ON α-SYMMETRIC MULTIVARIATE DISTRIBUTIONS*

by

Stamatis Cambanis, Robert Keener† and Gordon Simons

University of North Carolina

ABSTRACT

A random vector is said to have a 1-symmetric distribution if its characteristic function is of the form $\phi(|t_1| +\ldots+ |t_n|)$. 1-symmetric distributions are characterized through representations of the admissible functions ϕ and through stochastic representations of the random vectors, and some of their properties are studied.

AMS 1970 Subject Classification: Primary 60E05.

Key Words and Phrases: α-symmetric, multivariate, characteristic function, Laplace transform, conditional distribution, n-dimensional version, stable.

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*This research was supported by the Air Force Office of Scientific Research Grant AFOSR-80-0080 and by the National Science Foundation Grants MCS78-01240 and MCS81-00748.

†Now at the University of Michigan.

FOOTNOTES

[1] This proof is due to Hari Mukerjee. We are aware of several other proofs, including one of our own and two distinct geometric proofs produced by Norman L. Johnson and (again) by Mukerjee.

[2] For the remainder of this section, X_1 and X_2 will denote *vectors* rather than single components of X, and in Theorems 4.1 and 4.2 below, they will denote subvectors of X.

1. INTRODUCTION

We shall say that a random vector $X = (X_1, \ldots, X_n)$ has an α-*symmetric distribution* ($\alpha > 0$) if its characteristic function is of the form

$$E \exp\{i(t_1X_1 + \ldots + t_nX_n)\} = \phi(|t_1|^\alpha + \ldots + |t_n|^\alpha) \tag{1.1}$$

for all $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, and we shall write $X \sim S_n(\alpha,\phi)$. We shall also denote by $\Phi_n(\alpha)$ the class of all functions $\phi\colon [0,\infty) \to \mathbb{R}$ which are such that $\phi(|t_1|^\alpha + \ldots + |t_n|^\alpha)$ is a characteristic function on $\mathbb{R}^n$. We are interested here in studying α-symmetric distributions through representations of the classes $\Phi_n(\alpha)$, through stochastic representations of the random vectors X, and, where feasible, through representations of their probability density functions.

It is clear that α-symmetric distributions in one dimension are simply symmetric distributions, and that the marginals of α-symmetric distributions are α-symmetric. It follows, for given α, that the classes $\Phi_n(\alpha)$ are nonincreasing in n. We shall let $\Phi_\infty(\alpha) = \cap_n \Phi_n(\alpha)$ so that $\Phi_n(\alpha) \downarrow \Phi_\infty(\alpha)$ as $n \to \infty$.

2-symmetric distributions are naturally called spherically symmetric, and have been studied extensively. Schoenberg (1938) showed that $\phi \in \Phi_n(2)$ if and only if

$$\phi(u) = \int_{[0,\infty)} \Omega_n(r^2 u)\,dF(r) \ , \qquad u \geq 0 \ , \tag{1.2}$$

for some distribution function F on $[0,\infty)$, where $\Omega_n(t_1^2 + \ldots + t_n^2)$ is the characteristic function of a random vector $U = (U_1, \ldots, U_n)$ which is uniformly distributed on the surface of the unit sphere in $\mathbb{R}^n$. It follows immediately that X has a spherically symmetric distribution, i.e., $X \sim S_n(2,\phi)$ if and only if it has the stochastic representation

$$X \overset{L}{=} RU \ , \tag{1.3}$$

where $R \geq 0$ is independent of U and has distribution function F. This makes precise the intuitively clear property that spherically symmetric distributions are scale mixtures of uniform distributions.

In this paper, we establish integral representations of $\phi \in \Phi_n(1)$ analogous to (1.2), and stochastic representations of X's with 1-symmetric distributions analogous to (1.3). Again, 1-symmetric distributions are scale mixtures of a specified "primitive distribution". We expect that analogous representations hold for other values of α besides 1 and 2. We have some evidence for this in the case $\alpha = \frac{1}{2}$ $(n = 2)$.

The classes $\Phi_\infty(\alpha)$ have been studied by Bretagnolle, Dacunha Castelle and Krivine (1966), and $\Phi_\infty(2)$ by Schoenberg (1938): For $\alpha \in (0,2]$, $\phi \in \Phi_\infty(\alpha)$ if and only if

$$\phi(u) = \int_{[0,\infty)} e^{-ur^\alpha}\, dF(r) \ , \qquad u \geq 0 \ ,$$

for some distribution function F on $[0,\infty)$. Equivalently, X is a finite segment of (or) an infinite dimensional vector whose finite dimensional distributions are α-symmetric if and only if

$$X \overset{L}{=} RY \ ,$$

where $R \geq 0$ is independent of Y which has independent and identically distributed symmetric stable components of index α and characteristic functions $E\exp(itY_k) = \exp(-|t|^\alpha)$. I.e., α-symmetric infinite sequences of random variables are scale mixtures of independent and identically distributed stable variables of index α (a consequence of deFinetti's theorem).

When $X \sim S_n(\alpha,\phi)$, its component random variables X_k are identically distributed with $X_k \sim S_1(\alpha,\phi)$. It follows from (1.1) that

$$t_1X_1 + \ldots + t_nX_n \overset{L}{=} (|t_1|^\alpha + \ldots + |t_n|^\alpha)^{1/\alpha} X_1 . \tag{1.4}$$

Recently, Eaton (1981) introduced the interesting notion of n-dimensional versions of one-dimensional symmetric distributions: The distribution of X is an n-*dimensional version* of the distribution of a symmetric random variable Z if $t_1X_1 + \ldots + t_nX_n \overset{L}{=} c(t)Z$ for $t = (t_1,\ldots,t_n) \in \mathbb{R}^n$ with $c(t) \geq 0$ for all t. Relationship (1.4) shows that the distribution of X is an n-dimensional version of the distribution of X_1 with $c(t) = (|t_1|^\alpha + \ldots + |t_n|^\alpha)^{1/\alpha}$. Thus the results of this paper provide specific constructions of some n-dimensional versions of distributions with characteristic functions of the form $\phi(|t|)$ $(t \in \mathbb{R})$, $\phi \in \Phi_n(1)$. However, these n-dimensional versions may constitute a small subclass of all n-dimensional versions. For instance, in the special case $\phi(u) = \exp(-u)$ $(\phi \in \Phi_\infty(\alpha),\ 0 < \alpha \leq 2)$, the n-dimensional versions of the symmetric stable distribution with index α defined through (1.1) have independent and identically distributed stable components of index α, while, as shown in Theorem 6.1, the class of all n-dimensional versions consists of all symmetric n-dimensional stable distributions of index α. (Half of this theorem is due to Eaton (1981).)

We do not know whether the classes $\Phi_n(\alpha)$, $2 \le n < \infty$, contain nondegenerate members ($\phi \not\equiv 1$) when $\alpha > 2$. They clearly do when $n = 1$, and do not when $n = \infty$. (See Bretagnolle, Dacunha Castelle and Krivine (1966).)

The paper is organized as follows: For $\alpha = 1$, we consider in Section 2 the bivariate case; in Section 3 the general multivariate case $n \ge 2$; and in Section 4 certain properties of 1-symmetric distributions, including a description of their conditional distributions, which are also 1-symmetric. We believe that clarity is enhanced by treating the cases $n = 2$ and $n > 2$ separately; there is virtually no overlap between Sections 2 and 3. The case "$\alpha = \frac{1}{2}$ and $n = 2$" is discussed (but not fully resolved) in Section 5. Finally, n-dimensional versions and a possible generalization of α-symmetry are discussed in Section 6.

It should be evident that much work remains to be done on α-symmetric distributions when $\alpha \ne 1,2$.

2. BIVARIATE 1-SYMMETRIC DISTRIBUTIONS

In this section, we derive integral representations of functions in $\Phi_2(1)$, and stochastic representations of bivariate random vectors with 1-symmetric distributions. Such distributions are absolutely continuous, except possibly at zero; we derive, as well, representations of their densities.

THEOREM 2.1. (a) $\phi \in \Phi_2(1)$ *if and only if*

$$\phi(u) = \int_{[0,\infty)} \phi_0(ru)dF(r) , \qquad u \ge 0 , \tag{2.1}$$

where F *is a distribution function on* $[0,\infty)$ *and* $\phi_0 \in \Phi_2(1)$ *assumes the* *form*

$$\phi_0(u) = \frac{2}{\pi} \int_u^\infty \frac{\sin v}{v} \, dv , \qquad u \ge 0 . \tag{2.2}$$

Expressed more simply,

$$\phi(u) = \frac{2}{\pi} \int_0^\infty \frac{\sin uv}{v} F(v) dv , \qquad u > 0 . \tag{2.3}$$

(b) *Equivalently, the characteristic function of* (X,Y) *has the form*

$$Ee^{i(sX+tY)} = \phi(|s|+|t|) , \qquad s,t \in \mathbb{R} , \tag{2.4}$$

if and only if

$$(X,Y) \stackrel{L}{=} R(X_0,Y_0) , \tag{2.5}$$

where R *is a nonnegative random variable with distribution function* F, *and* (X_0,Y_0) *is independent of* R *and has the characteristic function* $\phi_0(|s|+|t|)$. *A stochastic description of* (X_0,Y_0) *is given by*

$$(X_0,Y_0) \stackrel{L}{=} \left(\frac{U}{B^{\frac{1}{2}}}, \frac{V}{(1-B)^{\frac{1}{2}}}\right) , \tag{2.6}$$

where (U,V) *is uniformly distributed on the unit circle of* $\mathbb{R}^2$, *and* B *is distributed* Beta$(\frac{1}{2},\frac{1}{2})$ *independently of* (U,V).

(c) *Equivalently, a bivariate distribution is* 1*-symmetric if and only if it is absolutely continuous on* $\mathbb{R}^2 - \{(0,0)\}$ *with a density of the form*

$$g(x,y) = \int_{(0,\infty)} r^{-2}\, g_0(\tfrac{x}{r}, \tfrac{y}{r})\,dF(r) \tag{2.7}$$

(and has an atom of size F(0) *at* (0,0)*), where* F *is a distribution function on* $[0,\infty)$ *and* g_0 *is the bivariate density of* (X_0, Y_0) *given by*

$$g_0(x,y) = \frac{1_{[1,\infty)}(|x|) - 1_{[1,\infty)}(|y|)}{\pi^2(x^2 - y^2)}, \qquad x^2 \neq y^2 . \tag{2.8}$$

Expressed more simply,

$$g(x,y) = \frac{F(|x|) - F(|y|)}{\pi^2(x^2 - y^2)}, \qquad x^2 \neq y^2 . \tag{2.9}$$

We shall refer to ϕ_0 (and occasionally to (X_0, Y_0) and g_0) as a *primitive*. Part (a) of the theorem characterizes members of $\Phi_2(1)$ as scale mixtures of a primitive. The work of Schoenberg (1938) and of Bretagnolle, Dacunha Castelle and Krivine (1966) referred to can be viewed as providing the same kind of characterization of members of $\Phi_n(\alpha)$ for $\alpha = 2$, $1 \le n \le \infty$, and for $n = \infty$, $0 < \alpha < 2$, respectively. Theorem 3.1 and Section 5 below provide additional evidence that the members of every class $\Phi_n(\alpha)$ are expressible as scale mixtures of a primitive $\phi_0 \in \Phi_n(\alpha)$ $(0 < \alpha \le 2,\ 1 \le n \le \infty)$.

In the course of the proof of Theorem 2.1, we will use the following facts of independent interest contained in Propositions 2.1 and 2.2.

PROPOSITION 2.1. *If the random variable* B *is* Beta$(\frac{1}{2},\frac{1}{2})$, *then for all* $s,t \in \mathbb{R}$,

$$\frac{s^2}{B} + \frac{t^2}{1-B} \overset{L}{=} \frac{(|s|+|t|)^2}{B} . \tag{2.10}$$

Proof.[1] It is sufficient to consider the case $s,t \geq 0$ with $s + t > 0$, and to let $B = \sin^2 \theta$, where θ is uniformly distributed on the interval $(0, \pi/2)$. Then the left-hand side of (2.10) becomes $(s^2/\sin^2 \theta) + (t^2/\cos^2 \theta)$, which is expressible as $(s+t)^2 \, (1 + [T_x(\theta)]^2)$, where $x = (s-t)/(s+t)$ and $T_x(\theta) = (x + \cos 2\theta)/\sin 2\theta$. Thus it suffices to verify that the distribution of $|T_x(\theta)|$ is independent of x for $-1 \leq x \leq 1$, or, more specifically, that for all such x,

$$\Pr(|T_x(\theta)| \leq z) = \frac{2}{\pi} \tan^{-1} z \,, \qquad z > 0 \,. \tag{2.11}$$

This is easily shown for $x = \pm 1$. When $-1 < x < 1$, $T_x(\theta)$ is a strictly decreasing function of θ on $(0, \pi/2)$, and has range $\mathbb{R}$. Fix z, and let θ_0 and $\theta_1 > \theta_0$ be the two roots of $|T_x(\theta)| = z$. Then $x + \cos 2\theta_0 = z \sin 2\theta_0$ and $x + \cos 2\theta_1 = -z \sin 2\theta_1$, which yield $\cos 2\theta_0 - \cos 2\theta_1 = z(\sin 2\theta_0 + \sin 2\theta_1)$ and, in turn, $\theta_1 - \theta_0 = \tan^{-1} z$, from which (2.11) follows. $\square$

Proposition 2.1 is equivalent to a curious definite integral formula:

$$\int_0^{\pi/2} f\left(\frac{s^2}{\sin^2 \theta} + \frac{t^2}{\cos^2 \theta}\right) d\theta = \int_0^{\pi/2} f\left(\frac{(s+t)^2}{\sin^2 \theta}\right) d\theta \,,$$

which is valid for all $s \geq 0$, $t \geq 0$, and all functions f for which the integrals make sense. This surprisingly general identity appears to be new.

PROPOSITION 2.2. *Let* (U,V) *be uniformly distributed on the unit circle in* $\mathbb{R}^2$ *and* B *be distributed* Beta(½,½) *independently of* (U,V).

Then

$$(X_0, Y_0) = \left(\frac{U}{B^{\frac{1}{2}}}, \frac{V}{(1-B)^{\frac{1}{2}}}\right) \tag{2.12}$$

has characteristic function

$$E e^{i(sX_0 + tY_0)} = \frac{2}{\pi} \int_{|s|+|t|}^{\infty} \frac{\sin v}{v}\, dv\,, \qquad s,t \in \mathbb{R}\,, \tag{2.13}$$

joint density

$$g_0(x,y) = \begin{cases} \dfrac{1}{\pi^2 |x^2 - y^2|} & \textit{for } |x| < 1 \le |y| \textit{ or } |y| < 1 \le |x|\,, \\ 0 & \textit{otherwise}\,, \end{cases} \tag{2.14}$$

and common marginal densities

$$h_0(u) = \frac{1}{\pi^2 |u|} \ln\left|\frac{1+|u|}{1-|u|}\right|\,, \qquad u \ne 0\,. \tag{2.15}$$

Moreover,

$$sX_0 + tY_0 \overset{L}{=} (|s|+|t|)X_0\,, \qquad s,t \in \mathbb{R}\,. \tag{2.16}$$

Proof. (2.14) follows from (2.12) by a straightforward computation. Likewise, (2.15) follows from (2.14) by a straightforward, albeit messy, computation. (2.16) follows easily from (2.13), which remains to be shown. The characteristic function of (U,V) is well-known (see, for instance, Schoenberg (1938)) and is given by $E\,\exp[i(aU + bV)] = J_0([a^2 + b^2]^{\frac{1}{2}})$, where J_0 is a Bessel function of the first kind. Thus we obtain by (2.12) and Proposition 2.1,

$$Ee^{i(sX_0+tY_0)} = EJ_0([\frac{s^2}{B} + \frac{t^2}{1-B}]^{\frac{1}{2}}) = EJ_0(\frac{|s|+|t|}{B^{\frac{1}{2}}}) .$$

Putting $|s| + |t| = a$ and using the identity $J_0(ax) = 1 - x \int_0^a J_1(vx)dv$ (cf., Gradshteyn and Ryzhik (1980), 6.511 (7)), we have (with the help of Fubini's theorem):

$$EJ_0(\frac{a}{\sqrt{B}}) = \frac{1}{\pi} \int_0^1 J_0(\frac{a}{\sqrt{x}}) \frac{dx}{\sqrt{x(1-x)}} = \frac{2}{\pi} \int_1^\infty J_0(ax) \frac{dx}{x\sqrt{x^2-1}}$$

$$= \frac{2}{\pi} \int_1^\infty \frac{dx}{x\sqrt{x^2-1}} - \frac{2}{\pi} \int_0^a dv \int_1^\infty J_1(vx) \frac{dx}{\sqrt{x^2-1}}$$

$$= 1 - \frac{2}{\pi} \int_0^a \frac{\sin v}{v} dv = \frac{2}{\pi} \int_a^\infty \frac{\sin v}{v} dv$$

(cf., Gradshteyn and Ryzhik (1980), 6.552 (6), 8.464 (1,2), 8.465 (1), 3.721 (1)). □

Proof of Theorem 2.1. Let the characteristic function of (X,Y) have the form (2.4). We will show (2.1), (2.5) and (2.7).

Assume first that the characteristic function is $\mathbb{R}^2$-integrable:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty |\phi(|s|+|t|)|dsdt = 8 \int_0^\infty u|\phi(u)|du < \infty . \tag{2.17}$$

Then (X,Y) has a continuous density f(x,y) given by

$$g(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(sx+ty)} \phi(|s|+|t|)dsdt$$

$$= \frac{1}{\pi^2} \int_0^{\infty} \int_0^{\infty} \cos(sx)\cos(ty)\phi(s+t)dsdt$$

$$= \frac{1}{\pi^2} \int_0^{\infty} du\phi(u) \int_0^{u} dt \cos([u-t]x)\cos(ty)$$

$$= \frac{1}{\pi^2} \int_0^{\infty} \phi(u) \frac{x \sin(ux) - y \sin(uy)}{x^2 - y^2} du \text{ for } x^2 \neq y^2 ,$$

which is just the right side of (2.9) with the function F defined by

$$F(x) = x \int_0^{\infty} \sin(ux)\phi(u)du , \qquad x \geq 0 . \tag{2.18}$$

Since $g \geq 0$, (2.9) implies that F is nondecreasing. Thus the right side of (2.7) (with this F and g_0 defined in (2.8)) is well-defined, and equality (2.7) can be viewed as an alternative way of writing (2.9). Since g and g_0 are p.d.f.'s, we have

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)dxdy$$

$$= \int_{(0,\infty)} dF(r) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r^{-2} g_0(\frac{x}{r}, \frac{y}{r})dxdy = \int_{(0,\infty)} dF(r) ,$$

and consequently F is a d.f. on $[0,\infty)$. This shows (2.7); (2.1) and (2.5) follow from Proposition 2.2.

Now let ϕ in (2.4) be any member of $\Phi_2(1)$ and let $\phi_n(u) = \phi(u)\exp(-|u|/n)$, $u \geq 0$, $n \geq 1$. Then ϕ_n satisfies (2.17), and the conclusions of the theorem hold for $(X + n^{-1}W, Y + n^{-1}Z)$, where W and Z are independent Cauchy variables which are jointly independent of (X,Y). Thus

$$(X+\frac{W}{n}, Y+\frac{Z}{n}) \overset{L}{=} R_n(X_0,Y_0)$$

for some $R_n \geq 0$ which is independent of (X_0,Y_0), $n \geq 1$. From this, (2.5) is easily established, and then (2.1) and (2.7) follow from Proposition 2.2.

Conversely, if any of the equivalent relations (2.1), (2.5) or (2.7) holds, then (2.4) follows from Proposition 2.1 in the same way that (2.13) follows from (2.12). □

The correspondence between $\phi \in \Phi_2(1)$ and the distribution function F in (2.1) is one-to-one. Indeed, F determines ϕ uniquely by (2.1). Conversely, if ϕ satisfies (2.17), then F is determined by (2.18), and, in general, F is the weak limit of $\{F_n\}$, where $F_n(x) = x \int_0^\infty \sin(ux)\phi(u)\exp(-u/n)du$, $x \geq 0$, $n \geq 1$. Alternatively, one can derive (using (2.18) and a simple weak convergence argument) the following *general* form of the Laplace-Stieltjes transform $\hat{F}$ of F (and thereby establish the uniqueness of F):

$$\hat{F}(s) = \int_0^\infty (1+v)^{-2} \phi(s\sqrt{v})dv , \qquad s \geq 0 . \tag{2.19}$$

The following theorem based upon (2.19) is analogous to Theorem 2 of Cambanis, Huang and Simons (1981), and its proof is given in Section 3 in a more general setting.

THEOREM 2.2. *A bounded continuous function* ϕ *defined on* $[0,\infty)$ *is a member of* $\Phi_2(1)$ *if and only if the right-hand side of* (2.19) *is the Laplace transform of a nonnegative random variable.*

3. MULTIVARIATE 1-SYMMETRIC DISTRIBUTIONS

In this section, we continue the study of 1-symmetric distributions by deriving the analogue of Theorem 2.1 for $n > 2$.

Below, $U = (U_1, \ldots, U_n)$ denotes an n-dimensional random vector which is uniformly distributed on the unit sphere in $\mathbb{R}^n$, $D = (D_1, \ldots, D_n)$ denotes an n-dimensional random vector with Dirichlet distribution and parameters $(\frac{1}{2}, \ldots, \frac{1}{2})$, and $\Omega_n(t_1^2 + \ldots + t_n^2)$ is the characteristic function of U, which is given by

$$\Omega_n(r^2) = \Gamma(n/2)(2/r)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(r) \ , \qquad r > 0 \ .$$

THEOREM 3.1. $\phi \in \Phi_n(1)$ *(*$n \geq 2$*) if and only if*

$$\phi(u) = \int_{[0,\infty)} \phi_0(ru)dF(r) \ , \tag{3.1}$$

where F *is a distribution function on* $[0,\infty)$ *and* $\phi_0 \in \Phi_2(1)$ *is given by*

$$\phi_0(u) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})} \int_1^\infty \Omega_n(xu^2) x^{-\frac{n}{2}} (x-1)^{\frac{n-3}{2}} dx \ . \tag{3.2}$$

Equivalently, the joint characteristic function of $X = (X_1, \ldots, X_n)$ *has the form*

$$Ee^{i(t_1X_1 + \ldots + t_nX_n)} = \phi(|t_1| + \ldots + |t_n|) \tag{3.3}$$

if and only if

$$X \overset{L}{=} R\left(\frac{U_1}{\sqrt{D_1}}, \ldots, \frac{U_n}{\sqrt{D_n}}\right) , \tag{3.4}$$

where R *is a nonnegative random variable with distribution function* F, *and* R, $U = (U_1, \ldots, U_n)$, *and* $D = (D_1, \ldots, D_n)$ *are independent. In this case,* X *(has an atom of size* F(0) *at* 0 *and) is absolutely continuous on* $\mathbb{R}^n - 0$ *with density* g *given by*

$$g(x) = \int_{[0,\infty)} r^{-n} g_0(r^{-1} x) dF(r) , \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n , \tag{3.5}$$

$$= \frac{\Gamma^2(\frac{n}{2})}{(n-2)!\ \pi^n} \sum_{k=1}^{n} \frac{\int_{(0,|x_k|)} (x_k^2 - r^2)^{n-2}\, r^{-n+2}\, dF(r)}{\prod_{\substack{j=1 \\ j \neq k}}^{n} (x_k^2 - x_j^2)} ,$$

$$|x_k| \neq |x_j| , \qquad k \neq j ,$$

where

$$g_0(x) = \frac{\Gamma^2(\frac{n}{2})}{(n-2)!\ \pi^n} \sum_{k=1}^{n} \{(x_k^2 - 1)_+^{n-2} / \prod_{\substack{j=1 \\ j \neq k}}^{n} (x_k^2 - x_j^2)\} , \tag{3.6}$$

$$|x_k| \neq |x_j| , \qquad k \neq j .$$

In proving Theorem 3.1, we will use the following higher dimensional versions of Propositions 2.1 and 2.2.

PROPOSITION 3.1. *For all real* $t_1, \ldots, t_n$,

$$\frac{t_1^2}{D_1} + \ldots + \frac{t_n^2}{D_n} \overset{L}{=} \frac{(|t_1| + \ldots + |t_n|)^2}{D_1} \qquad (n \geq 2) . \tag{3.7}$$

PROPOSITION 3.2. *The random vector*

$$X_0 = \left(\frac{U_1}{\sqrt{D_1}}, \ldots, \frac{U_n}{\sqrt{D_n}} \right) \tag{3.8}$$

has characteristic function $\phi_0(|t_1| + \ldots + |t_n|)$, *where* ϕ_0 *is defined by* (3.2), *and probability density function* g_0 *described in* (3.6) *(*$n \geq 2$*).*

By Propositions 2.1 and 2.2, these claims are true when $n = 2$. Their proofs for general $n > 2$ require induction, and are based upon the following (easily justified) stochastic representations of D and U:

$$D \stackrel{L}{=} ((1-D_n)D_1', \ldots, (1-D_n)D_{n-1}', D_n) = ((1-D_n)D', D_n) \tag{3.9}$$

and

$$U \stackrel{L}{=} (\sqrt{1-U_n^2}\, U_1', \ldots, \sqrt{1-U_n^2}\, U_{n-1}', U_n) = (\sqrt{1-U_n^2}\, U', U_n) \ , \tag{3.10}$$

where $D' = (D_1', \ldots, D_{n-1}')$ is $\text{Dirichlet}_{n-1}(\frac{1}{2}, \ldots, \frac{1}{2})$ and independent of D, and $U' = (U_1', \ldots, U_{n-1}')$ is uniformly distributed on the unit sphere in $\mathbb{R}^{n-1}$ and independent of U. Hereafter we assume D, D', U and U' are jointly independent.

Proof of Proposition 3.1 *for* $n > 2$. Make the induction hypothesis that (3.7) holds for $n - 1 \geq 1$. Using this twice and using the symmetry of the distribution of D, we obtain

$$\frac{t_1^2}{D_1} + \ldots + \frac{t_n^2}{D_n} \stackrel{L}{=} \frac{1}{1-D_n}\left(\frac{t_1^2}{D_1'} + \ldots + \frac{t_{n-1}^2}{D_{n-1}'}\right) + \frac{t_n^2}{D_n}$$

$$\stackrel{L}{=} \frac{(|t_1| + \ldots + |t_{n-1}|)^2}{(1-D_n)D_1'} + \frac{t_n^2}{D_n} \stackrel{L}{=} \frac{(|t_1| + \ldots + |t_{n-1}|)^2}{D_1} + \frac{t_n^2}{D_2}$$

$$\stackrel{L}{=} \frac{1}{1-D_n}\left\{\frac{(|t_1| + \ldots + |t_{n-1}|)^2}{D_1'} + \frac{t_n^2}{D_2'}\right\}$$

$$\stackrel{L}{=} \frac{1}{1-D_n} \cdot \frac{(|t_1| + \ldots + |t_n|)^2}{D_1'} \stackrel{L}{=} \frac{(|t_1| + \ldots + |t_n|)^2}{D_1} \ . \qquad \square$$

Proof of Proposition 3.2 *for* $n > 2$. We need the fact that D_n is Beta(½, (n-1)/2). Putting $\tau = |t_1| + \ldots + |t_n|$ and $X_0 = (X_1, \ldots, X_n)$, we obtain from Proposition 3.1,

$$\mathcal{E} e^{i(t_1X_1 + \ldots + t_nX_n)} = \mathcal{E}\mathcal{E}\{e^{i(t_1X_1 + \ldots + t_nX_n)} | D^{(n)}\}$$

$$= \mathcal{E}\Omega_n\left(\frac{t_1^2}{D_1} + \ldots + \frac{t_n^2}{D_n}\right) = \mathcal{E}\Omega_n\left(\frac{\tau^2}{D_1}\right)$$

$$= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\ \Gamma(\frac{n-1}{2})} \int_0^1 \Omega_n(\frac{\tau^2}{x}) x^{-\frac{1}{2}}(1-x)^{\frac{n-3}{2}}\, dx$$

$$= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\ \Gamma(\frac{n-1}{2})} \int_1^\infty \Omega_n(r\tau^2) r^{-\frac{n}{2}}(r-1)^{\frac{n-3}{2}}\, dr$$

$$= \phi_0(\tau) \ .$$

Turning our attention now to g_0, we assume that (3.6), with n replaced by n-1, gives the p.d.f. of $X_0' = (X_1', \ldots, X_{n-1}') = (U_1'/\sqrt{D_1'}, \ldots, U_{n-1}'/\sqrt{D_{n-1}'})$ for some $n - 1 \geq 1$. To complete the induction step, we must show that (3.6) is the p.d.f. of

$$X_0 = (X_1,\dots,X_n) \stackrel{L}{=} \left(\sqrt{\frac{1-U_n^2}{1-D_n}}\, X_0',\ \frac{U_n}{\sqrt{D_n}}\right) .$$

Making the transformation

$$(x_1',\dots,x_{n-1}', u_n, d_n) \to (x_1,\dots,x_n, d_n) ,$$

where

$$x_k = \sqrt{\frac{1-u_n^2}{1-d_n}}\, x_k' , \qquad k = 1,\dots,n-1 , \qquad x_n = \frac{u_n}{\sqrt{d_n}} ,$$

whose Jacobian is $d_n^{-\frac{1}{2}}[(1-u_n^2)/(1-d_n)]^{(n-1)/2}$, and using the facts that D_n is Beta($\frac{1}{2}$, $(n-1)/2$) and U_n has p.d.f.

$$\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\;\Gamma(\frac{n-1}{2})}(1-u_n^2)^{(n-3)/2} , \qquad -1 < u_n < 1 ,$$

we find that the ((n+1)-dimensional) random vector (X_0,D_n) has the p.d.f.

$$\frac{\Gamma^2(\frac{n}{2})}{(n-3)!\;\pi^n} \sum_{k=1}^{n-1} \frac{[x_k^2 - 1 + d_n(x_n^2 - x_k^2)]^{n-3}}{\prod_{\substack{j=1\\ j\neq k}}^{n-1} (x_k^2 - x_j^2)}$$

$$\times\, 1_{(1,\infty)}\left(\frac{(1-d_n)x_k^2}{1 - d_n x_n^2}\right) 1_{(-1,1)}(x_n\sqrt{d_n})\, 1_{(0,1)}(d_n) .$$

Integrating out the variable d_n, we obtain the p.d.f. of X_0:

$$\frac{\Gamma^2(\frac{n}{2})}{(n-3)!\ \pi^n} \sum_{k=1}^{n-1} \frac{1}{\prod_{\substack{j=1 \\ j\neq k}}^{n-1} (x_k^2 - x_j^2)} \int_{I_{kn}} [x_k^2 - 1 + w(x_n^2 - x_k^2)]^{n-3}\, dw \tag{3.11}$$

$$= \frac{\Gamma^2(\frac{n}{2})}{(n-2)!\ \pi^n} \left\{ \sum_{k=1}^{n-1} \frac{(x_k^2 - 1)_+^{n-2}}{\prod_{\substack{j=1 \\ j\neq k}}^{n} (x_k^2 - x_j^2)} - \frac{(x_n^2 - 1)_+^{n-2}}{x_n^{2(n-2)}} \sum_{k=1}^{n-1} \frac{x_k^{2(n-2)}}{\prod_{\substack{j=1 \\ j\neq k}}^{n} (x_k^2 - x_j^2)} \right\},$$

where

$$I_{kn} = \begin{cases} [0, x_n^{-2}] & \text{for } x_k^2, x_n^2 > 1\ , \\ [(1 - x_k^2)/(x_n^2 - x_k^2),\ x_n^{-2}] & \text{for } x_k^2 \le 1 < x_n^2\ , \\ [0,\ (x_k^2 - 1)/(x_k^2 - x_n^2)] & \text{for } x_n^2 \le 1 < x_k^2\ , \\ \emptyset \text{ (the empty set)} & \text{for } x_k^2, x_n^2 \le 1\ . \end{cases}$$

The right side of (3.11) reduces to g_0, as described in (3.6), since

$$\sum_{k=0}^{n} \frac{x_k^{2(n-2)}}{\prod_{\substack{j=1 \\ j\neq k}}^{n} (x_k^2 - x_j^2)} = 0\ , \qquad x_k^2 \neq x_j^2\ , \qquad k \neq j\ .$$

The latter is seen as follows. With $y_k = x_k^2$ and $h(y) = y^{n-2}$, the sum becomes the (n-1)-st divided difference of h at the points $y_1, \ldots, y_n$, which is zero, since the (n-1)-st derivative of h vanishes. □

In terms of a Bessel function, the primitive $\phi_0 \in \Phi_n(1)$ is given by

$$\phi_0(u) = \frac{2^{\frac{n}{2}} \Gamma^2(\frac{n}{2})}{\sqrt{\pi}\, \Gamma(\frac{n-1}{2}) u^{\frac{n-2}{2}}} \int_1^\infty J_{\frac{n-2}{2}}(ur) \frac{(r^2-1)^{\frac{n-3}{2}}}{r^{\frac{3n-4}{2}}}\, dr , \qquad u > 0 \ (n \geq 2) .$$

For $n = 3$, $J_{\frac{1}{2}}(z) = (2/\pi z)^{\frac{1}{2}} \sin z$, and thus

$$\phi_0(u) = \frac{1}{2u} \sin u + \tfrac{1}{2} \cos u - \frac{u}{2} \int_u^\infty \frac{\sin x}{x}\, dx , \qquad u > 0 .$$

Similar expressions for ϕ_0 can be obtained for $n = 2k + 3$ using the corresponding expressions for $J_{k+\frac{1}{2}}$ (see Gradshteyn and Ryzhik (1980), 8.463).

Proof of Theorem 3.1. Let the characteristic function of X have the form (3.2), and assume first that it is $\mathbb{R}^n$-integrable:

$$\int_{\mathbb{R}^n} |\phi(|t_1| + \ldots + |t_n|)| dt_1 \ldots dt_n = \frac{2^n}{(n-1)!} \int_0^\infty u^{n-1} |\phi(u)| du < \infty . \quad (3.12)$$

Then X has a continuous p.d.f. given by

$$g(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(t_1x_1 + \ldots + t_nx_n)} \phi(|t_1| + \ldots + |t_n|) dt_1 \ldots dt_n \quad (3.13)$$

$$= \frac{1}{\pi^n} \int_{\mathbb{R}^n_+} \cos(t_1x_1) \ldots \cos(t_nx_n)\, \phi(t_1 + \ldots + t_n) dt_1 \ldots dt_n$$

$$= \frac{1}{\pi^n} \text{ Even} \int_{\mathbb{R}^n_+} \phi(t_1 + \ldots + t_n) e^{i(t_1 + \ldots + t_n)} dt_1 \ldots dt_n ,$$

where Even $h(x_1,\ldots,x_n) = 2^{-n} \sum h(\pm x_1,\ldots,\pm x_n)$, the sum being taken over all 2^n possible choices of the signs. If we denote by $[y_1,\ldots,y_n;\ \beta(\cdot)]$ the (n-1)-st divided difference of $\beta(\cdot)$ at the (distinct) points $y_1,\ldots,y_n$, we have

$$[y_1,\ldots,y_n;\ \beta(\cdot)] = \sum_{k=1}^{n} \frac{\beta(y_k)}{\prod_{\substack{j=1 \\ j\neq k}}^{n} (y_k - y_j)}\ , \tag{3.14}$$

and it easily can be shown by induction (under modest assumptions) that

$$\int_{\mathbb{R}_+^n} \alpha(t_1 +\ldots+ t_n)\beta^{(n-1)}(t_1x_1 +\ldots+ t_nx_n)dt_1\ldots dt_n \tag{3.15}$$

$$= \int_0^\infty \alpha(u)\,[x_1,\ldots,x_n;\ \beta(u\cdot)]\,du\ .$$

(It is sufficient that α and β be measurable functions on $\mathbb{R}_+$ and $\mathbb{R}$ respectively, that β have $n-2$ absolutely continuous derivatives, and that the left-hand integrand in (3.15) be Lebesgue integrable on $\mathbb{R}_+^n$.) It follows from (3.13) and (3.15) that

$$\pi^n\ g(x) = \text{Even} \int_0^\infty \phi(u) i^{-(n-1)} [x_1,\ldots,x_n;\ \exp(iu\cdot)]\,du\ ,$$

and since

$$\text{Even}[x_1,\ldots,x_n;\ \exp(iu\cdot)] = \begin{cases} i\,[x_1^2,\ldots,x_n^2;\ (\cdot)^{\frac{n-1}{2}} \sin(u\sqrt{\cdot})]\ , & n \text{ even}\ , \\ [x_1^2,\ldots,x_n^2;\ (\cdot)^{\frac{n-1}{2}} \cos(u\sqrt{\cdot})]\ , & n \text{ odd}\ , \end{cases}$$

we obtain

$$\pi^n g(x) = [x_1^2,\dots,x_n^2;\ B_n(\cdot)] \qquad (|x_j| \neq |x_k|,\ j \neq k)\ , \tag{3.16}$$

where

$$B_n(\cdot) = \begin{cases} (-1)^{\frac{n-2}{2}} (\cdot)^{\frac{n-1}{2}} \int_0^\infty \sin(u\sqrt{\cdot})\phi(u)du\ , & n \text{ even}\ , \\ (-1)^{\frac{n-1}{2}} (\cdot)^{\frac{n-1}{2}} \int_0^\infty \cos(u\sqrt{\cdot})\phi(u)du\ , & n \text{ odd}\ . \end{cases}$$

It follows from (3.12) that B_n is $(n-1)$-times continuously differentiable on $(0,\infty)$ and, moreover, $B^{(k)}(0+)$, $k = 1,\dots,n-2$, exist and are finite. In fact, from the nonnegativity of (3.16) (g is a p.d.f.) follow nearly as strong assertions concerning B_n and the nonnegativity of $B^{(n-1)}$. (See Roberts and Varberg (1973), page 238.) Substituting

$$B_n(y) = \frac{1}{(n-2)!} \int_0^y (y-z)^{n-2} B_n^{(n-1)}(z)dz + \sum_{k=0}^{n-2} \frac{y^k}{k!} B_n^{(k)}(0+)$$

into (3.16), and using the fact that $(n-1)$-divided differences of polynomials of order $n-2$ vanish, we obtain

$$\pi^n g(x) = \frac{1}{(n-2)!} \int_0^\infty [x_1^2,\dots,x_n^2;\ (\cdot-z)_+^{n-2}]\, B_n^{(n-1)}(z)dz\ .$$

Finally, setting

$$f(r) = 2\Gamma^{-2}(n/2) r^{n-1} B_n^{(n-1)}(r^2)\ , \qquad r > 0\ ,$$

it follows from (3.6) and (3.14) that

$$g(x) = \int_0^\infty r^{-n} g_0(r^{-1}x) f(r)dr\ .$$

Since g and g_0 are p.d.f.'s on $\mathbb{R}^n$,

$$1 = \int_{\mathbb{R}^n} g(x)dx = \int_0^\infty dr \cdot r^{-n} f(r) \int_{\mathbb{R}^n} dx\, g_0(r^{-1} x) = \int_0^\infty f(r)dr ,$$

so that f is a p.d.f. on $[0,\infty)$ with d.f. F (appearing in (3.5)). The proof is now completed in a manner similar to that used in the proof of Theorem 2.1. □

The following property will be useful in the sequel.

THEOREM 3.2. *The random vector* X *defined by* (3.4) *has independent standard Cauchy components if and only if* R^2 *is* F-*distributed with* n *and* n *degrees of freedom, i.e.,* R *has the p.d.f.*

$$f_n(r) = 2\Gamma(n)\{\Gamma(n/2)\}^{-2} r^{n-1}(1+r^2)^{-n} , \qquad r \geq 0 . \tag{3.17}$$

Proof. For the "if" part, it suffices to show that $\phi(u) = e^{-u}$, or equivalently that $(RU_1)/\sqrt{D_1}$ is standard Cauchy. Using 3.197 (4) of Gradshteyn and Ryzhik (1980), one first shows that $R^2/D_1 \overset{L}{=} B/(1-B)$, where B is beta-distributed with parameters $n/2$ and $\frac{1}{2}$, and then that $U_1(R^2/D_1)^{\frac{1}{2}}$ is standard Cauchy. The "only if" part follows from the one-to-one correspondence between ϕ and F in (3.1), established in Theorem 3.3, which uses only the "if" part of Theorem 3.2. □

The following result provides an alternative characterization of the class $\Phi_n(1)$, similar to that given in Theorem 2 of Cambanis, Huang and Simons (1981) for $\Phi_n(2)$, which together with Bernstein's theorem (cf., Feller (1971), page 439) provides an effective way to check whether a function ϕ belongs to the class $\Phi_n(1)$. It also establishes that *the correspondence between* $\phi \in \Phi_n(1)$ *and the distribution function* F *in* (3.1) *is one-to-one.*

THEOREM 3.3. $\phi \in \Phi_n(1)$ *(*$n \geq 1$*) if and only if* ϕ *is continuous and bounded, and*

$$\int_0^\infty \phi(sr)f_n(r)dr \ , \qquad s \geq 0 \ , \tag{3.18}$$

is the Laplace transform of a nonnegative random variable. The distribution function of this random variable is the distribution function F *appearing in* (3.1).

Proof. The "only if" part, as well as the claim concerning F, follow immediately from (3.1) and the "if" part of Theorem 3.2, which implies that

$$e^{-s} = \int_0^\infty \phi_0(sr)f_n(r)dr \ , \qquad s \geq 0 \ .$$

Conversely, let (3.18) be the Laplace transform of a nonnegative random variable whose distribution function is F, and define $\phi^* \in \Phi_n(1)$ using the right side of (3.1). We will show that $\phi = \phi^*$ by showing that the condition $H(s) \triangleq \int_0^\infty h(sr)f_n(r)dr = 0$ for $s \geq 0$ occurs for a bounded continuous function h (in our case, $h = \phi - \phi^*$) only when h is identically zero on $[0,\infty)$. But for $s > 0$, using (3.17) and $x^{-n} = \{\Gamma(n)\}^{-1} \int_0^\infty v^{n-1} \exp(-vx)dv$, we obtain

$$H(s) = 2\{\frac{\Gamma(n)}{\Gamma(n/2)}\}^2 s^n \int_0^\infty dv\, e^{-s^2 v} \cdot v^{n-1} \int_0^\infty du\, e^{-vu^2} \cdot u^{n-1} h(u) \ , \tag{3.19}$$

which, in words, is a multiple of a Laplace transform (in s^2) of a multiple of a Laplace transform (in v) of the function $(\cdot)^{(n/2)-1}\, h(\sqrt{\cdot})$. Because of the uniqueness property of Laplace transforms, it follows from (3.19) that $H(s) = 0$, $s > 0$, only if $h = 0$ a.e. Since h is continuous, it must be identically zero. □

4. PROPERTIES OF 1-SYMMETRIC DISTRIBUTIONS

In this section, we study certain properties of random vectors $X = (X_1, \ldots, X_n)$ with 1-symmetric distributions. Analogous and more extensive properties of 2-symmetric distributions were established in Cambanis, Huang and Simons (1981).

It is apparent from (3.4) that *unless* $X = 0$ *with probability one* (corresponding to $R \equiv 0$), *the components of* X *have finite* p^{th} *moments if and only if* $-1 < p < 1$ *and* $ER^p < \infty$. Thus the components of X fail to have finite first moments except in the degenerate case.

We now show that *if two random vectors have a joint* 1-*symmetric distribution, then the conditional distribution of one of them given the other is also* 1-*symmetric*. The following lemma, whose proof is straightforward and therefore omitted, will be needed.[2]

LEMMA 4.1. *Let* R_1 *and* R_2 *be nonnegative random variables,* Z_1 *and* Z_2 *be random vectors,* $X_i = R_i Z_i$ *(i = 1,2), and assume the random vectors* (R_1, R_2), Z_1, *and* Z_3 *are independent. Then the conditional distribution of* X_1 *given* $X_2 = x_2$ *is given by*

$$(X_1 | X_2 = x_2) \stackrel{L}{=} R_{x_2} Z_1 , \tag{4.1}$$

where $R_{x_2} \stackrel{L}{=} (R_1 | X_2 = x_2)$ *and* R_{x_2}, Z_1 *are independent.*

A regular conditional distribution of X_1 given $X_2 = x_2$ can easily be described, but for the sake of brevity it will be omitted.

In order to simplify the notation of the stochastic representation (3.4), we will write

$$\left(\frac{U_1}{\sqrt{D_1}}, \ldots, \frac{U_n}{\sqrt{D_n}}\right) = \frac{U^{(n)}}{\sqrt{D^{(n)}}} .$$

THEOREM 4.1. *Let* $X \stackrel{L}{=} R(U^{(n)}/\sqrt{D^{(n)}}) \sim S_n(1,\phi)$ *and* $X = (X_1, X_2)$, *where* X_1 *is* m*-dimensional,* $1 \le m < n$. *Then the conditional distribution of* X_1 *given* $X_2 = x_2$ *is given for almost every value of* X_2 *by*

$$(X_1|X_2=x_2) \stackrel{L}{=} R_{x_2} \frac{U^{(m)}}{\sqrt{D^{(m)}}} , \tag{4.2}$$

where the distribution of R_{x_2} *is defined implicitly by* (4.4) *below, and the right side of* (4.2) *is a stochastic representation of the type described in* (3.4).

A regular conditional distribution of X_1 given $X_2 = x_2$ can be defined which properly reflects (4.2) and (4.4). The distribution function F_{x_2} of R_{x_2} can be expressed in terms of the distribution function F of R, but the resulting expression is too complicated to be useful. Thus $(X_1|X_2=x_2) \sim S_m(1,\phi_{x_2})$, where ϕ_{x_2} is defined by (3.1) with F replaced by F_{x_2}, and with ϕ_0 (in (3.1)) defined by (3.2) with n replaced by m.

Proof of Theorem 4.1. The trick is to express X stochastically as

$$X = (X_1, X_2) \stackrel{L}{=} R\left(\frac{R_{mn}}{S_{mn}} \frac{U^{(m)}}{\sqrt{D^{(m)}}}, \left(\frac{1-R_{mn}^2}{1-S_{mn}^2}\right)^{\frac{1}{2}} \frac{U^{(n-m)}}{\sqrt{D^{(n-m)}}}\right) \tag{4.3}$$

and to apply Lemma 4.1 with $R_1 = RR_{mn}/S_{mn}$, $R_2 = R(1-R_{mn}^2)^{\frac{1}{2}}(1-S_{mn}^2)^{-\frac{1}{2}}$, $Z_1 = U^{(m)}/\sqrt{D^{(m)}}$ and $Z_2 = U^{(n-m)}/\sqrt{D^{(n-m)}}$, so that one has

$$R_{x_2} \stackrel{L}{=} \left(\frac{RR_{mn}}{S_{mn}} \;\middle|\; R\left(\frac{1-R_{mn}^2}{1-S_{mn}^2}\right)^{\frac{1}{2}} \frac{U^{(n-m)}}{\sqrt{D^{(n-m)}}} = x_2\right) . \tag{4.4}$$

In (4.3) and (4.4), R_{mn}^2 and S_{mn}^2 are both Beta(m/2, (n-m)/2), and R, R_{mn}, S_{mn}, $U^{(m)}$, $U^{(n-m)}$, $D^{(m)}$, and $D^{(n-m)}$ are jointly independent. The bases for (4.3) are the stochastic representations

$$U^{(n)} \stackrel{L}{=} (R_{mn} U^{(m)}, (1-R_{mn}^2)^{\frac{1}{2}} U^{(n-m)}), \quad D^{(n)} \stackrel{L}{=} (S_{mn} D^{(m)}, (1-S_{mn}^2)^{\frac{1}{2}} D^{(n-m)}) ,$$

of which the first is validated in Cambanis, Huang and Simons ((1981), Lemma 1) and the second follows similarly from the fact that $D^{(n)} = (D_1, \ldots, D_n) \stackrel{L}{=} (U_1^2, \ldots, U_n^2)$. □

If $X = (X_1, X_2) \sim S_n(1,\phi)$ with $\phi(u) = \exp(-u)$, then as noted previously, the components of X are i.i.d. and standard Cauchy. It follows that the random vectors X_1 and X_2, referred to in Theorem 4.1, are independent, and thus the conditional distribution of X_1 given X_2 does not depend upon X_2. This motivates the following theorem, which is a direct analogue of Theorem 6 in Cambanis, Huang and Simons (1981); it has an identical proof which is thus omitted.

THEOREM 4.2. *If* $X = (X_1, X_2)$ *has a* 1-*symmetric distribution, then the conditional distribution of* X_1 *given* X_2 *does not depend on* X_2 *with probability one if and only if the components of* X *are i.i.d. scaled Cauchy random variables.*

It is unnecessary in Theorem 4.2 to say that "the components of X are" *both* "i.i.d. and scaled Cauchy random variables". Suppose X is 1-symmetric with characteristic function of the form $\phi(|t_1| + \ldots + |t_n|)$. If X even has pairwise independent components, then it easily follows that ϕ satisfies the functional relationship $\phi(u+v) = \phi(u)\phi(v)$ for $u,v \geq 0$, from which one can show that the components of X are (i.i.d.) scaled Cauchy random variables. Conversely, if the components of X are scaled Cauchy random variables, then $\phi(u) = \exp(-cu)$, $u \geq 0$, for some $c \geq 0$, and it easily follows that they are i.i.d.

5. BIVARIATE ½-SYMMETRIC DISTRIBUTIONS

We have found the class $\Phi_2(\frac{1}{2})$ to be an enigma. For a long time, we had what we thought was its primitive. Everything in the class *is* a scale mixture of this "pseudo-primitive". But it eventually became apparent that, for certain members of $\Phi_2(\frac{1}{2})$, the distribution function F in (2.1) must be replaced by a *signed* measure (still on $[0,\infty)$). We interpret this as encouraging evidence that there does exist a true primitive ϕ_0 for $\Phi_2(\frac{1}{2})$, one for which (2.1) holds, with F a genuine distribution function for each $\phi \in \Phi_2(\frac{1}{2})$.

Our pseudo-primitive is closely related to the primitive for $\Phi_2(1)$; it is stochastically representable as $(X_0,Y_0) \overset{L}{=} (U/(B_1B_2^{\frac{1}{2}}), V/(\bar{B}_1\bar{B}_2^{\frac{1}{2}}))$ (cf., (2.6)), where $\bar{B}_i = (1-B_i)$ $(i = 1,2)$, (U,V) is uniformly distributed on the unit circle of $\mathbb{R}^2$, $B_i \sim \text{Beta}(\frac{1}{2},\frac{1}{2})$ $(i = 1,2)$, and (U,V), B_1, and B_2 are independent. That (X_0,Y_0) has a characteristic function of the right form is easily shown with two applications of Proposition 2.1.

Not only are we uncertain about the existence of a primitive for $\Phi_2(\frac{1}{2})$, but we are even uncertain that (X_0,Y_0) is "factorable", i.e., that $(X_0,Y_0) \overset{L}{=} R(X_1,Y_1)$ for some (X_1,Y_1) which has a characteristic function of the right kind for $\Phi_2(\frac{1}{2})$, and some R, independent of (X_1,Y_1), which is a *nonconstant* nonnegative random variable.

6. n-DIMENSIONAL VERSIONS

In this section, we characterize the class of n-dimensional versions of the symmetric one-dimensional stable laws and address some related questions. The connection between n-dimensional versions and α-symmetric distributions is discussed in Section 1.

Recall that, according to Eaton (1981), the distribution of an n-dimensional random vector $X = (X_1,\ldots, X_n)$ is an n-dimensional version of the distribution of a symmetric random variable Z if $t_1X_1 +\ldots+ t_nX_n \overset{L}{=} c(t)Z$ for $t = (t_1,\ldots, t_n) \in \mathbb{R}^n$ with $c(\underset{\sim}{t}) \geq 0$ for all t. We prefer here to delete Eaton's requirement that $c(t)$ be strictly positive for all nonzero t, which has the effect of excluding X's with linearly dependent components. We note in passing that $c(t)$ has to be a homogeneous function of degree one, i.e., $c(st) = |s|c(t)$ for each scalar s and $t \in \mathbb{R}^n$. Beyond this, little is known, in general, about its properties.

Eaton completely characterized the class of n-dimensional versions when Z has a finite positive variance: Let $\phi(u^2)$ denote the characteristic function of Z. Then the n-dimensional versions are the distributions of n-dimensional vectors AX_0' arising from $k \times n$ matrices A of rank k, $1 \leq k \leq n$, where $X_0 \sim S_k(2,\phi)$. The function $c(t)$ assumes the form $c(t) = (t\Sigma t')^{\frac{1}{2}}$, $t \in \mathbb{R}^n$, where $\Sigma = AA'$. Thus, there exists an n-dimensional version, with linearly independent components, associated with such a Z if and only if $\phi \in \Phi_n(2)$.

Eaton showed, as well, that the assumption of finite variance is critical. In particular, he observed that when Z is a symmetric stable random variable of index α, having characteristic function $\exp(-|u|^{\alpha})$, the symmetric n-dimensional stable laws of index α are n-dimensional versions. When $0 < \alpha < 2$, Z has an infinite second moment, and the class includes many examples which cannot arise from a linearly transformed 2-symmetric X_0. The characteristic functions of symmetric n-dimensional stable laws of index α assume the form

$$E \exp\{itX'\} = \exp\{-\int |tu'|^{\alpha} \mu(du)\}, \qquad t \in \mathbb{R}^n , \tag{6.1}$$

where μ is a finite (symmetric) measure on the unit sphere in $\mathbb{R}^n$ ($u \in \mathbb{R}^n$, $uu' = 1$). Thus

$$t_1X_1 + \ldots + t_nX_n \stackrel{L}{=} c(t)Z , \qquad t \in \mathbb{R}^n , \tag{6.2}$$

where

$$c(t) = (\int |tu'|^{\alpha} \mu(du))^{1/\alpha} , \qquad t \in \mathbb{R}^n . \tag{6.3}$$

The following theorem shows that there are no other n-dimensional versions.

THEOREM 6.1. *Let Z be a symmetric one-dimensional stable random variable of index* α. *The class of* n-*dimensional versions of* Z *coincides with the class of symmetric* n-*dimensional stable laws of index* α.

Proof. Because of the previous discussion, it is sufficient to show that a random vector X which satisfies (6.2) is symmetric n-dimensional stable of index α. It will be enough to show that if Y is an independent copy of X, then for all real r and s, $rX + sY \overset{L}{=} (|r|^\alpha + |s|^\alpha)^{1/\alpha} X$. Equivalently, we will show that $(rX + sY)t' \overset{L}{=} (|r|^\alpha + |s|^\alpha)^{1/\alpha} Xt'$, $t \in \mathbb{R}^n$. But for real u,

$$\mathcal{E}e^{iu(rX+sY)t'} = \mathcal{E}e^{iurXt'}\, \mathcal{E}e^{iusYt'} = \mathcal{E}e^{iurc(t)Z}\, \mathcal{E}e^{iusc(t)Z}$$

$$= e^{-|uc(t)|^\alpha (|r|^\alpha + |s|^\alpha)} = \mathcal{E}e^{iu(|r|^\alpha + |s|^\alpha)^{1/\alpha} c(t)Z}$$

$$= \mathcal{E}e^{iu(|r|^\alpha + |s|^\alpha)^{1/\alpha} Xt'} . \qquad \square$$

The foregoing suggests a (perhaps natural) generalization of the α-symmetric distributions to those whose characteristic functions assume the form

$$\mathcal{E}\exp\{itX'\} = \phi\left(\int |tu'|^\alpha \mu(du)\right) , \qquad t \in \mathbb{R}^n ,$$

with the attendant classes $S_n(\alpha,\mu,\phi)$ and $\Phi_n(\alpha,\mu)$ (obvious analogues of $S_n(\alpha,\phi)$ and $\Phi_n(\alpha)$). The new parameter μ plays the role of a scaling parameter and corresponds to a covariance matrix Σ when $\alpha = 2$ via $\int |tu'|^2 \mu(du) = t\Sigma t'$. Except for a few special cases, no information seems to be available on these distributions and classes.

ACKNOWLEDGEMENT

The authors wish to thank Professors Norman Johnson and Hari Mukerjee for helpful conversations pertaining to the contents of this paper.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) UNCLASSIFIED

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) On α-Symmetric Multivariate Distributions	5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
	6. PERFORMING ORG. REPORT NUMBER Mimeo Series No. 1350
7. AUTHOR(s) Stamatis Cambanis, Robert Keener and Gordon Simons	8. CONTRACT OR GRANT NUMBER(s) AFOSR Grant AFOSR-80-0080; NSF Grant MCS78-01240; NSF Grant MCS81-00748
9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research, Bolling Air Force Base, Washington, DC 20332	12. REPORT DATE June 1981
	13. NUMBER OF PAGES 32
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release -- Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

α-symmetric, multivariate, characteristic function, Laplace transform, conditional distribution, n-dimensional version, stable.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A random vector is said to have a 1-symmetric distribution if its characteristic function is of the form $\phi(|t_1| + \dots + |t_n|)$. 1-symmetric distributions are characterized through representations of the admissible functions ϕ and through stochastic representations of the random vectors, and

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20. some of their properties are studied.