

REPRESENTATIONS FOR EIGENFUNCTIONS OF
EXPECTED VALUE OPERATORS IN THE WISHART DISTRIBUTION

by

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ABSTRACT

Kushner *et al* [9], [10] have posed the problem of characterising the "EP" function $f(S)$ for which $E(f(S)) = \lambda_n f(\Sigma)$, whenever the $m \times m$ matrix S has the Wishart distribution $W(m, n, \Sigma)$. In this article, we obtain integral representations for all non-negative EP functions. It is also shown that any bounded EP function is harmonic, and that EP polynomials may be used to approximate the functions in certain L^p spaces.

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1. INTRODUCTION

In two recent articles, Kushner and Meisner [9] and Kushner, Lebow and Meisner [10] have formulated and treated the problem of classifying certain "EP" functions. Specifically, a scalar-valued function $f(S)$ of matrix argument is EP with respect to the Wishart distribution if

$$(1) \quad E(f(S)) = \lambda_n f(\Sigma)$$

whenever the $m \times m$ random matrix S has a Wishart distribution $W(m, n, \Sigma)$ on n degrees of freedom, with covariance matrix Σ . Thus, $f(S)$ is EP if (1) holds for all positive definite symmetric Σ , all $n \geq$ some n_0 , and for some sequence of eigenvalues $\{\lambda_n\}$ which are independent of Σ .

For $m \leq 2$, detailed solutions are provided in [9], while [10] in the main, discusses EP polynomials. It is seen there that the zonal polynomials of James [6] play a central role. Richards [11] has obtained a characterization of EP functions in terms of invariant differential operators. More recently, Davis [2] has independently, and *via* different methods, obtained some of the main results of [10].

In this article, the intent is to derive general descriptions of *non-negative* EP functions. For EP polynomials, Kushner *et al* [10] show that the zonal polynomials may be used as generating functions, so that the results given here may be regarded as an extension of their results.

Actually, we go one step further. If $f(S)$ satisfies the EP condition (1) for a fixed n , we shall say that $f(S)$ is an n -EP function. Using results of Furstenberg [5], we obtain representations (in Section 2) for all non-negative n -EP and EP functions.

Section 3 extends certain connections between EP and harmonic functions which appeared earlier in [9]. Finally, Section 4 shows that the EP polynomials may be used to approximate the functions in certain L^p spaces,

extending some results of [10].

On a more personal note, it is a pleasure to acknowledge numerous stimulating conversations on these topics with V. Mandrekar and G. Kallianpur. Thanks are also due to A. W. Davis whose preprint [2] partly motivated this work, and to H. B. Kushner for making available his unpublished thesis [8].

2. INTEGRAL REPRESENTATIONS

Throughout, we denote by $GL(m)$ (respectively, $O(m)$) the group of $m \times m$ real non-singular (respectively, orthogonal) matrices, and by $P(m)$ the cone in $GL(m)$ of positive definite symmetric matrices. We shall also have to deal with $SL(m)$ (respectively, $SO(m)$) the subgroup of $GL(m)$ (respectively, $O(m)$) consisting of the matrices with determinant unity.

Now, the n -EP criterion (1) may be written in the form

$$(2) \quad \int_{P(m)} f(S) k(\Sigma, S) d\mu(S) = \lambda_n f(\Sigma) \quad , \quad \Sigma \in P(m) \quad .$$

Here, $k(\Sigma, S) |S|^{-(m+1)/2}$ is the Wishart density, and $d\mu(S) = |S|^{-(m+1)/2} (dS)$ is the invariant measure on $P(m)$.

Since n is temporarily fixed, we suppress the suffix on λ_n , and denote by $V(\lambda)$ the cone of non-negative solutions of (2). The following lemma collects certain properties of $V(\lambda)$ which we shall need later.

Definition. A cone C of non-negative functions on $P(m)$ is *translation-invariant* if for any $f \in C$, the function $f_{\Lambda}(S) = f(\Lambda S \Lambda')$ ($\Lambda \in GL(m)$, $S \in P(m)$) also belongs to C .

LEMMA 1. *The cone $V(\lambda)$ consists of continuous functions, is translation-invariant, and is closed in the topology of pointwise convergence on $\mathbb{R}_+^{P(m)}$.*

Proof. That $V(\lambda)$ is translation-invariant is proven in [9] and [10]. Further, the infinite differentiability of the functions in $V(\lambda)$ may be deduced by regarding (2) as a multi-dimensional Laplace transform. Indeed, this view of (2) also implies that $V(\lambda)$ is closed; a detailed proof can be obtained by suitably modifying the proof of Proposition 9.5 in Berg and Forst [1]. \square

Definition. A closed, translation-invariant cone $C \subseteq \mathbb{R}_+^{P(m)}$ is *irreducible* if C contains no proper, closed, translation-invariant sub-cone.

Except for $m = 1$, $V(\lambda)$ is not irreducible. Before we construct an irreducible sub-cone of $V(\lambda)$, recall that in [10], irreducible subspaces of EP *polynomials* were shown to be generated by the functions $f_\alpha(S) = \prod_{i=1}^m |S_i|^{\alpha_i - \alpha_{i+1}}$, where $\alpha = (\alpha_1, \dots, \alpha_m)$ is an m -tuple of non-negative integers, $\alpha_{m+1} = 0$, and S_i is the $i \times i$ principal minor of S . Thus, it is reasonable to expect that the f_α 's should also generate irreducible sub-cones of $V(\lambda)$. To construct such a sub-cone, let $M_+(O(m))$ denote the space of positive Borel measures on $O(m)$, and

$$S_n = \{ \alpha = (\alpha_1, \dots, \alpha_m) : \alpha_i > - (n+1-i)/2, 1 \leq i \leq m, \text{ and} \\ \lambda = 2^{\alpha_1 + \dots + \alpha_m} \prod_{i=1}^m [\Gamma(\alpha_i + \frac{1}{2}(n+1-i)) / \Gamma(\frac{1}{2}(n+1-i))] \} .$$

For any $\alpha \in S_n$, define

$$V(\lambda, \alpha) = \{ f_{\alpha, \tau}(S) = \int_{O(m)} f_\alpha(HSH') d\tau(H) : \tau \in M_+(O(m)) \} .$$

In Furstenberg's [5] terminology, $f_\alpha(S)$ is an *irreducible* $O(m)$ -*multiplier*. That is,

- (i) for any $\Lambda \in GL(m)$, $f_\alpha(H\Lambda S'H')$ is proportional to $f_\alpha(H_1 S H_1')$,
 $H_1 \in O(m)$;
- (ii) $f_\alpha(HSH') = 1$ for $S = I_m$;
- (iii) the cone $V(\lambda; \alpha)$ is an irreducible sub-cone of $V(\lambda)$.

Strictly speaking, the results of [5] are formulated not for $GL(m)$, but for $SL(m)$. In any case, (i) is not difficult to establish, (ii) is trivial, and (iii) is a consequence of (i) and the fact that up to multiplication by positive constants, $V(\lambda; \alpha)$ contains precisely one $O(m)$ -invariant function.

The converse of (iii), that every irreducible sub-cone of $V(\lambda)$ is a $V(\lambda; \alpha)$ is the content of Theorem 12.2 in [5]. It allows us to formulate our main result.

THEOREM 1. For $f(S)$ to be a non-negative n -EP function with eigenvalue λ_n , it is necessary and sufficient that there exist a positive measure ν , concentrated on $O(m) \times S_n$, such that

$$(3) \quad f(S) = \int_{O(m) \times S_n} f_\alpha(HSH') d\nu(H, \alpha), \quad S \in P(m).$$

Proof. If $f(S)$ is of the form (3), it is clearly n -EP with eigenvalue λ_n . For the converse, let V^* be the closure, in the topology of pointwise convergence, of the cone generated by all functions of the form (3). If there exists $g \in V(\lambda_n) \setminus V^*$ then $\langle g \rangle$, the pointwise closure of the cone consisting of all non-negative multiples of g , is clearly an irreducible sub-cone of $V(\lambda)$. By a remark above, $\langle g \rangle$ contains $f_\alpha(S)$ for some $\alpha \in S_n$, so that $f_\alpha(S) \notin V^*$; a contradiction. \square

The measure ν in (3) is uniquely determined by $f(S)$. From Theorem 1, we obtain a description of the non-negative EP functions. Let

$$S = \bigcap_{n \geq n_0} S_n.$$

THEOREM 2. For $f(S)$ to be a non-negative EP function with eigenvalues $\{\lambda_n\}$, $n \geq n_0$, it is necessary and sufficient that $f(S)$ be of the form (3), where the positive measure ν has support on $O(m) \times S$.

The proof is immediate. In addition, a number of results previously obtained in [9] and [11] follow directly from Theorem 2.

COROLLARY 1. *Let $f(S)$ be a non-trivial non-negative EP function. Then, the sequence of eigenvalues $\{\lambda_n\}$ converges, $f(S)$ is homogeneous, and for any invariant differential operator D , $Df(S)$ is also EP.*

Proof. Since $f(S)$ is non-trivial, the set S in Theorem 2 is non-empty. Now, $\alpha = (\alpha_i) \in S$ if and only if for all $n \geq n_0$,

$$\lambda_n / 2^{\sum \alpha_i} = \prod_{i=1}^m [\Gamma(\alpha_i + \frac{1}{2}(n+1-i)) / \Gamma(\frac{1}{2}(n+1-i))] .$$

Since the product of gamma functions converges to 1, we see that

$\lambda_n \rightarrow 2^{\sum \alpha_i}$ as $n \rightarrow \infty$; necessarily, the function $\alpha \rightarrow \sum \alpha_i$ is constant on S , which shows that $f(S)$ is homogeneous.

Finally, the last assertion is implied by the well-known result that $f_\alpha(S) = \prod_{i=1}^m |S_i|^{\alpha_i - \alpha_{i+1}}$ is an eigenvalue of every invariant operator. \square

An interesting special case is $m = 1$. Then in Theorem 1, we have $S_n = \{\alpha : \alpha > -n/2, \lambda_n = 2^\alpha \Gamma(\alpha + \frac{1}{2}n) / \Gamma(\frac{1}{2}n)\}$. Since the function $\alpha \rightarrow 2^\alpha \Gamma(\alpha + \frac{1}{2}n)$, $\alpha > -n/2$, intersects any horizontal line in at most a finite number of points, the set S_n is finite. We should also mention that for this case, Theorem 1 can be derived using the Choquet-Deny Theorem [3] (cf. the introduction in [5]).

3. HARMONIC FUNCTIONS

In this section, we show that certain connections between harmonic and EP functions which were first established in [9] have appropriate generalisations.

To begin, we give a brief account of certain results on the theory of harmonic functions on $SL(m)$ as treated in [4]. Since we have seen that all

non-negative EP functions are homogeneous, we shall restrict our attention to the class of homogeneous n -EP functions; for simplicity, we even assume $\lambda_n = 1$.

Let (dH) denote the normalised Haar measure on $SO(m)$, $P_1(m) = P(m) \cap SL(m)$, and $d\mu_1(\cdot)$ be the invariant measure on $P_1(m)$.

Definition ([4], Definition 4.1) A function g on $P_1(m)$ is *harmonic* if

$$(4) \quad \int_{SL(m)} \int_{SO(m)} g(\Lambda H S H' \Lambda') k_1(S) (dH) d\mu_1(S) = g(\Lambda \Lambda'), \quad \Lambda \in SL(m),$$

where $k_1(S) d\mu_1(S)$ is a probability measure on $P_1(m)$.

The article [4] is a *tour de force* in the modern theory of harmonic functions. There, integral representations and partial differential equations for the harmonic functions are only two of a large number of properties developed. We are particularly interested in the bounded harmonic functions; for any such g , the results of [4] show that the definition (4) is independent of the choice of the density $k_1(\cdot)$, and that with Δ denoting the Laplace-Beltrami operator on $P_1(m)$, (4) is equivalent to $\Delta g = 0$.

We apply these results to the homogeneous n -EP functions, noting that the result below was initiated by formulae in Davis [2].

THEOREM 3. *Let $f(S)$ be a bounded, homogeneous n -EP function, with $\lambda_n = 1$. Then, the restriction, g , of f to $P_1(m)$ satisfies $\Delta g = 0$.*

Proof. We must show that g satisfies (4). Since f is homogeneous, we can assume that in (2), $\Sigma \in P_1(m)$. In (2), set $\Sigma = \Lambda \Lambda'$, $\Lambda \in SL(m)$, and replace S by $\Lambda S \Lambda'$ to obtain:

$$(5) \quad \int_{P(m)} f(\Lambda S \Lambda') k(S) d\mu(S) = f(\Lambda \Lambda').$$

Setting $S = tS_1$, $t = |S|^{-1/m}$ in (5) and integrating over t reduces (5) to

$$\int_{P_1(m)} g(\Lambda S_1 \Lambda') k_1(S_1) d\mu_1(S_1) = g(\Lambda \Lambda') .$$

Here, $k_1(S_1) d\mu_1(S_1)$ is the distribution of $|S|^{-1/m} S$ where S is a Wishart matrix $W(m, n, I_m)$. Replacing S_1 by HS_1H' , $H \in SO(m)$, and averaging over $SO(m)$, we obtain (4), since the density $k_1(\cdot)$ is invariant under rotations. By previous remarks, we have $\Delta g = 0$. \square

It is worth noting that our remarks above also imply that the n -EP functions in Theorem 3 remain unchanged if the kernel $k(\Sigma, S)$ in (2) is replaced by any other point-pair invariant function k^* , viz., $k^*(\Sigma, S) = k^*(\Lambda \Sigma \Lambda', \Lambda S \Lambda')$ for all $\Lambda \in GL(m)$. Further, known results on the solutions of $\Delta g = 0$ may be used to obtain the representation in Theorem 1.

3. APPROXIMATION BY EP FUNCTIONS

So far, nothing has been said about the representation of arbitrary EP functions. From the results obtained before, it is reasonable to expect that the real-valued EP functions have similar integral representations, with signed measures as their representing measures.

Instead of pursuing this train of thought, we adopt another approach based on the result [10] that any polynomial may be expressed as a linear combination of EP polynomials. We denote by $dW_n(S)$ the $W(m, n, I_m)$ distribution.

LEMMA 2. *Let $f \in L^p(dW_n)$, $1 \leq p < \infty$. Then, for any $\epsilon > 0$, f can be approximated by a linear combination of translates of EP polynomials to within ϵ , in the L^p norm.*

Proof. The proof is almost standard. First, we can find a compactly supported continuous function g such that $\|f - g\|_p < \epsilon/2$. Next, there exists a polynomial P such that $\sup_S |g(S) - P(S)| < \epsilon/2$. Since dW_n is a probability measure, the last estimate implies $\|g - P\|_p < \epsilon/2$. Hence,

$\|f-P\|_p < \epsilon$. But by [10], $P(S)$ is a linear combination of translates of EP polynomials. \square

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