

ABSTRACT

A TEST OF SYMMETRY ASSOCIATED
WITH THE HODGES-LEHMANN ESTIMATOR

One version of the Hodges-Lehmann estimator, $\hat{\theta} = \text{median}\{(X_i + X_j)/2, 1 \leq i, j \leq n\}$, is also the minimum distance estimator derived from a particular Cramer-von Mises distance. This distance evaluated at $\hat{\theta}$, i.e., the minimized distance, provides a natural statistic for testing symmetry of the underlying distribution.

1. Introduction. Let X_1, \dots, X_n be a sample from a continuous distribution $F(x) = F_0((x-\theta)/\sigma)$. The Hodges-Lehmann (H-L) estimator for location $\hat{\theta}_1 = \text{median}\{(X_i+X_j)/2, 1 \leq i \leq j \leq n\}$ is well-known to have excellent robustness and efficiency properties under the usual assumption of symmetry $F_0(x) = 1 - F_0(-x)$. For example, the asymptotic relative efficiency (ARE) of $\hat{\theta}_1$ to the sample mean \bar{X} is $\geq .864$ for all symmetric F_0 and $= .955$ for $F_0 = \text{normal}$. However, Bickel and Lehmann (1975) have shown that this ARE could in fact be as low as 0 for asymmetric F_0 . Thus symmetry is important for efficiency considerations as well as for identifying the location parameter θ . In this paper we propose a natural test of the symmetry hypothesis which is based on a minimum distance estimator characterization of the Hodges-Lehmann estimator.

Let F_n be the usual empirical distribution function and define

$$(1.1) \quad d_n(\theta) = \int_{-\infty}^{\infty} [1 - F_n(\theta+x) - F_n(\theta-x)]^2 dx .$$

Simple calculations allow (1.1) to be reexpressed as

$$(1.2) \quad d_n(\theta) = -\frac{2}{n^2} \sum_{i < j} |X_i - X_j| + \frac{2}{n^2} \sum_{i=1}^n \sum_{j=1}^n \left| \frac{X_i + X_j}{2} - \theta \right| .$$

Therefore $d_n(\theta)$ is minimized by $\hat{\theta}_2 = \text{median}\{(X_i+X_j)/2, 1 \leq i, j \leq n\}$, which is another version of the H-L estimator. (We use the term "H-L estimator" generically for any asymptotically equivalent version.) It appears that Knüsel (1969) first noticed this characterization. As our test statistic we propose the minimized distance $nd_n(\hat{\theta}_2)$ scaled by Gini's mean difference

$$\hat{\sigma} = \binom{n}{2}^{-1} \sum_{i < j} |X_i - X_j|$$

$$(1.3) \quad nd_n(\hat{\theta}_2)/\hat{\sigma} = (n-1) \left[-1 + \frac{\sum_{i=1}^n \sum_{j=1}^n |X_i + X_j - 2\hat{\theta}_2|}{\sum_{i < j} |X_i - X_j|} \right].$$

Gini's mean difference arises naturally in (1.2) and its use in (1.3) produces computational ease as well as an intuitive ratio of absolute differences. The idea of course is to compute (1.3) as an auxiliary statistic along with the H-L estimator in order to give confidence in the symmetry assumption. If symmetry rather than location is emphasized, (1.3) may be viewed as a modified form of the Cramer-von Mises statistic $nd_n(\theta)$; others proposed for the "known" case include Rothman and Woodroffe (1972), Gregory (1977), and Hill and Rao (1977). Whatever the emphasis, Corollary 1 of Section 2 indicates the advantage of using the minimum distance estimator (or any asymptotically equivalent version) when θ is unknown: the limiting distribution of $nd_n(\hat{\theta}_2)$ is the difference of two terms, the first gives the limiting distribution of $nd_n(\theta)$ and the second reflects the estimation of θ by $\hat{\theta}_2$. Moreover, when F_0 is the logistic distribution, the first term reduces simply to a weighted sum of chi-square random variables $\sum_{i=1}^{\infty} Z_i^2 / (i - \frac{1}{2})$ and the second term is just $2Z_1^2$. Monte Carlo results indicate that this limiting distribution based on the logistic distribution is fairly close to that based on a variety of other symmetric distributions.

The paper is organized as follows. Section 2 gives a general theorem for finding the limiting distribution of $nd_n(\hat{\theta})$, $\hat{\theta}$ arbitrary. Corollaries 1 and 2 then relate specifically to the H-L estimator. Section 3 presents the Monte Carlo results and Section 4 outlines the use of an alternative distance $d_n^*(\theta)$. An appendix contains some technical lemmas and the proof of Corollary 1.

2. Asymptotic distributions. The limiting distribution of (1.3) is handled in several steps. First, Theorem 1 approximates $nd_n(\hat{\theta})$ by a suitable functional of $\Delta_n(x) = F_n(x) - F(x)$. Corollary 1 then finds the limiting distribution of this latter functional when $\hat{\theta}$ is the H-L estimator. Corollary 2 restricts to the logistic distribution and characterizes the limiting distribution as a weighted sum of chi-square random variables.

THEOREM 1. Let X_1, \dots, X_n be independent random variables having distribution $F(x) = F_0((x-\theta)/\sigma)$. Suppose that F_0 has a $(2+\epsilon)$ th moment and a bounded density f_0 symmetric about 0 and continuous a.e. Lebesgue. If $\hat{\theta}$ satisfies

$$(2.1) \quad |\hat{\theta} - \theta| \leq n^{-1/2} \log n \quad \text{w.p.1 for all } n \text{ sufficiently large,}$$

$$(2.2a) \quad n^{1/2} \{ \hat{\theta} - \theta - T(F; \Delta_n) \} \xrightarrow{P} 0,$$

and

$$(2.2b) \quad n^{1/2} \{ T(F; \Delta_n) \} = O_p(1),$$

then

$$(2.3) \quad nd_n(\hat{\theta}) - n \int_{-\infty}^{\infty} \{ [f(\theta+x) + f(\theta-x)] T(F; \Delta_n) + \Delta_n(\theta+x) + \Delta_n(\theta-x) \}^2 dx \xrightarrow{P} 0,$$

REMARK. Theorem 1 is fairly general since most commonly used estimators satisfy (2.1) and (2.2), where $T(F; \Delta_n)$ is usually an average. The $(2+\epsilon)$ th moment is surely not necessary; in fact only a first moment is required for $E d_n(\theta) < \infty$.

PROOF. The difference (2.3) may be written as

$$(2.4) \quad n \int_{-\infty}^{\infty} \{f(\theta+x)T(F;\Delta_n) - [F_n(\hat{\theta}+x) - F_n(\theta+x)]\} Q_n(\hat{\theta}, \theta, x) dx \\ + n \int_{-\infty}^{\infty} \{f(\theta-x)T(F;\Delta_n) - [F_n(\hat{\theta}-x) - F_n(\theta-x)]\} Q_n(\hat{\theta}, \theta, x) dx ,$$

where $Q_n(\hat{\theta}, \theta, x) = \{1 - F_n(\hat{\theta}+x) - F_n(\hat{\theta}-x) - [f(\theta+x) + f(\theta-x)]T(F;\Delta_n) - \Delta_n(\theta+x) - \Delta_n(\theta-x)\}$.

The first term of (2.4) is bounded in absolute value by

$$(2.5) \quad n \int_{-\infty}^{\infty} |f(\theta+x)T(F;\Delta_n) - [F(\hat{\theta}+x) - F(\theta+x)]| \cdot |Q_n(\hat{\theta}, \theta, x)| dx \\ + n \sup_{-\infty < x < \infty} |\Delta_n(\hat{\theta}+x) - \Delta_n(\theta+x)| \cdot \int_{-\infty}^{\infty} |Q_n(\hat{\theta}, \theta, x)| dx .$$

We first show that $\int |Q_n(\hat{\theta}, \theta, x)| dx = O_p(n^{-1/2})$.

$$\int_{-\infty}^{\infty} |Q_n(\hat{\theta}, \theta, x)| dx \leq \int_{-\infty}^{\infty} |F(\hat{\theta}+x+2(\theta-\hat{\theta})) - F(\hat{\theta}+x)| dx + 2 |T(F;\Delta_n)| + 4 \int_{-\infty}^{\infty} |\Delta_n(x)| dx \\ = 2|\hat{\theta}-\theta| + 2|T(F;\Delta_n)| + 4 \int_{-\infty}^{\infty} |\Delta_n(x)| dx .$$

The first two terms of this last expression are $O_p(n^{-1/2})$ using (2.2), and taking expectations, the third term can be shown to be $O_p(n^{-1/2})$ if $\int [F(x)(1-F(x))]^{1/2} dx < \infty$, which in turn is satisfied if F has a finite $(2+\varepsilon)$ th moment. Thus the second term of (2.5) $\xrightarrow{P} 0$ because of the convergence $n^{1/2} \sup_x |\Delta_n(\hat{\theta}+x) - \Delta_n(\theta+x)| \xrightarrow{P} 0$ which follows from (2.1) and Theorem 4.2 of Sen and Ghosh (1971). Using the mean value theorem, the first term of (2.5) is bounded by

$$n \|f\|_{\infty} \cdot |\hat{\theta} - \theta - T(F; \Delta_n)| \cdot \int_{-\infty}^{\infty} |Q_n(\hat{\theta}, \theta, x)| dx$$

$$+ n |\hat{\theta} - \theta| \cdot \|Q_n(\hat{\theta}, \theta, \cdot)\|_{\infty} \cdot \int_{-\infty}^{\infty} |f(\theta+x) - f(\theta^*(x)+x)| dx,$$

where $\theta^*(x)$ lies between θ and $\hat{\theta}$ and $\|h\|_{\infty}$ means $\sup_x |h(x)|$. Note that $\|Q_n(\hat{\theta}, \theta, \cdot)\|_{\infty} = O_p(n^{-1/2})$ follows from well-known properties of the Kolmogorov-Smirnov statistic $\|\Delta_n\|_{\infty}$ and from (2.2) and the fact that f is bounded. Scheffé's convergence theorem for densities handles the last integral. The second term of (2.4) is of course bounded in the same manner as the first. \square

Another asymptotically equivalent version of the Hodges-Lehmann estimator is $\hat{\theta}_3 = \text{median}\{(X_i + X_j)/2, 1 \leq i < j \leq n\}$. The following corollary applies to any of the three versions mentioned. Let $U(t)$ be the usual Brownian Bridge on $C[0,1]$.

COROLLARY 1. *Let X_1, \dots, X_n be independent random variables having distribution $F(x) = F_0((x-\theta)/\sigma)$. Suppose that F_0 has a $(2+\epsilon)$ th moment and a bounded density f_0 symmetric about 0 and continuous a.e. Lebesgue. Then for $i = 1, 2, \text{ or } 3$*

$$(2.6) \quad nd_n(\hat{\theta}_i) \xrightarrow{d} \sigma \left\{ \int_0^1 [U(t) + U(1-t)]^2 dF_0^{-1}(t) - \left(\int_0^1 [U(t) + U(1-t)] dt \right)^2 / \int_{-\infty}^{\infty} f_0^2(x) dx \right\}.$$

The proof of Corollary 1 requires some technical lemmas to verify (2.1) and (2.2) and is postponed until the Appendix. Note that the first term on the right-hand side of (2.6) is just the limiting distribution of $nd_n(\theta)$, and the second term is the adjustment due to estimation of θ . Similar decompositions are found in Pyke (1970) and Boos (1979).

When F_0 is logistic, the Hodges-Lehmann estimator is asymptotically efficient and the right-hand side of (2.6) reduces conveniently to a weighted sum of chi-square random variables. The maximum likelihood estimate of θ for this situation would also produce the following result.

COROLLARY 2. Let X_1, \dots, X_n be independent random variables having distribution $F(x) = [1 + \exp((x-\theta)/\sigma)]^{-1}$. If $\hat{\sigma} = \binom{n}{2}^{-1} \sum_{i < j} |X_i - X_j|$, then for $i = 1, 2, \text{ or } 3$

$$(2.7) \quad n d_n(\hat{\theta}_i)/\hat{\sigma} \xrightarrow{d} \sum_{i=1}^{\infty} \frac{Z_i^2}{(2i+1)(i+1)},$$

where Z_1, Z_2, \dots are independent standard normal random variables.

PROOF. Consider the representation for $U(t)$ found in Anderson and Darling (1952):

$$U(t) = [t(1-t)]^{\frac{1}{2}} \sum_{j=1}^{\infty} \frac{Z_j}{[j(j+1)]^{\frac{1}{2}}} p_j(t),$$

where the p_j are the Ferrer associated Legendre polynomials.

Then, since $p_j(t) = (-1)^{j+1} p_j(1-t)$ and $f_0(F_0^{-1}(t)) = t(1-t)$, we have

$$\begin{aligned} \int_0^1 [U(t) + U(1-t)]^2 dF_0^{-1}(t) &= \int_0^1 \frac{[U(t) + U(1-t)]^2}{t(1-t)} dt \\ &= \sum_{j=1}^{\infty} \frac{Z_{2j-1}^2}{j(j-\frac{1}{2})}. \end{aligned}$$

Likewise, since $\int_0^1 f_0^2(x) dx = 1/6$ and $p_1(t) = [6t(1-t)]^{\frac{1}{2}}$, we have

$$\left(\int_0^1 [U(t) + U(1-t)] dt \right)^2 / \int_0^1 f_0^2(x) dx = 6 \left(Z_1 \int_0^1 \frac{2p_1^2(t) dt}{6^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}} \right)^2 = 2Z_1^2.$$

Finally, $\hat{\sigma} \xrightarrow{p} 2\sigma$ yields

$$\begin{aligned} nd_n(\hat{\theta}_i)/\hat{\sigma} &\xrightarrow{d} \sigma \left\{ \sum_{j=2}^{\infty} \frac{z_{2j-1}^2}{j(j-\frac{1}{2})} \right\} / 2\sigma \\ &= \sum_{j=2}^{\infty} \frac{z_{2j-1}^2}{j(2j-1)} \\ &\stackrel{d}{=} \sum_{i=1}^{\infty} \frac{z_i^2}{(2i+1)(i+1)} \quad . \quad \square \end{aligned}$$

Approximate percentiles of this limiting distribution were calculated by inverting numerically the characteristic function of $Q = \sum_{i=1}^{40} z_i^2 / (2i+1)(i+1)$ using a version of the standard Imhoff procedure found in Koerts and Abrahamse (1969). Test runs with the Cramer-von Mises statistic $\int U^2(t)dt$ indicated that 40 eigenvalues were sufficient to get within .002 of the percentiles listed in Anderson and Darling (1952). We expect similar accuracy for the percentiles listed in Table 1.

TABLE 1. Percentiles of $Q = \sum_{i=1}^{40} z_i^2 / (2i+1)(i+1)$.

P(Q≤x)	.80	.85	.90	.95	.975	.99
x	.523	.595	.700	.887	1.081	1.348

3. Monte Carlo Results. A small Monte Carlo study was carried out to determine the power of the proposed test for various asymmetric alternatives and also to check the stability of level over a variety of symmetric distributions. Included in the study for comparison purposes was a version of a statistic suggested by David and Johnson (1956) and later studied by Resek (1974)

$$J = \{X_{([\cdot 08n] + 1)} + X_{(n - [\cdot 08n])} - \text{median}\{X_i\}\} / \hat{\sigma},$$

where $X_{(1)} \leq \dots \leq X_{(n)}$ are the order statistics, $[\]$ is the greatest integer function, and $\hat{\sigma}$ is Gini's mean difference. The calculations were based on samples of size 10,000 generated by the McGill "Super-Duper" random number generator and further analysed via a Monte Carlo package written by David Dickey (1978). The results are listed below in Table 2 where GS refers to our statistic (1.3). All cut-off points were estimated by Monte Carlo samples from the logistic distribution except GS* which used instead the asymptotic value .887 from Table 1. The J** row shows the improved performance of J when testing for only right skewness.

TABLE 2. Rejection Probabilities for $\alpha = .05$.

	χ_4^2	χ_2^2	Log-normal	χ_1^2	Uniform	Normal	Logistic	DEXP	Cauchy
<u>n=20</u>									
GS	.38	.68	.82	.93	.090	.047	.050	.079	.31
GS*	.35	.65	.80	.92	.079	.039	.043	.068	.28
J	.31	.57	.68	.83	.038	.037	.050	.090	.26
J**	.43	.68	.77	.89	.040	.039	.050	.076	.16
<u>n=50</u>									
GS	.82	.99	1.00	1.00	.107	.046	.050	.077	.33
GS*	.81	.99	1.00	1.00	.105	.045	.049	.076	.33
J	.59	.88	.93	.99	.064	.043	.050	.086	.12
J**	.71	.93	.965	.995	.061	.044	.050	.075	.13

* Cut-off value: .887 from Table 1.

** Cut-off value: the estimated 95th percentile for logistic data.

Several conclusions:

- 1) The convergence of (1.3) is fast enough so that the asymptotic values are acceptable even for $n=20$.
- 2) The .05 levels for the Cauchy are much too large, but the uniform and double exponential (DEXP) are only moderately above .05. The normal is expectedly close to .05.
- 3) The power of GS is better than J and certainly comparable to other tests in the literature (see, e.g., Table 5 of Doksum, Fenstad and Aaberge (1977)).

4. A Related Test. Consider the statistic

$$d_n^*(\theta) = \int_{-\infty}^{\infty} [1 - F_n(\theta+x) - F_n(\theta-x)]^2 dF_n(\theta+x),$$

which is similar to ones proposed by Rothman and Woodroffe (1972) and Gregory (1977) for testing symmetry about θ . Under regularity conditions on f_0 including symmetry, the estimator $\hat{\theta}$ obtained by minimizing $d_n^*(\theta)$ has asymptotic approximation

$$T(F; \Delta_n) = -\sigma \int_{-\infty}^{\infty} [\Delta_n(\theta+\sigma y) + \Delta_n(\theta-\sigma y)] f_0^2(y) dy / 2 \int_{-\infty}^{\infty} f_0^3(y) dy.$$

This reduces to the same approximation obtained by Parr and Schucany (1978) for the $\hat{\theta}$ which minimizes $\int [F_n(x) - F_0(x-\theta)]^2 dF_0(x-\theta)$ when X_1, \dots, X_n have distribution $F_0(x-\theta)$. Similar to Corollary 1 we can show

$$nd_n^*(\hat{\theta}) \xrightarrow{d} \int_0^1 [U(t) + U(1-t)]^2 dt - \left(\int_0^1 [U(t) + U(1-t)] f_0(F_0^{-1}(t)) dt \right)^2 / \int_{-\infty}^{\infty} f_0^3(x) dx.$$

One advantage of this statistic is that it is automatically scale invariant. Moreover, the impact of f_0 on the limiting distribution is restricted to the second term.

Further analysis similar to Corollary 2 shows that if

$$f_0(x) = \frac{2}{\pi} \cdot \frac{1}{e^x + e^{-x}} = \frac{1}{\pi} \cdot \operatorname{sech}(x) ,$$

then

$$\begin{aligned} \operatorname{nd}_n^*(\hat{\theta}) &\xrightarrow{d} \sum_{j=1}^{\infty} \frac{z_{2j-1}^2}{(j-\frac{1}{2})^2 \pi^2} - \frac{4z_1^2}{\pi^2} \\ &\stackrel{d}{=} \sum_{i=1}^{\infty} \frac{z_i^2}{(i+\frac{1}{2})^2 \pi^2} . \end{aligned}$$

Thus f_0 could be used as a base distribution just as the logistic was used for the statistic of Section 2.

Appendix. In order to prove Corollary 1 we list several lemmas. The first two are general and straightforward to prove. The second two relate to the H-L estimator. Let $F^{-1}(p) = \inf\{x: F(x) \geq p\}$ and denote the convolution of F and G by $F*G$.

LEMMA 1. Let F_1 and F be distribution functions. If F has a positive derivative $F'(x) > 0$ at $x = F^{-1}(p)$ and $\|F_1 - F\|_{\infty} < \delta, \delta$ sufficiently small, then there exists a constant C_1 depending only on δ and F such that

$$|F_1^{-1}(p) - F^{-1}(p)| \leq C_1 \|F_1 - F\|_{\infty} .$$

LEMMA 2. Let F_1, F_2, G_1 , and G_2 be distribution functions. If G_1 has at most a finite number of discontinuities, then there exists a constant C_2 depending only on G_1 such that

$$\|F_1 * F_2 - G_1 * G_2\|_{\infty} \leq C_2 \|F_1 - G_1\|_{\infty} + \|F_2 - G_2\|_{\infty} .$$

For the next two lemmas it is useful to write the three forms of the H-L estimator as functionals $\hat{\theta}_i = F_{in}^{-1}(\frac{1}{2})$, where

$$F_{1n}(x) = \frac{2}{n(n+1)} \sum_{i < j} I(X_i + X_j \leq 2x) ,$$

$$F_{2n}(x) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n I(X_i + X_j \leq 2x) = F_n * F_n(2x) ,$$

and

$$F_{3n}(x) = \frac{2}{n(n-1)} \sum_{i < j} I(X_i + X_j \leq 2x) .$$

LEMMA 3. Let X_1, \dots, X_n be independent random variables having distribution F . If $G(x) = F * F(2x)$ has a positive derivative $G'(\theta) > 0$ at its median $\theta = G^{-1}(\frac{1}{2})$ and F is continuous, then for n sufficiently large

$$|\hat{\theta}_i - \theta| \leq n^{-\frac{1}{2}} \log n \quad \text{w.p.1, } i = 1, 2, 3 .$$

PROOF. By Lemmas 1 and 2 and for n sufficiently large

$$\begin{aligned} |\hat{\theta}_2 - \theta| &\leq \frac{1}{2} C_1 \|F_n * F_n - F * F\|_{\infty} \\ &\leq \frac{1}{2} C_1 (C_2 + 1) \|F_n - F\|_{\infty} . \end{aligned}$$

The result for $\hat{\theta}_2$ then follows from a theorem of Dvoretzky, Kiefer, and Wolfowitz (1956). The result for $\hat{\theta}_1$ and $\hat{\theta}_3$ follow likewise since $\|F_{1n} - F_{2n}\|_\infty \leq 2(n+1)^{-1}$ and $\|F_{3n} - F_{2n}\|_\infty \leq 2n^{-1}$. \square

In the proof of Corollary 1 we also need the following approximation of $\hat{\theta}_i - \theta$ by a sum of independent random variables.

LEMMA 4. *Under the assumptions of Lemma 3, we have*

$$(1) \quad \hat{\theta}_i = \theta + \frac{\frac{1}{2} - F_{in}(\theta)}{G'(\theta)} + R_{in},$$

where $n^{\frac{1}{2}}R_{in} \xrightarrow{P} 0$, $i = 1, 2, 3$. In addition, we have

$$(2) \quad \hat{\theta}_i = \theta + \frac{2}{n} \sum_{i=1}^n \frac{\frac{1}{2} - F(2\theta - X_i)}{G'(\theta)} + R_{in}^*,$$

where $n^{\frac{1}{2}}R_{in}^* \xrightarrow{P} 0$, $i = 1, 2, 3$.

PROOF. The proof of (1) for $\hat{\theta}_3$ is almost exactly the same as the proof of Theorem 1 of Ghosh (1971) except that F_{3n} replaces F_n . The crucial step,

$$n E\{F_{3n}(\theta + tn^{-\frac{1}{2}}) - G(\theta + tn^{-\frac{1}{2}}) - F_{3n}(\theta) + G(\theta)\}^2 \longrightarrow 0,$$

is verified easily from known variance formula calculations for U-statistics.

Representation (2) follows directly from (1) since the *projection* of $\frac{1}{2} - F_{3n}(\theta)$ is just $2n^{-1} \sum [\frac{1}{2} - F(2\theta - X_i)]$. The results for $\hat{\theta}_1$ and $\hat{\theta}_2$ follow since F_{1n} and F_{2n} are suitably close to F_{3n} . \square

PROOF OF COROLLARY 1. We need to verify that the second term of (2.3) converges to the right-hand side of (2.6). As in Pyke and Shorack (1968, Theorem 2.1), construct $U_n^*(t)$ to have the same distribution as the empirical process $U_n(t) = n^{\frac{1}{2}} \Delta_n(\sigma F_0^{-1}(t) + \theta)$ and such that for some $\delta > 0$

$$d(U_n^*, U_n) = \sup_{0 < t < 1} \left| \frac{U_n^*(t) - U_n(t)}{[t(1-t)]^{\frac{1}{2}-\delta}} \right| \xrightarrow{p} 0 .$$

Note that the appropriate $T(F; \Delta_n)$ comes from (2)

$$\begin{aligned} T(F; \Delta_n) &= \frac{- \int_{-\infty}^{\infty} [\Delta_n(\theta+x) + \Delta_n(\theta-x)][f(\theta+x) + f(\theta-x)] dx}{\int_{-\infty}^{\infty} [f(\theta+x) + f(\theta-x)]^2 dx} \\ &= \frac{- n^{\frac{1}{2}} \int_0^1 [U_n(t) + U_n(1-t)] dt}{2/\sigma \int_{-\infty}^{\infty} f_0^2(x) dx} \\ &= \frac{2}{n} \sum_{i=1}^n \left[\frac{\frac{1}{2} - F(2\theta - X_i)}{G'(\theta)} \right] . \end{aligned}$$

Thus the second term of (2.3) may be expanded and written as

$$T(U_n) = \sigma \left\{ \int_0^1 [U_n(t) + U_n(1-t)]^2 dF_0^{-1}(t) - \left(\int_0^1 [U_n(t) + U_n(1-t)] dt \right)^2 / \int_{-\infty}^{\infty} f_0^2(x) dx \right\} .$$

The last step is to show that $T(U_n^*) - T(U) \xrightarrow{p} 0$. The first term of $T(U_n^*) - T(U)$ is bounded in absolute value by

$$\left| \int_{-\infty}^{\infty} [U_n^*(F_0(x)) + U_n^*(F_0(-x))]^2 dx - \int_{-\infty}^{\infty} [U(F_0(x)) + U(F_0(-x))]^2 dx \right|$$

$$\leq 2d(U_n^*, U_n) \{d(U_n^*, U_n) + 2d(U, 0)\} \int_{-\infty}^{\infty} [F_0(x)(1-F_0(x))]^{1-2\delta} dx \xrightarrow{P} 0 .$$

The moment condition insures that the latter integral is finite. The second term of $T(U_n^*) - T(U)$ is handled similarly and thus both $T(U_n^*)$ and $T(U_n)$ have limiting distribution $T(U)$. \square

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