DEPENDENT RELEVATIONS

N.L. Johnson, University of North Carolina at Chapel Hill

and

S. Kotz, University of Maryland

ABSTRACT

Krakowski (1973) introduced the concept of the *relevation*, which is the conditional distribution of a random variable X_2, given that it exceeds an independent random variable X_1. This is relevant to failure time distributions when replacement is made from an aging stock.

We develop analysis appropriate to relevation for dependent variables, and suggest it might be relevant if there is batch-to-batch variation, producing association between lifetimes of initial and replacement components.

KEY WORDS AND PHRASES:

Relevation; Farlie-Gumbel-Morgenstern; Multivariate Burr; Multivariate Pareto; Replacement; Survival Distributions.

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1. Introduction

Krakowski (1973) (see also Grosswald *et al.* (1980), Johnson and Kotz (1979)) has studied life-time distributions which arise in the following circumstances. When a component fails, a replacement is taken from a stock which has been stored for some time and so has been subject to an aging process. It is desired to find the distribution of time to failure (T) of the replacement component.

Denote by T_1, T_2 the times to failure of randomly chosen first and second components respectively, and their survival distribution functions (SDF's) by

$$S_i(t) = \Pr[T_i > t] \ , \ (i = 1,2) \text{ respectively .}$$

The SDF of time to failure of the second component, given that it is still in working order at the time of failure $T_1 = t_1$ of the first component, and *assuming T_1 and T_2 to be independent*, is

$$S_{2|1}(t|t_1) = \Pr[T_2 > t | T_2 > t_1] = S_2(t)/S_2(t_1) \quad (t \geq t_1) \ .$$

Hence the SDF of the time to failure (T) of the second component is

$$S_T(\tau) = \Pr[T>\tau] = \Pr[T_1>\tau] + \Pr[(T_1\le\tau)\cap(T_2>\tau)]$$

$$= S_1(\tau) + \int_0^\tau f_1(t)S_{2|1}(\tau|t)dt$$

$$= S_1(\tau) + S_2(\tau)\int_0^\tau \frac{f_1(t)}{S_2(t)}\,dt \qquad (1)$$

where $f_1(t) = -dS_1(t)/dt$ is the density function of T_1. Krakowski terms the distribution (1) the *relevation* of $S_1(t)$ and $S_2(t)$.

There are circumstances in which it is desirable to relax the assumption of independence - for example, the second component may be taken from the same batch of product as the first, and batch-to-batch variation would result in some dependence. Formula (1) then generalizes to

$$S_T(\tau) = S_1(\tau) + \int_0^\tau f_1(t)\,\frac{S_{2|1}(\tau|t)}{S_{2|1}(t|t)}\,dt \qquad (2)$$

where $S_{2|1}(t_2|t_1) = \Pr[T_2>t_2|T_1=t_1]$.

In the circumstances just described it seems reasonable to assume that failure times for components in the same batch have an "exchangeable" joint distribution. This implies, *inter alia*, that the marginal distributions $S_1(t)$, $S_2(t)$ are identical. We now discuss some special cases.

2. Farlie-Gumbel-Morgenstern Distributions

We will now examine the consequences of supposing the joint distribution of T_1 and T_2 is of Farlie-Gumbel-Morgenstern (FGM) form with joint SDF

$$\Pr[(T_1>t_1)\cap(T_2>t_2)] = S_1(t_1)S_2(t_2)[1+\alpha(1-S_1(t_1))(1-S_2(t_2))] \qquad (3)$$

with $|\alpha| \le 1$. For the moment we do not impose the condition $S_1(t) = S_2(t)$.

The conditional SDF of T_2, given $T_1 = t_1$ is

$$S_{2|1}(t_2|t_1) = S_2(t_2)[1+\alpha(1-2S_1(t_1))(1-S_2(t_2))]$$

and hence (2) becomes

$$S_T(\tau) = S_1(\tau)+S_2(\tau) \int_0^\tau \frac{f_1(t)[1+\alpha(1-S_2(\tau))(1-2S_1(t))]}{S_2(t)[1+\alpha(1-S_2(t))(1-2S_1(t))]} dt \; . \tag{4}$$

If $S_1(t) = S_2(t) = S(t)$, (4) becomes

$$S_T(\tau) = S(\tau)\{1 + \int_0^\tau \frac{f(t)[1+\alpha(1-S(\tau))(1-2S(t))]}{S(t)[1+\alpha(1-S(t))(1-2S(t))]} dt\}$$

with $f(t) = -dS(t)/dt$. Making the substitution $s = S(t)$, we obtain

$$S_T(\tau) = S(\tau)\{1 + \int_{S(\tau)}^1 \frac{1+\alpha(1-S(\tau))(1-2s)}{s[1+\alpha(1-s)(1-2s)]} ds\} \; . \tag{5}$$

The integrand on the right can be written

$$As^{-1} - \{A(s - \tfrac{3}{4})-c\}\{(s - \tfrac{3}{4})^2 + \tfrac{1}{2}\alpha^{-1} - \tfrac{1}{16}\}^{-1}$$

with $A = \{1+\alpha(1-S(\tau))\}(1+\alpha)^{-1}$; $c = \frac{1}{4}\{3 - (4+\alpha)(1-S(\tau))\}(1+\alpha)^{-1}$, leading to the following results.

If $\alpha = 0$ so that T_1 and T_2 are independent

$$S_T(\tau) = S(\tau)[1 - \ln S(\tau)] \; .$$

For $\alpha = -1$

$$S_T(\tau) = S(\tau)[1 + \int_{S(\tau)}^1 \frac{S(\tau) + 2(1-S(\tau))s}{s^2(3-2s)} ds]$$

$$= S(\tau)[1 + \tfrac{1}{3}(1-S(\tau)) - \frac{6-4S(\tau)}{9} \ln \frac{S(\tau)}{3-2S(\tau)}]$$

$$= S(\tau)[\frac{4-3S(\tau)}{3} + \frac{2(3-2S(\tau))}{9} \ln \frac{3-2S(\tau)}{S(\tau)}] \; .$$

For $-1 < \alpha \le 1$

$$S_T(\tau) = S(\tau)\left[1 + \frac{1+\alpha(1-S(\tau))}{1+\alpha}\,\ell n\left\{\frac{1+\alpha(1-S(\tau))(1-2S(\tau))}{(S(\tau))^2}\right\} + \frac{3-(4+\alpha)(1-S(\tau))}{1+\alpha}\,g(S(\tau))\right]$$

where

$$g(S(\tau)) = \begin{cases} \dfrac{1}{2\sqrt{(1-8\alpha^{-1})}}\,\ell n\left[\left\{\dfrac{4S(\tau)-3+\sqrt{(1-8\alpha^{-1})}}{1+\sqrt{(1-8\alpha^{-1})}}\right\}^2\left\{\dfrac{1}{1+\alpha(1-S(\tau))(1-2S(\tau))}\right\}\right] & \text{for } \alpha<0 \\ \dfrac{1}{\sqrt{(8\alpha^{-1}-1)}}\,\tan^{-1}\left\{\dfrac{(1-S(\tau))\sqrt{(8\alpha^{-1}-1)}}{2\alpha^{-1}-(1-S(\tau))}\right\} & \text{for } \alpha > 0\ . \end{cases}$$

Some numerical values are given in Table 1.

Table 1

Values of $S_T(\alpha)$

α \ $S(\tau)$	0.2	0.4	0.5	0.6	0.8
-1.0	0.54972	0.81337	0.89139	0.94355	0.99261
-0.8	0.54494	0.80456	0.88294	0.93664	0.98990
-0.6	0.53973	0.79548	0.87425	0.92942	0.98714
-0.4	0.53414	0.78612	0.86530	0.92200	0.98432
-0.2	0.52818	0.77647	0.85608	0.91435	0.98144
0	0.52189	0.76651	0.84657	0.90649	0.97851
0.2	0.51524	0.75476	0.83676	0.89839	0.97552
0.4	0.50824	0.74286	0.82662	0.89004	0.97246
0.6	0.50090	0.73173	0.81615	0.88141	0.96934
0.8	0.49319	0.71830	0.80530	0.87250	0.96603
1.0	0.48510	0.70552	0.79407	0.86329	0.96288

Note that the value of $S_T(\tau)$ depends only on the value of $S(\tau)$ (the common $S_i(\tau)$) - not on the mathematical form of $S(\tau)$ as a function of τ.

That is to say, the relevation distribution value is defined by the value of the marginal SDF. This will be so whenever $S_1(\tau) = S_2(\tau)$, provided $S_{2|1}(\tau|t)/S_{2|1}(t|t)$ is a function of $S(\tau)$ and $S(t)$ only. The substitution $s = S(t)$ then results in expressing the integrand in (2) as a function of $S(\tau)$ and s only.

3. Multivariate Burr and Pareto Distributions

If t_1, t_2 have a bivariate Burr distribution (see e.g. Johnson and Kotz (1972), p. 289) with joint density function

$$k(k+1)(1+t_1^c + t_2^c)^{-(k+2)} \quad (k,c > 0;\ t_1,t_2 > 0) \tag{6}$$

the common marginal density and survival functions are

$$f(t) = kct^{c-1}(1+t^c)^{-(k+1)}; \quad S(t) = (1+t^c)^{-k}$$

respectively. Also

$$S_{2|1}(t_2|t_1) = \{1 + t_2^c(1+t_1^c)^{-1}\}^{-(k+1)}$$

and (4) becomes

$$\begin{aligned} S_T(\tau) &= (1+\tau^c)^{-k}[1 + \int_0^\tau \frac{kct^{c-1}}{(1+t^c)^{k+1}}\{\frac{1+\tau^c(1+t^c)^{-1}}{1+t^c(1+t^c)^{-1}}\}^{-(k+1)}dt] \\ &= (1+\tau^c)^{-k}[1 + k\int_0^\tau ct^{c-k}\{\frac{2t^c+1}{(t^c+1+\tau^c)(1+t^c)}\}^{k+1}dt] \\ &= (1+\tau^c)^{-k}[1 + k\int_0^{\tau^c}\{\frac{2u+1}{(u+1)(u+1+\tau^c)}\}^{k+1}du] \quad \text{(putting } t^c=u). \end{aligned} \tag{7}$$

Since $\tau^c = [S(\tau)]^{-\frac{1}{k}} - 1$ and the right hand side of (7) is a function of τ^c only (not of τ and c separately) we see that $S_T(\tau)$ can be expressed as a function of $S(\tau)$, depending on k, but not on c.

If t_1, t_2 have a bivariate Pareto distribution (e.g. Johnson and Kotz (1972), p. 285) with joint density function

$$a(a+1)\theta(t_1+t_2-\theta)^{-(a+2)} \quad (a,\theta > 0;\quad t_1,t_2 > \theta) \tag{8}$$

the common marginal density and survival functions are

$$f(t) = a\theta^a t^{-(a+1)};\quad S(t) = \theta^a t^{-a}$$

respectively. Also

$$S_{2|1}(t_2|t_1) + \left(\frac{t_1}{t_2+t_1-\theta}\right)^{a+1} \quad (t_2 > \theta)$$

and ( ) becomes

$$S_T(\tau) = \left(\frac{\theta}{\tau}\right)^a\left[1 + a\theta^a \int_\theta^\tau \left\{\frac{2t-\theta}{t(t+\tau-\theta)}\right\}^{a+1} dt\right]$$

$$= \left(\frac{\theta}{\tau}\right)^a\left[1 + a \int_1^{\tau/\theta} \left\{\frac{2w-1}{w(w+\tau\theta^{-1}-1)}\right\}^{a+1} dw\right] \quad \text{(putting } t=\theta w\text{)}. \tag{9}$$

Since $\tau\theta^{-1} = [S(\tau)]^{-\frac{1}{\alpha}}$ we see that, similarly to the bivariate Burr, $S_T(\tau)$ is a function of $S(\tau)$, depending on a, but not on θ

Comparison of (7) and (9) (and making the substitution u = w-1, or *vice versa*) leads to the more remarkable result that if k = a, $S_T(\tau)$ is the *same* function of $S(\tau)$ for the bivariate Pareto as for the bivariate Burr.

The integrals in (7) and (9) can be evaluated explicitly in terms of elementary functions if k (or a) are integers. For even moderately small values of these parameters (4 or more) the expressions are quite lengthy.

It is interesting to note that if the joint distribution of T_1 and T_2 is of the "fatal shock" type described by Marshall and Olkin (1967) (for the special case of exponential marginal distributions) wherein there is a non-zero probability that $T_1 = T_2$ (i.e. both fail simultaneously) but otherwise

T_1 and T_2 are independent, the relevation distribution is the same as if T_1 and T_2 were fully independent. This is because the replacement component must be selected from those which have not as yet failed.

4. Multifold Relevations

Formula (2) may be generalized to relevations of three of more variables $T_1, T_2, T_3, \ldots$ with SDP's $S_1(t)$, $S_2(t), S_3(t) \ldots$. Denoting by T_n^* the time of failure of the n-th component used (so that T would now be T_2^*) we have

$$S_{T_3^*}(\tau) = S_{T_2^*}(\tau) + \int_0^\tau \int_0^{t_2} f_{12}^*(t_1,t_2) \frac{S_{3|12}(\tau|t_1,t_2)}{S_{3|12}(t_2|t_1,t_2)} dt_1 dt_2 \tag{10}$$

where $f_{12}^*(t_1,t_2) = -f_1(t_1)\{\frac{d}{dt_2} S_{2|1}(t_2|t_1)\}/S_{2|1}(t_1|t_1)$, $(t_1<t_2)$, is the joint probability density function of T_1^* and T_2^*. The last term in (10) is the contribution to $S_{T_3^*}(\tau)$ from those occasions when the third component is in service at time τ.

Extension of the argument leads to the general recurrence formula

$$S_{T_n^*}(\tau) = S_{T_{n-1}^*}(\tau) + (-1)^{n+1} \int_0^\tau \int_0^{t_{n-1}} \cdots \int_0^{t_2} \left\{ \prod_{i=1}^{n-1} \frac{\frac{d}{dt_i} S_{i|<i}(t_i|t_1,\ldots,t_{i-1})}{S_{i|<i}(t_{i-1}|t_1,\ldots,t_{i-1})} \right\}$$

$$\times \frac{S_{n|<n}(\tau|t_1,\ldots,t_{n-1})}{S_{n|<n}(t_{n-1}|t_1,\ldots,t_{n-1})} dt_1 \ldots dt_{n-1} \tag{11}$$

where $S_{i|<i}(\cdot)$ denotes $S_{i|12\ldots\overline{i-1}}(\cdot)$.

Analytic evaluation of the multiple integral in (11) rapidly becomes very complex as n increases - even if the joint distribution of $T_1, T_2, \ldots$ is of simple form, for example generalized Farlie-Gumbel-Morgenstern (Johnson and Kotz (1975)). In this case, it is, however, still true that if $S_1(\tau) = S_2(\tau) = \ldots = S_n(\tau) = S(\tau)$, then $S_{T_n^*}(\tau)$ can be expressed as a function of α and $S(\tau)$ only.

If the joint distribution of T_1, T_2, T_3 is of the symmetrical generalized Farlie-Gumbel Morgenstern form

$$S_{T_1,T_2,T_3}(t_1,t_2,t_3)$$
$$= S(t_1)S(t_2)S(t_3)[1+\alpha\{(1-S(t_1))(1-S(t_2)) + (1-S(t_1))(1-S(t_3))$$
$$+ (1-S(t_2))(1-S(t_3))\} + \alpha'(1-S(t_1))(1-S(t_2))(1-S(t_3))] \quad (12)$$

$(|\alpha|\le 1;\ |\alpha'|\le 1 + 3\alpha;\ \alpha-\alpha'\le 1)$, then

$$S_{\ |1,2}(t_3|t_1,t_2)$$
$$= \{1+\alpha(1-2S(t_1))(1-2S(t_2))\}^{-1}S(t_3)[1+\alpha(1-2S(t_1))(1-2S(t_2))$$
$$+ 2\alpha(1-S(t_3))(S(t_1)+S(t_2)-1) + \alpha'(1-S(t_3))(1-2S(t_1))(1-2S(t_2))] \quad (13)$$

Also, since the joint distribution of any two of the T's is of form (3), the SDF of T_2^* is given by (4), and

$$\frac{\frac{d}{dt_2} S_{2|1}(t_2|t_1)}{S_{2|1}(t_1|t_1)} = -f(t_2) \cdot \frac{1+\alpha(1-2S(t_2))(1-2S(t_2))}{S(t_2)[1+\alpha(1-S(t_1))(1-2S(t_1))]} \quad (14)$$

After substitutions in (11), and putting $s_1 = S(t_1)$, $s_2 = S(t_2)$, we find

$$S_{T_3^*}(\tau) = S_{T_2^*}(\tau) + S(\tau) \int_0^{S(\tau)} \int_0^{s_2} \frac{1+\alpha(1-2s_1)(1-2s_2)}{s_1[1+\alpha(1-s_1)(1-2s_1)]} \times$$
$$\times \frac{1+\alpha(1-2s_1)(1-2s_2)+(1-S(\tau))[2\alpha(s_1+s_2-1)+\alpha'(1-2s_1)(1-2s_2)]}{s_2[1+\alpha(1-2s_1)(1-2s_2)+(1-s_2)\{2\alpha(s_1+s_2-1)+\alpha'(1-2s_1)(1-2s_2)\}]}\, ds_1 ds_2 \ .$$

The integral could be evaluated by expanding the integrand in partial fractions, but we do not proceed further.

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