THE CLOSENESS OF UNIFORM RANDOM POINTS IN A SQUARE AND A CIRCLE

by

D. J. Daley

Department of Statistics (I.A.S.)
Australian National University
Canberra, 2600, Australia

Abstract

It is far from clear how best to approximate one random vector (for example, a stochastic process observed at n consecutive time-points) by another random vector which may have for example a simpler stochastic structure. This problem is illustrated by seeking constructions of points P and Q uniformly distributed over concentric square and circle respectively and of unit area, so as to minimize $D_1 \equiv E|P-Q|$ and $D_2 \equiv (E|P-Q|^2)^{\frac{1}{2}}$.

Key Words and Phrases: Distance of random vectors, optimal mappings of random vectors, approximating stochastic processes.

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THE CLOSENESS OF UNIFORM RANDOM POINTS IN A SQUARE AND A CIRCLE

D. J. Daley, Australian National University

1. STATEMENT OF PROBLEM

Let the points P and Q be uniformly distributed over a unit square S and a circle C, concentric with S and having the same unit area. What joint distribution(s) for P and Q minimize(s) $D_1 \equiv E|P-Q|$ and $D_2 \equiv (E|P-Q|^2)^{\frac{1}{2}}$? ($|\cdot|$ denotes Euclidean distance.)

2. ORIGIN OF PROBLEM

Given distribution functions (d.f.s) F and G of R^1-valued random variables (r.v.s), there exists a mapping in terms of a r.v. U uniformly distributed on (0,1) such that r.v.s X^* and Y^* have F and G as their d.f.s and $E|X^*- Y^*|$ and $E|X^*- Y^*|^2$ are least: the mapping $(X^*,Y^*) = (F^{-1}(U), G^{-1}(U))$ suffices for both minimization problems. It is essentially unique for minimizing $E|X^*- Y^*|^2$, but need not be unique for minimizing $E|X^*- Y^*|$.

Considering the analogous problem for R^2-valued r.v.s, the optimal strategy is no longer clear. The strategy is needed in approximating one sequence of r.v.s by another, in that the approximating sequence should be close to the original sequence, especially when the r.v.s are structural elements in a stochastic process.

3. PARTIAL SOLUTION OF PROBLEM, I:D_1

For definiteness, let the square S have vertices at $(\pm 2^{-\frac{1}{2}},0)$, $(0,\pm 2^{-\frac{1}{2}})$, so that the circle has centre at the origin and radius $\pi^{-\frac{1}{2}}$. Symmetry considerations show that it is enough to consider 1-1 maps of P and Q which lie in the 45° wedge $0 \le y \le x$ intersecting S and C respectively.

In minimizing D_1, it is asserted first that it is enough to consider mappings $T: S \to C$ for which $Q = TP$ has $P = Q$ if P and Q are in the common part of S and C, and hence region A is then mapped 1-1 into region B (see Figure 1). For suppose not; let S_0 be the measurable subset of A that is not mapped into B (S_0 is measurable because the mapping T, being defined via r.v.s, is measurable). Using $\lambda(\cdot)$ to denote Lebesgue measure, if $\lambda(S_0) = 0$, then $\Pr\{P: TP \in B\} = \Pr\{P \in A\}$ and there is nothing more to prove. Otherwise, set $C_0 = TS_0$, and let S_1 be the part of C_0 (if any) that is not mapped into B.

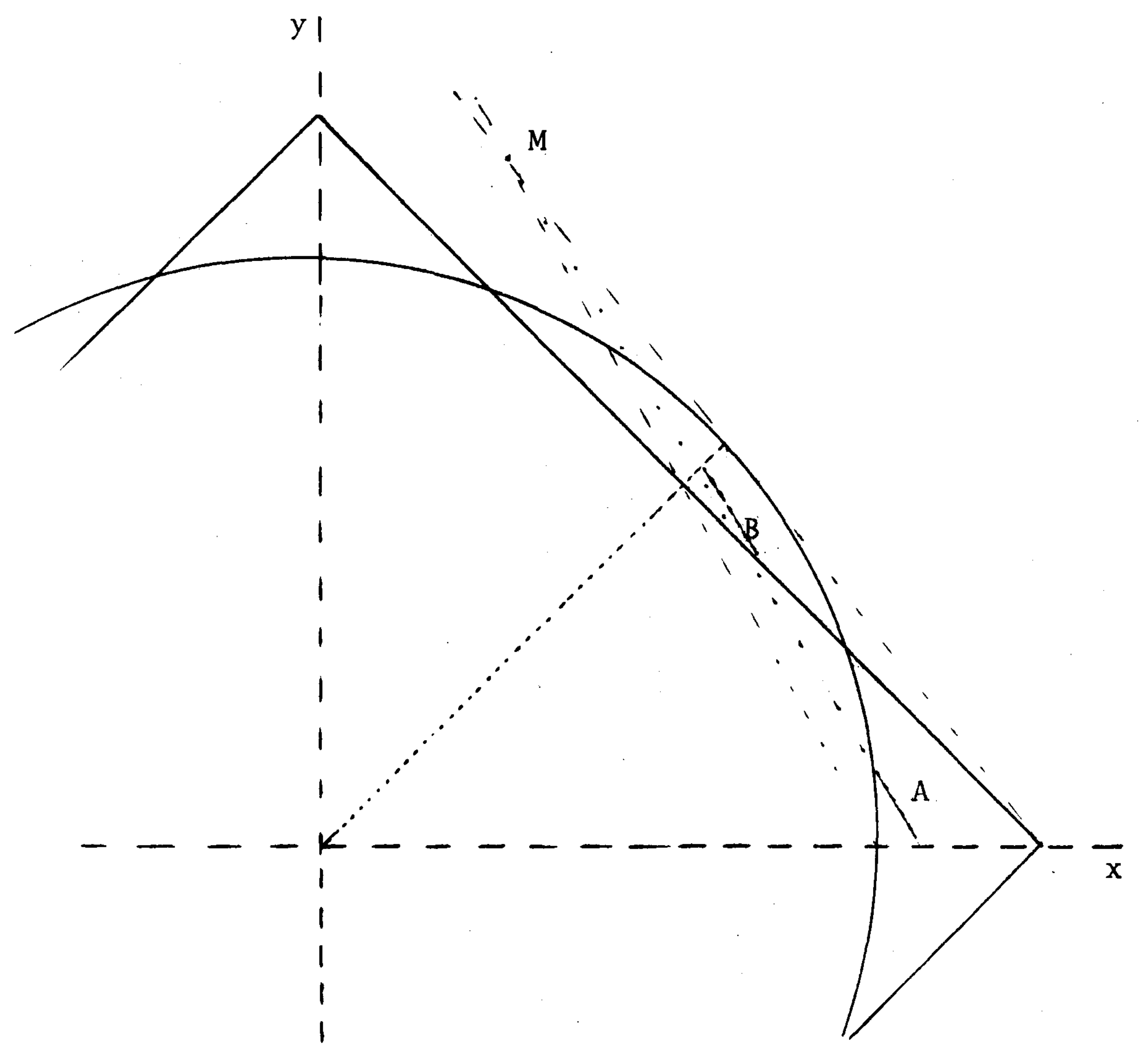

Figure 1

If $\lambda(S_1) = 0$, then $\Pr\{TP \text{ or } T^2P \in B | P \in A\} = 1$, and we can consider the mapping T^* defined by

$$T^*P = \begin{cases} Q & (P \notin A) \\ T^{n(P)}P & (P \in A) \end{cases}$$

where $n(P) = \inf\{n: T^nP \in B\}$ $(P \in A)$; when $\lambda(S_1) = 0$, $n(P) = 1$ or 2. If $\lambda(S_1) > 0$, continue sequentially defining S_{n+1} equal to the subset of $C_n = TS_n$ which is not in B. Since all $\{S_j: j = 0,1,\ldots\}$ are disjoint (because the mapping is 1-1), and $1 = \lambda(S) \geq \sum_{j=0}^{\infty} \lambda(S_j)$, we conclude that

$$\Pr\{P \in A: n(P) < \infty\} = \lambda(A).$$

Consequently, given any $P \in A$, we can a.s. find points $P_j \equiv T^jP$ $(j < n(P))$, $Q = T^{n(P)}P$, such that

$$|P - T^*P| \leq |P - P_1| + |P_1 - P_2| + \ldots + |P_{n(P)-1} - Q|,$$

and therefore, for any mapping T that does not map A into B, there is a mapping T^* taking A into B and setting $T^*P = P$ for $P \notin A$ for which $E|P - T^*P| \leq E|P - TP|$.

We assert next that for any P_1, P_2 in A, the optimal mapping T cannot have the two straight line segments $[P_1, TP_1]$, $[P_2, TP_2]$ intersecting, for if they did, the triangular inequality again shows that

$$|P_1 - TP_2| + |P_2 - TP_1| \leq |P_1 - TP_1| + |P_2 - TP_2|.$$

To describe a mapping having such a property, construct tangents to the regions A and B as indicated by the dotted lines in Figure 1, intersecting in M say. With rays centered on M, sweep out segments of the rays through B and A so that equal areas of B and A are cut off by corresponding segments, and map points of one segment onto the other in proportion to the areas of the two truncated cones formed by the ray segments and differentially perturbed ray segments.

I believe this mapping as just outlined is optimal, though a complete proof is lacking. Whether there exists a unique optimal mapping (strictly, an equivalence class of such mappings, since any mapping may be altered on a set of zero Lebesgue measure), I do not know either. What is easy to show (and tractable to compute), by rotation of axes so that one is then parallel to the line through the centroids μ^A and μ^B of A and B respectively, is that

$$D_1 \geq |\mu^A - \mu^B| \Pr\{P \neq Q\}.$$

Under the mapping as described,

$$\Pr\{P \neq Q\} = 8.\tfrac{1}{2}(\pi^{-1}\arccos(\pi^{\frac{1}{2}}/2) - \tfrac{1}{2}(\pi^{-1} - 4^{-1})^{\frac{1}{2}})$$

$$= 4\pi^{-1}\arccos(\pi^{\frac{1}{2}}/2) - (4\pi^{-1} - 1)^{\frac{1}{2}} \approx .090546.$$

By further calculation, $\mu^A = (.606448, .032890)$, $\mu^B = (.441662, .301939)$, so

$$D_1 \geq .028567 .$$

It is also worth remarking that any mapping that identifies P and Q inside the common part of S and C, yields

$$D_0 \equiv \inf_{\text{maps}} \lim_{\alpha \downarrow 0} E|P - Q|^{\alpha} = \Pr\{P \neq Q\} .$$

4. PARTIAL SOLUTION OF PROBLEM, II: D_2

It will be convenient to retain the same axes as in section 3, but to write r for the radius of C: later we shall consider circles of different radii. Let W, Z be independent r.v.s uniformly distributed over (0,1), and let

$$P = (Z(W/2)^{\frac{1}{2}}, (1-Z)(W/2)^{\frac{1}{2}}),$$

$$Q = (rW^{\frac{1}{2}} \sin \tfrac{1}{2}\pi Z, rW^{\frac{1}{2}} \cos \tfrac{1}{2}\pi Z)$$

It can be verified that P and Q are then uniformly distributed over the first quadrant in the square S and a concentric circle of radius r. Further, the mapping corresponds to letting a radial arm sweep out equal areas from the y-axis in a clockwise direction, being a proportion Z of the total area of each region of S and C in the quadrant, and then mapping the points on each arm into one another in proportion $W^{\frac{1}{2}}$ of their respective total lengths.

It is conjectured that this mapping minimizes D_2; if so,

$$D_2^2 = E[W(Z/2^{\frac{1}{2}} - r \sin \tfrac{1}{2}\pi Z)^2 + W((1-Z)/2^{\frac{1}{2}} - r \cos \tfrac{1}{2}\pi Z)^2]$$

$$= \tfrac{1}{2} E[\tfrac{1}{2}(Z^2+(1-Z)^2) + r^2 - 2^{\frac{1}{2}}r(Z \sin \tfrac{1}{2}\pi Z + (1-Z) \cos \tfrac{1}{2}\pi Z)]$$

$$= \frac{1}{6} + \frac{r^2}{2} - \frac{(4\sqrt{2})r}{\pi^2} .$$

With $r = \pi^{-\frac{1}{2}}$, this expression equals

$$\frac{1}{6} + \frac{1}{2\pi} - (\frac{2}{\pi})^{5/2} \approx .002451 \approx (.04951)^2$$

while choosing r so as to minimize the mean square distance, i.e., putting $r = (4\sqrt{2})\pi^{-2}$, the mean square distance equals

$$\frac{1}{6} - \frac{16}{\pi^4} \approx .002411 .$$

Certainly we must have

$$D_2 \leq .04951,$$

while by writing $P - Q = P - \mu_8^S + \mu_8^S - \mu_8^C + \mu_8^C - Q$ where μ_8^S and μ_8^C are the centroids of S and C in the wedge $0 \leq y \leq x$, we must have

$$D_2 \geq |\mu_8^S - \mu_8^C| = .02693 .$$

By considering the motion of points on the perimeter of the circle under the mapping described in this section, it can be deduced that the mean distance moved is at least as large as

$$\frac{2}{3}(\frac{\pi}{4})^{-1} \left[\int_0^{\alpha} (\pi^{-\frac{1}{2}} \cos \theta - \tfrac{1}{2})\, d\theta + 2^{\frac{1}{2}} \int_{\alpha}^{\pi/4} (\tfrac{1}{2} - \pi^{-\frac{1}{2}} \cos \theta)\, d\theta \right] \approx .03458$$

where $\alpha = \arccos(\pi^{\frac{1}{2}}/2)$.

On the other hand, any mapping that leaves P inside $S \cap C$ invariant has

$$(E|P-Q|^2)^{\frac{1}{2}} \geq |\mu^A - \mu^B|(\Pr\{P \neq Q\})^{\frac{1}{2}} \approx .094936.$$

5. CONCLUDING REMARKS

The argument showing that an optimal mapping as measured by D_1 (or D_0) leaves invariant points in the common part of S and C, extends to any two bounded sets in R^d of the same d-dimensional volume. The other property of the map of the non-intersection of line-segments $[P_1, TP_1]$ and $[P_2, TP_2]$ extends to other figures in R^2 of the same area: the higher-dimensional analogue is harder to visualize.

Note that the suggested (class of) optimal mappings, leaving P invariant inside $S \cap C$, has some suggestions of the probabilistic notion of coupling of stochastic processes on discrete state space, or of yielding the variation metric of two probability distributions.

There may well be a physical principle underlying the mapping which is suggested as minimizing D_2: connected regions in S remain connected when mapped into C. (This property is not held by neighbourhoods on the boundary of A interior to S under the mapping of Section 3 for D_1.) Observe that if $P = (x,y)$, then $Q = (r2^{\frac{1}{2}}(x+y)\ \sin(\frac{1}{2}\pi x/(x+y),\ r2^{\frac{1}{2}}(x+y)\ \sin(\frac{1}{2}\pi y/(x+y)))$, which is not an analytic function of $x + iy$. The mapping has the obviously appealing property of being defined for the class of similar circles S. And its higher-dimensional analogue can also be visualized: map the surface of an orthant onto the surface of the hyper-sphere optimally, and the rest is scaled by the d-th root of the radial distance; the harder part is to determine the mapping of the surfaces.

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