APPLICATIONS OF INVARIANT DIFFERENTIAL OPERATORS TO MULTIVARIATE DISTRIBUTION THEORY*

by

Donald St. P. Richards
University of North Carolina at Chapel Hill

Abstract

The invariant differential operators are applied to derive partial differential equations for the zonal polynomials, to the calculation of generalized binomial coefficients and certain multivariate integrals, and to deducing a characterization of EP functions (Kushner, Lebow and Meisner, *J. Multivariate Anal.*, 1981).

Key words and phrases: Invariant differential operator, zonal polynomial, generalized binomial coefficient, multivariate integral, expectation property.

AMS 1970 subject classification: Primary 62H10, 62E15.

---

*Part of this work was done while the author was at the University of the West Indies.

1. Introduction

Let $\kappa = (k_1, k_2, \ldots, k_m)$ be a partition of the nonnegative integer k, and $C_\kappa(S)$ be the zonal polynomial (James (1964)) corresponding to κ, of the positive definite (symmetric) m×m matrix S. If $S = (s_{ij})$, let $\partial_S \equiv \partial = (\omega_{ij}\partial/\partial s_{ij})$ be a symmetric matrix of differential operators, where $\omega_{ij} = \frac{1}{2}(1+\delta_{ij})$ with δ_{ij} denoting Kronecker's delta.

Definition. A differential operator $D = D(S,\partial)$ in the entries of S is *invariant* if $D(S,\partial_S) = D(S^*,\partial_{S^*})$, $S^* = RSR'$, where R is an arbitrary nonsingular m×m matrix with transpose R'.

Thus, a differential operator is invariant if it is unchanged by the action of the general linear group on the space of positive definite matrices.

Although the general theory ensures that the zonal polynomials are eigenfunctions of the class of invariant operators, James (1968, 1973) appears to be the only author to have used this result. In developing further applications, we find it convenient to use explicit descriptions of the operators and also the methods employed by Maass (1971).

Important for us is Maass' result that every invariant operator can be represented as a polynomial in the operators $\mathrm{tr}(S\partial)^h$, h=1,2,...,m. Using this, we propose a method which in principle will produce partial differential equations (pde's), similar to those obtained by Fujikoshi (1970) and Sugiura (1973), for $C_\kappa(S)$. This constitutes one of our more important applications since it raises the question of whether the new pde's may be used to extend James' method for calculating the coefficients of $C_\kappa(S)$.

Section 2 presents several other applications. We show how the operators may be used to evaluate weighted sums of zonal polynomials similar to those of Fujikoshi-Sugiura (1969-1971), to represent certain eigenvalues associated with $C_\kappa(S)$ in terms of the generalized binomial coefficients (Constantine (1966), Bingham (1974)), and to provide explicit evaluations for certain multivariate integrals. Finally, we give a characterization of "EP" functions (Kushner and Meisner (1980)) by showing that if a certain growth condition holds, then for a function f to be EP, it is necessary and sufficient that Df be EP for every invariant operator D.

2. Applications

1) Partial differential equations for $C_\kappa(S)$ In what follows, all indices run independently from 1 to m, unless indicated otherwise. In deriving the pde's for $C_\kappa(S)$, we need to compute the $\lambda_h(\kappa)$ defined by

$$\mathrm{tr}(S\partial)^h C_\kappa(S) = \lambda_h(\kappa) C_\kappa(S), \; h=1,2,\ldots,m. \tag{1}$$

To do this, we evaluate both sides of (1) at S = Y, where $Y = \mathrm{diag}(y_1, y_2, \ldots, y_m)$ is a diagonal matrix containing the latent roots of S. We give a detailed exposition when h=2,3.

Case: h=2. A direct expansion shows that

$$\mathrm{tr}(S\partial)^2 = \tfrac{1}{2}(m+1)\mathrm{tr}(S\partial) + \sum \omega_{\alpha j}\omega_{\beta i} s_{\alpha i} s_{\beta j} \frac{\partial^2}{\partial s_{\alpha j}\partial s_{\beta i}} \quad .$$

By Lemma 3.1 of Sugiura (1973), we have

$$\mathrm{tr}(S\partial)^2 C_\kappa(S)\big|_{S=Y} = \tfrac{1}{2}(m+1)kC_\kappa(Y) + \left\{ \sum y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{i\neq j} \frac{y_i\, y_j}{y_i - y_j}\frac{\partial}{\partial y_i} \right\} C_\kappa(Y),$$

which shows (cf. James (1968)) that $\lambda_2(\kappa) = \frac{1}{2}(m+1)k + \sum k_i(k_i - i)$.

This result should not be surprising since it is known (Maass (1955)) that $\mathrm{tr}(S\partial)^2$ is the Laplace-Beltrami operator on the space of positive definite matrices.

Case: h=3. Proceeding as before, we have

$$\mathrm{tr}(S\partial)^3 = \tfrac{1}{2}(2m+3)\mathrm{tr}(S\partial)^2 - \tfrac{1}{4}(m+1)(m+2)\mathrm{tr}(S\partial)$$

$$+ \tfrac{1}{2}\sum \omega_{i\alpha}\omega_{j\beta}s_{i\alpha}s_{j\beta} \frac{\partial^2}{\partial s_{i\alpha}\partial s_{j\beta}}$$

$$+ \sum \omega_{\alpha j}\omega_{\beta\ell}\omega_{\gamma i}s_{i\alpha}s_{j\beta}s_{\ell\gamma} \frac{\partial^3}{\partial s_{\alpha j}\partial s_{\beta\ell}\partial s_{\gamma i}} \qquad (2)$$

Thus, $C_\kappa(S)$ is an eigenfunction of the sum of the last two terms on the right side of (2). Evaluating these at $S = Y$ as before, we have that $C_\kappa(Y)$ is an eigenfunction of the operator

$$\sum y_i^3 \frac{\partial^3}{\partial y_i^3} + \tfrac{1}{2}\sum y_i y_j \frac{\partial^2}{\partial y_i \partial y_j} + \frac{3}{2}\sum_{i\neq j} \frac{y_i^2 y_j}{y_i - y_j}\left(\frac{\partial^2}{\partial y_i^2} - \frac{\partial^2}{\partial y_i \partial y_j}\right)$$

$$- \frac{3}{2}\sum_{i\neq j} \frac{y_i y_j}{y_i - y_j}\frac{\partial}{\partial y_i} + \frac{3}{4}\sum_{i\neq j\neq \ell} \frac{y_i y_j y_\ell}{(y_i - y_j)(y_i - y_\ell)}\frac{\partial}{\partial y_i}\,. \qquad (3)$$

It can be shown by an argument paralleling James (1968) that the eigenvalue corresponding to the operator in (3) is $(a_2(\kappa) - 3a_1(\kappa) - k^2)/4$, where the $a_i(\kappa)$ are those of Fujikoshi-Sugiura. Hence,

$$\lambda_3(\kappa) = (a_2(\kappa) + (4m+3)a_1(\kappa) - k^2 + (m+1)k)/4.$$

Remarks

(1) It is evident that the method given above will provide pde's for higher values of h. However, the computations quickly become involved. We note that an alternative method for calculating $\lambda_h(\kappa)$ is the following: denoting by S_i the principal minor of S of order i ($1\leq i\leq m$), then James (1973) has shown that with $k_{m+1} \equiv 0$,

$$C_\kappa(S) = C_\kappa(I_m)\int_{O(m)} \prod_{i=1}^{m} |(HSH')_i|^{k_i - k_{i+1}} dH.$$

Here, I_m is the m×m identity matrix, and dH is the normalized invariant measure on

$O(m)$, the group of $m\times m$ orthogonal matrices. Thus, to compute $\lambda_h(\kappa)$, it is sufficient to show that $\Pi_{i=1}^m |S_i|^{k_i - k_{i+1}}$ is an eigenfunction of $\mathrm{tr}(S\partial)^h$ and to determine the corresponding eigenvalue. Both of these have been thoroughly treated by Maass ((1971), page 69 ff).

(2) With $\mathrm{tr}_i(S)$ denoting the i-th elementary symmetric function of S, Maass ((1971), page 67) remarks that the operators $\mathrm{tr}_i(S\partial)$, $i=1,2,\ldots,m$, also form a basis for the class of invariant operators. Here again, we can obtain pde's for $C_\kappa(S)$ using the method given above.

(3) Our result that $C_\kappa(S)$ is an eigenfunction of the operator in (2) may be restated as

$$(\mathrm{tr}(Y\partial)^3 + \tfrac{1}{2}(\mathrm{tr}\ Y\partial)^2)\ C_\kappa(S)|_{S=Y}$$

$$= \tfrac{1}{4}(a_2(\kappa) - 3a_1(\kappa) - k^2)\ C_\kappa(Y),$$

which is similar to the pde's obtained by Fujikoshi and Sugiura (cf. Sugiura (1973)). We do not know whether our pde's can be deduced from theirs or vice versa.

2) <u>Sums and integrals of zonal polynomials</u> We begin with some lemmas.

<u>Lemma 1</u>. *If* $A = A(S)$ *is an* $m\times m$ *matrix function of* S, *then*

$$(S\partial)'(AS) = ((S\partial)'A)S + \tfrac{1}{2}A'S + \tfrac{1}{2}(\mathrm{tr}\ AS)I_m, \tag{4}$$

$$(S\partial)(AS) = \tfrac{1}{2}(m+1)SA + S(S\partial)'A, \tag{5}$$

and as an operator identity,

$$(S\partial)^j|S|^t = \sum_{i=0}^{j} \binom{j}{i}\ t^i|S|^t(S\partial)^{j-i} \tag{6}$$

Both (4) and (5) can be established in a straightforward manner while (6) requires induction on j. We remark that (4) and (6) appear in Maass (1955).

Lemma 2. *Let* $A(S) = \exp(\mathrm{tr}\ S)I_m$. *Then*

$$(S\partial)^h A = \begin{cases} (S^2 + \tfrac{1}{2}(m+1)S)A, & h=2; \quad (7) \\ (S^3 + \tfrac{1}{2}(2m+3)S^2 + \tfrac{1}{2}(\mathrm{tr}\ S)S + \tfrac{1}{4}(m+1)^2 S)A, & h=3. \quad (8) \end{cases}$$

Lemma 2 follows directly from (4) and (5). Using (7), we find that

$$\sum_{\kappa} \lambda_2(\kappa)\ C_\kappa(S) = \mathrm{tr}(S\partial)^2(\mathrm{tr}\ S)^k$$

$$= 2\binom{k}{2}s_1^{k-2}s_2 + \tfrac{1}{2}(m+1)ks_1^k\ , \quad (9)$$

where $s_i = \mathrm{tr}(S^i)$. Expanding (9) in a series of zonal polynomials via Bingham ((1974), Theorem 2), we obtain

$$\lambda_2(\kappa) = 2\binom{\kappa}{s_2} + \tfrac{1}{2}(m+1)k. \quad (10)$$

By virtue of the result previously obtained for $\lambda_2(\kappa)$, (10) is equivalent to Bingham's expression for $\binom{\kappa}{s_2}$. This method can be also used along with (8) to show that

$$\lambda_3(\kappa) = 6\binom{\kappa}{s_3} + (2m+3)\binom{\kappa}{s_2} + \binom{k}{2} + \tfrac{1}{4}k(m+1)^2,$$

which is also consistent with Bingham's results. This shows not only how the λ's can be expressed in terms of the generalized binomial coefficients, but also how various weighted sums of zonal polynomials can be evaluated.

We turn to the computation of some multivariate integrals. As a simple example, consider

$$f(R) = \int_{S>0} \exp(-\mathrm{tr}\ S)|S|^t(\mathrm{tr}\ S^2)\ C_\kappa(RS)d\mu$$

where R is symmetric m×m, and $d\mu = |S|^{-(m+1)/2}dS$. Previously, the standard method of computing f(R) was to observe that f is a symmetric function of R and hence $f(R) = (C_\kappa(S)/C_\kappa(I_m))\ f(I_m)$. Expanding $(\mathrm{tr}\ S^2)\ C_\kappa(S)$ in a series of zonal polynomials (cf. Bingham (1974), Richards (1981a,b)), term-by-term integration gives us the value of $f(I_m)$. The only difficulty here is that the coefficients appearing in the expansion of $(\mathrm{tr}\, S^2)\, C_\kappa(S)$ are generally known only in principle. On the other hand, for suitably large t,

$$\lambda_2(\kappa) \int_{S>0} \exp(-\mathrm{tr}\, S)|S|^t C_\kappa(S)d\mu$$

$$= \int_{S>0} \exp(-\mathrm{tr}\, S)|S|^t (\mathrm{tr}(S\partial)^2 C_\kappa(S))d\mu$$

$$= \int_{S>0} (\mathrm{tr}(S\partial)^2 \exp(-\mathrm{tr}\, S)|S|^t)\, C_\kappa(S)d\mu, \tag{11}$$

since $\mathrm{tr}(S\partial)^2$ is *self-adjoint* (cf. Maass ((1971), page 57 ff)). Using (6) and (7) to expand $\mathrm{tr}(S\partial)^2 \exp(-\mathrm{tr}\, S)|S|^t$, then term-by-term integration gives us, explicitly,

$$f(R) = (\lambda_2(\kappa) + (2t+\tfrac{1}{2}(m+1))(k+mt) - mt^2)\, \Gamma_m(t,\kappa)\, C_\kappa(R)/C_\kappa(I_m).$$

We have evaluated other integrals using this method.

3. A characterization of EP functions

Suppose that S is a random matrix which follows the Wishart distribution $W(m,n,\Sigma)$ on n degrees of freedom with covariance matrix Σ. Kushner and Meisner (1980) (cf. Kushner, Lebow and Meisner (1981)) posed the problem of classifying all (infinitely differentiable) scalar-valued "EP" functions f(S), having the property that

$$E(f(S)) = \lambda_{n,m} f(\Sigma), \tag{12}$$

where $E(\cdot)$ denotes expectation. When $m \le 2$, Kushner and Meisner (1980) obtain a complete solution to the problem, and their recent article treats the general case.

When $m \le 2$, Kushner and Meisner (1980) prove that the EP condition (12) is equivalent to f being an eigenfunction of certain differential operators. We reformulate their result in the following way.

Lemma 3. *When* $m \leq 2$, $f(S)$ *is an eigenfunction of every invariant differential operator* D *if and only if* $f(S)$ *is an* EP *function.*

Proof. The case $m = 1$ is trivial. When $m = 2$, it is enough to show that the EP hypothesis is equivalent to $f(S)$ being an eigenfunction of the operators $tr(\hat{S}\partial_{\hat{S}})$ and $|S||\partial_{\hat{S}}|$, $\hat{S} = S^{-1}$. (Indeed, since $\hat{S}\partial_{\hat{S}} = -(S\partial_S)'$, it is not difficult to check that these two latter operators are a basis for the class of invariant operators.) Then the equivalence of the two criteria follows from Kushner and Meisner (1980), Section 6.

In the general case, it is again true that Lemma 3 holds for any *polynomial* f. For the proof of this, we note that Kushner, Lebow and Meisner (1981) prove that for any EP polynomial f,

$$f(S) = \sum a_i \; C_\kappa(T_i S T_i'), \tag{13}$$

where the sum in (13) is finite, the a_i are constants, κ is a (uniquely determined) partition of $d = \deg.(f)$, and the T_i are non-singular $m \times m$ matrices. It is now easy to check that if $f(S)$ is EP, then $f(S)$ is an eigenfunction of any invariant D. As to the converse, this follows from Maass (1971), Section 6, whose result actually holds for any *function* $f(S)$ satisfying the growth condition (14) given below.

We conjecture that Lemma 3 is valid for general m. What we are able to prove is the following result, valid for all functions $f(S)$ satisfying the conditions that for every $h=1,2,\ldots$, and any choice of subscripts $i,j,\ldots,k,\ell$, there exist constants c_h, d_h such that

$$|\partial^h f(S)/\partial s_{ij} \ldots \partial s_{k\ell}| < c_h (\operatorname{tr} S)^{d_h}, \; S > 0, \; |S| = 1. \tag{14}$$

We then have the following result

Theorem 1. *Let* $f(S)$ *satisfy* (14). *Then,* $f(S)$ *is an* EP *function if and only if for every invariant differential operator* D, $Df(S)$ *is also* EP.

Proof. First, we write (12) in the form

$$\int_{S>0} \exp(-\tfrac{1}{2}\mathrm{tr}\ \Sigma^{-1}S)\,|\Sigma^{-1}S|^{n/2} f(S)\,d\mu = \rho_{n,m}\ f(\Sigma), \tag{15}$$

where $\rho_{n,m} = \lambda_{n,m}/C_{n,m}$, and $C_{n,m}$ is the normalizing constant in the Wishart density. Since the term $\exp(-\tfrac{1}{2}\mathrm{tr}\ \Sigma^{-1}S)|\Sigma^{-1}S|^{n/2}$ is a point-pair invariant function of (Σ,S) (Maass (1971), page 54), it follows that if $\hat{D}$ is the adjoint of the operator D (loc. cit., page 55), then

$$D_\Sigma \exp(-\tfrac{1}{2}\mathrm{tr}\ \Sigma^{-1}S)\,|\Sigma^{-1}S|^{n/2} = \hat{D}_S \exp(-\tfrac{1}{2}\mathrm{tr}\ \Sigma^{-1}S)\,|\Sigma^{-1}S|^{n/2}. \tag{16}$$

From (15) and (16), we get

$$\rho_{n,m} D_\Sigma\ f(\Sigma) = \int_{S>0} \{\hat{D}_S \exp(-\tfrac{1}{2}\mathrm{tr}\ \Sigma^{-1}S)\,|\Sigma^{-1}S|^{n/2}\}\ f(S)\,d\mu$$

$$= \int_{S>0} \exp(-\tfrac{1}{2}\mathrm{tr}\ \Sigma^{-1}S)\,|\Sigma^{-1}S|^{n/2}\ \{D_S\ f(S)\}\,d\mu,$$

which is the same as (15) with f replaced by Df. The converse follows by reversing these steps.

The significance of the conditions (14) is discussed by Maass (1971), page 100.

References

Bingham, C. (1974) An identity involving partitional generalized binomial coefficients. *J. Multivariate Anal.*, *4*, 210-223.

Constantine, A. G. (1963) Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.*, *34*, 1270-1285.

Constantine, A. G. (1966) The distribution of Hotelling's generalized T_0^2. *Ann. Math. Statist.*, *37*, 215-225.

Fujikoshi, Y. (1970) Asymptotic expansions of the distributions of test statistics in multivariate analysis. *J. Sci. Hiroshima Univ., Ser. A-I*, *34*, 73-144.

James, A. T. (1964) Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.*, *35*, 475-501.

James, A. T. (1968) Calculation of zonal polynomial coefficients by use of the Laplace-Beltrami operator. *Ann. Math. Statist.*, *39*, 1711-1718.

James, A. T. (1973) The variance information manifold and the functions on it. In *Multivariate Analysis-III* (ed. P. R. Krishnaiah), Academic Press, New York.

Kushner, H. B. and Meisner, M. (1980) Eigenfunctions of expected value operators in the Wishart distribution. *Ann. Statist.*, *8*, 977-988.

Kushner, H. B., Lebow, A. and Meisner, M. (1981) Eigenfunctions of expected value operators in the Wishart distribution, II. *J. Multivariate Anal.*, *11*, 418-433.

Maass, H. (1955) Die Bestimmung der dirichletreihen mit Grössencharakteren zu den Modulformen n-ten Grades. *J. Indian Math. Soc.*, *19*, 1-23.

Maass, H. (1971) *Siegel's Modular Forms and Dirichlet Series*. Lecture Notes in Math., Vol. 216, Springer-Verlag, Berlin.

Richards, D. St. P. (1981 a,b) Differential operators associated with zonal polynomials. I, II. *Ann. Inst. Statist. Math.*, to appear.

Sugiura, N. and Fujikoshi, Y. (1969) Asymptotic expansions of the non-null distribution of the likelihood ratio criteria for multivariate linear hypothesis and independence. *Ann. Math. Statist.*, *40*, 942-952.

Sugiura, N. (1971) Note on some formulas for weighted sums of zonal polynomials. *Ann. Math. Statist.*, *42*, 768-772.

Sugiura, N. (1973) Derivatives of the characteristic root of a symmetric or a Hermitian matrix with two applications in multivariate analysis. *Comm. Statist.*, *1*, 393-417.