A Note on the Campbell Sampling Theorem

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ABSTRACT. Campbell's 1968 sampling theorem is examined and a more explicit formula for the truncation error is given. The result is shown to apply to random processes bandlimited in a general sense.

§1. INTRODUCTION. Many versions of the Shannon sampling series have been proposed; see e.g. [1] for a recent review. In 1968 Campbell [2] proposed the sampling series

$$(1) \qquad f(t) = \sum_{n=-\infty}^{\infty} f(nh) \frac{\operatorname{Sin} \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \hat{\psi}(\beta(t-nh)) \qquad -\infty < t < \infty$$

valid for $h^{-1} > 2w$, $\beta < h^{-1}/2 - w$ where $\hat{\psi}$ is the Fourier transform of a C^{∞} function ψ supported by $[-1,1]$ with $\int \psi dx = 1$ and f is any function whose fourier transform (in the distributional sense) is supported by $[-w,w]$.

The series (1) is also valid for stationary random processes. Campbell also gives an expression for the truncation error committed in using $2N + 1$ terms of (1) to evaluate f. In this note we give a more explicit form of the truncation error and show that the sampling theorem is valid for a general type of bandlimited random process, giving an expression for the truncation error in this case also.

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§2. BANDLIMITED FUNCTIONS. Call a function f bandlimited to w if $\|f\|_k^2 = \int |f(t)|^2(1+t^2)^{-k}dt < \infty$ for some $k \geq 0$ and if the Fourier transform of f is supported by $[-w,w]$. The following form of the Paley-Wiener theorem will be useful.

<u>Theorem</u>. For all real t, a function f bandlimited to w satisfying $\|f\|_k < \infty$ satisfies

(2) $$|f(t)| \leq C_k(w)(1+|t|)^k \|f\|_k$$

where $C_k(w) \leq 4(W+1)K_k$ for constants K_k defined below.

<u>Proof</u>. Let ϕ be any C^∞ function with compact support, i.e. a testfunction. Then if F is the Fourier transform of f,

$$|F(\phi)| = |\int f(u)\hat{\phi}(u)du| \leq \left\{\int |f(u)|^2(1+u^2)^{-k}du . \int |\hat{\phi}(u)|^2(1+u^2)^k du\right\}^{\frac{1}{2}}$$
$$= \|f\|_k \left\{\int |\hat{\phi}(u)|^2(1+u^2)^k du\right\}^{\frac{1}{2}}$$

Now the distribution F can be extended to a continuous linear functional on the space E of all C^∞ functions topologized by the usual family of semi norms (see e.g. [3] p 88) and if X is a testfunction equal to 1 on $[-w,w]$ then for all $\xi \in E$ $F(\xi) = F(\xi X)$, so in particular, setting $\xi(x) = e^{2\pi itx}$ we obtain

$$|f(t)| = |F(e^{2\pi ixt})| = |F(e^{2\pi ixt}X(x))| \leq \|f\|_k \left\{\int |\hat{X}(u-t)|^2(1+u^2)^k du\right\}^{\frac{1}{2}} .$$

Now let $$\gamma(x) = \begin{cases} K\exp\left(1/(x^2-1)\right) & |x| \leq 1 ; \\ 0 & |x| > 1 ; \end{cases}$$

where $K^{-1} = \int_{|x|\leq 1} \exp\left(1/(x^2-1)\right)dx = 2.2523$; then $\int \gamma(x)dx = 1$

and $\gamma(x)$ is a testfunction supported by $[-1,1]$. Also let for $\delta > 0$

$$I_{w+\delta}(x) = \begin{cases} 1 & |x| \le w+\delta , \\ 0 & |x| > w+\delta . \end{cases}$$

Then choose for χ the function $\chi(x) = \frac{1}{\delta} \int_{-\infty}^{\infty} I_{w+\delta}(v)\, \gamma\left(\frac{x-v}{\delta}\right) dv$

then $\chi(x)$ is 1 on $[-w,w]$ and is supported by $(-w-\delta, w+\delta)$, and is C^{∞}. Then denoting the Fourier transform of a function λ by $\hat{\lambda}(u) = \int_{-\infty}^{\infty} e^{-2\pi iux}\lambda(x)dx$ we have

$$\chi(u) = \gamma(u\delta)\,\frac{\text{Sin}\, 2\pi(w+\delta)u}{\pi u} \quad \text{and so}$$

$$|f(t)| \leqq \|f\|_k \left\{ \int_{-\infty}^{\infty} |\hat{\gamma}(u\delta)|^2 \left(\frac{\text{Sin}\, 2\pi(w+\delta)(u-t)}{\pi(u-t)} \right)^2 (1+u^2)^k du \right\}^{\frac{1}{2}}$$

$$= \|f\|_k \left\{ \int_{-\infty}^{\infty} |\hat{\gamma}(u\delta)|^2 \left\{ \frac{\text{Sin}\, 2\pi(w+\delta)u}{\pi u} \right\} \left(1+(u+t)^2\right)^k du \right\}^{\frac{1}{2}}$$

$$\le \|f\|_k (1+|t|)^k \left\{ \int |\hat{\gamma}(u\delta)|^2 \frac{\text{Sin}^2 2\pi(w+\delta)u}{(\pi u)^2} \left(1+|u|^2\right)^k du \right\}^{\frac{1}{2}}$$

using the inequality $\left(1+(t+u)^2\right) \le (1+|t|)^2(1+|u|)^2$.

Then (2) is true with $C_k^2(w) = \inf_{\delta>0} \int |\hat{\gamma}(u\delta)|^2 \text{Sin}^2 \frac{2\pi(w+\delta)u}{(\pi u)^2}(1+|u|^2)^k du$.

For an upper bound on $C_k^2(w)$, consider setting $\delta = 1$, then since $\left|\frac{\text{Sin}\, x}{x}\right| \le \frac{2}{1+|x|}$

$$\begin{aligned} C_k^2(w) &\le 16(w+1)^2 \int |\hat{\gamma}(u)|^2 \left(1+2\pi(w+1)|u|\right)^{-2} (1+|u|^2)^k \\ &\le 16(w+1)^2 \int |\hat{\gamma}(u)|^2 (1+u^2)^{k-1} du \\ &= 16(w+1)^2 K_k^2 \quad \text{say, and hence} \end{aligned}$$

(3) $$|f(t)| \le 4K_k(w+1)(1+|t|)^k \|f\|_k .$$

The constant K_k can be calculated from

$$K_k^2 = \int |\hat{\gamma}(u)|^2 (1+u^2)^{k-1} du = \sum_{j=0}^{k-1} \binom{k-1}{j} \int |\hat{\gamma}(u)|^2 u^{2j} \, du$$

$$= \sum_{j=0}^{k-1} \binom{k-1}{j} \int \left| \frac{\gamma^{(j)}(x)}{(2\pi)^j} \right|^2 dx \, .$$

The derivatives of γ can be generated by a simple recursive scheme described in [2], a numerical integration then allows the calculation of the K_k, given below in Table 1 for $k = 1,2,3,4,5$.

Table 1. Values of K_k to 4 significant figures.

k	K_k
1	8.217×10^{-1}
2	1.003×10^{0}
3	1.649×10^{0}
4	8.446×10^{0}
5	1.252×10^{2}

We note in passing that for $k = 0$, the inequality takes the simple form

$$|f(t)| = \left| \int_{-W}^{W} e^{2\pi i t x} \hat{f}(x) dx \right| \leq \left\{ \int_{-W}^{W} 1 \, dx \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx \right\}^{\frac{1}{2}}$$

$$= (2w)^{\frac{1}{2}} \|f\|_0$$

so (2) is valid for $k = 0$ with $C_0(w) = (2w)^{\frac{1}{2}}$.

For convenience, and following [2] we propose to take for ψ in the series (1) the function γ defined above. A version of (2) appropriate for $\hat{\gamma}$ is obtained simply by integrating the f.t. of $\gamma^{(\nu)} \left(= \frac{d^\nu \gamma}{\partial x^\nu} \right)$ by parts, obtaining

$$|(2\pi iu)^{\nu}\hat{\gamma}(u)| = |\int e^{-2\pi iux}\gamma^{(\nu)}(x)dx| \leq \int|\gamma^{(\nu)}(x)|dx$$

and so for $u \neq 0$ we obtain

$$(4) \qquad |\hat{\gamma}(u)| \leq \frac{1}{(2\pi)^{\nu}} \int|\gamma^{(\nu)}(x)|dx\,|u|^{-\nu}$$

$$= c_{\nu}|u|^{-\nu}, \text{ say, for any integer } \nu \geq 0 .$$

Again the constants c_{ν} may be calculated simply by numerical quadrature. An upper bound for the c_{ν} may be obtained by the methods of [2]. Table 2 below gives the first few values of c_{ν}:

Table 2. Values of c_{ν}.

ν	c_{ν}
0	1.000
1	.2637
2	.1822
3	.3237
4	1.5550
5	13.6874

By means of (3) and (4) we can obtain a precise inequality on the truncation error of (1) in terms of the norm $\|f\|_k$ of f, using the method of [2]:

<u>Theorem</u>. Let f be a function bandlimited to w and satisfying $\|f\|_k < \infty$. Then if γ is the function defined in the proof of theorem 1 and if $\rho_N(t)$ denotes the truncation error using the function γ in (1) i.e. if

$$\rho_N(t) = f(t) - \sum_{|n| \leq N} f(nh) \frac{\operatorname{Sin} \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \hat{\gamma}\big(\beta(t-nh)\big)$$

$$\text{then for } \nu > k \quad |\rho_N(t)| \leq \frac{2\, C_k(w)\, C_\nu \|f\|_k \operatorname{Sin} \pi h^{-1} t\, (1+Nh)^k}{\beta^\nu \pi \big| Nh - |t| \big|^\nu}$$

$$\leqq \frac{8(w+1) K_k C_\nu \|f\|_k |\operatorname{Sin} \pi h^{-1} t|\, (1+nh)^k}{\beta^\nu \pi \big| Nh - |t| \big|^\nu}$$

for $|t| < Nh$, $\beta < h^{-1}/2 - w$ and $h^{-1}/2 > w$.

Proof. Similar to §4 of [2].

§3. BANDLIMITED PROCESSES. Consider a zero mean second order process $x(t)$, and let $R(t,s) = E\big(x(t)\, \overline{x(s)}\big)$ be the covariance function of $x(t)$. Such a process will be termed band-limited to w , if for some non negative integer k , $\int R(t,t)(1+t^2)^{-k} dt < \infty$ and the Fourier transform of $R(t,s)$ is a distribution in the plane supported by $[-w,w] \times [-w,w]$.

For the properties of such processes see [4].

A key property is the following. Consider the Hilbert space H_k consisting of all functions f satisfying $\int |f|^2 (1+t^2)^{-k} dt$. Then the operator R defined by

$$R\, f(s) = \int R(t,s) f(t) (1+t^2)^{-k} dt$$

is a trace-class operator from H_k to H_k . Let λ_j , f_j , $j = 1,2,3, \cdots$ be the eigenvalues and eigenvectors of this operator. Then the following results are true (details may be found in [4] and [5])

1. There exist random variables e_j satisfying $E(e\, \bar{e}_j) = \delta_{jj}\, \lambda_j$ such that

$$x(t) = \sum_{j=1}^{\infty} f_j(t)\, e_j \quad , \tag{5}$$

the convergence of (5) being in mean square;

2. Each function f_j satisfies $\|f_j\|_k = 1$ and is bandlimited to w ;

3. For each t, s $R(t,s) = \sum_{j=1}^{\infty} \lambda_j f_j(t)\overline{f_j(s)}$, the series converging absolutely.

Using these we may prove the following

Theorem 3. Let $x(t)$ be a zero mean process whose covariance function R satisfies $\int R(t,t)(1+t^2)^{-k}dt < \infty$ and that is bandlimited to w. Then $x(t)$ satisfies the sampling expansion (1) and the root mean square truncation error

$$\left\{E\left|x(t) - \sum_{|n| \leq N} x(nh)\frac{\operatorname{Sin} \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)}\hat{\gamma}\big(p(t-nh)\big)\right|^2\right\}^{\frac{1}{2}}$$

is bound by $\{\int R(t,t)(1+t^2)^{-k}dt\}^{\frac{1}{2}} \dfrac{8(w+1)K_k C_\nu |\operatorname{Sin} \pi h^{-1} t|(1+Nh)^k}{\beta^\nu \pi(Nh - |t|)^\nu}$

for $k < \nu$, $\beta < h^{-1}/2 - w$, $h^{-1}/2 > w$, and $|t| < Nh$.

Proof. Using (5) above,

$$E\left|x(t) - \sum_{|n| \leq N} x(nh) \frac{\operatorname{Sin} \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \hat{\gamma}\big(\beta(t-nh)\big)\right|^2$$

$$\leq E\left|\sum_{j=1}^{\infty} f_j(t) - \sum_{|n| \leq N} f(nh) \frac{\operatorname{Sin} \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)} \hat{\gamma}\big(\beta(t-nh)\big)e_j\right|^2$$

$$= \sum_{j=1}^{\infty} \lambda_j \left|f_j(t) - \sum_{|n| \leq n} f_j(nh)\frac{\operatorname{Sin} \pi h^{-1}(t-nh)}{\pi h^{-1}(t-nh)}\hat{\gamma}\big(\beta(t-nh)\big)\right|^2$$

$$\leq \sum_{j=1}^{\infty} \lambda_j \left|\frac{8(w+1)K_k C_\nu \|f_j\|_k |\operatorname{Sin} \pi h^{-1} t|(1+Nh)^k}{\beta^\nu \pi(Nh - |t|)^\nu}\right|^2 .$$

Now consider $R(t,t) = \sum_{j=1}^{\infty} \lambda_j |f_j(t)|^2$. Integrating term by term with respect to $(1+t^2)^{-k}dt$ we obtain

$$\int R(t,t)(1+t^2)^{-k}dt = \sum_{j=1}^{\infty} \lambda_j \int |f_j(t)|^2 (1+t^2)^{-k}dt$$

$$= \sum_{j=1}^{\infty} \lambda_j \|f_j\|_k^2$$

$$= \sum_j \lambda_j .$$

For $k < \nu$, we see that the truncation error converges to zero as $N \to \infty$, thus proving the theorem.

The series (1) converges in mean square for a bandlimited process $x(t)$; since the sample paths of $x(t)$ are bandlimited to w with probability 1, (see [4]); it follows that $x(t)$ satisfies the series (1) with probability 1.

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