	1.178	1.677	2.203	2.967	3.609	4.318	5.375
S_U(i)	0.490	1.162	1.686	2.244	3.051	3.721	4.451	5.513
S_U(ii)	0.546	1.166	1.574	1.955	2.432	2.781	3.123	3.571

(i) moment fit; (ii) $-1.8576 + 3.5383 \sinh^{-1} \{(X + 1.5498)/2.6940\}$ unit normal

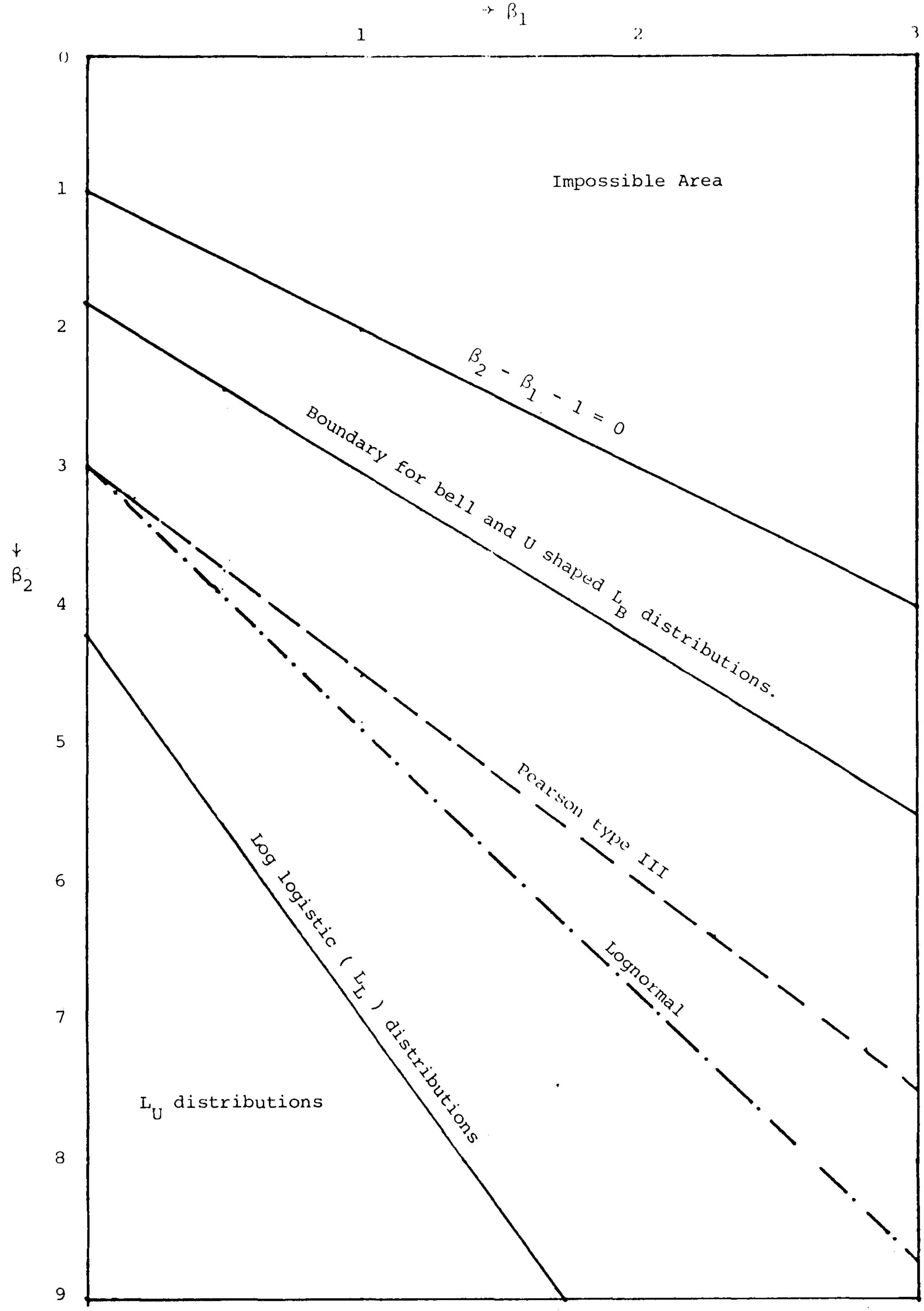

FIGURE A. β_1, β_2 Region for L_U, L_L, L_B Distributions

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