ON SHRINKAGE R-ESTIMATORS OF MULTIVARIATE LOCATION

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Summary. For a continuous and diagonally symmetric multivariate distribution, incorporating the idea of preliminary test estimators, a variant form of the James-Stein type estimation rule is used to formulate some admissible R-estimators of location. Asymptotic admissibility (and inadmissibility) results pertaining to the proposed and classical R-estimators are studied.

1. Introduction.

For the mean (vector) of a p-variate normal distribution, Stein (1956) established the inadmissibility of the maximum likelihood estimator (the sample mean vector) under a squared error risk measure when $p \geq 3$; a simple (non-linear) admissible estimator was later proposed by James and Stein (1961). Since then, this theory has been extensively studied, in the parametric case, by a host of workers; a detailed account of these developments is given by Berger (1980). The object of the present investigation is to consider some shrinkage estimators based on rank statistics wherein the theory of preliminary test estimation (PTE) and the James-Stein rule estimation are incorporated in the formulation of the proposed estimators.

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Let $\underset{\sim}{X}_i = (X_{i1},\ldots,X_{ip})'$, $i = 1,\ldots,n$ be n independent and identically distributed (i.i.d.) random vectors (r.v.) having a p-variate continuous distribution function (d.f.) $F_{\underset{\sim}{\theta}}$, defined on the Euclidean space E^p, for some $p \geq 1$. $F_{\underset{\sim}{\theta}}$ is assumed to be diagonally symmetric about its location (vector) $\underset{\sim}{\theta}(=(\theta_1,\ldots,\theta_p)')$, so that

$$(1.1) \qquad F_{\underset{\sim}{\theta}}(\underset{\sim}{x}) = F(\underset{\sim}{x} - \underset{\sim}{\theta}) \;, \quad \underset{\sim}{x} \in E^p \;,$$

where F is diagonally symmetric about $\underset{\sim}{0}$. It is desired to estimate $\underset{\sim}{\theta}$ using an estimator $\underset{\sim}{\delta}_n = (\delta_{n1},\ldots,\delta_{np})'$ based on $(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n)$, where one considers a quadratic loss function

$$(1.2) \qquad L(\underset{\sim}{\delta}_n,\underset{\sim}{\theta}) = n(\underset{\sim}{\delta}_n - \underset{\sim}{\theta})'\underset{\sim}{Q}(\underset{\sim}{\delta}_n - \underset{\sim}{\theta}) \;,$$

for some given positive definite (p.d.) matrix $\underset{\sim}{Q}$. The risk is then given by

$$(1.3) \qquad \rho_n(\underset{\sim}{\delta},\underset{\sim}{\theta}) = EL(\underset{\sim}{\delta}_n,\underset{\sim}{\theta}) = \mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{V}_n) \;,$$

where

$$(1.4) \qquad \underset{\sim}{V}_n = nE(\underset{\sim}{\delta}_n - \underset{\sim}{\theta})(\underset{\sim}{\delta}_n - \underset{\sim}{\theta})' \;.$$

For normal F and for $p \geq 3$, it is well known that $\underset{\sim}{\bar{X}}_n = n^{-1}\Sigma_{i=1}^n \underset{\sim}{X}_i$ is not admissible (in the sense that there exists some James-Stein rule estimators having smaller risk than $\bar{X}_n$ for every $\underset{\sim}{\theta}$). For possible non-normal F, the sample mean $\underset{\sim}{\bar{X}}_n$ may not be very robust and may even be quite inefficient for distributions with heavy tails. Robust rank based (R-) estimators of location in multivariate simple linear models have been studied by Sen and Puri (1969), Puri and Sen (1971) and others. PTE theory for such R-estimators has also been developed by Saleh and Sen (1978), Sen and Saleh (1979) and Saleh and Sen (1983), among others.

The object of the present study is to incorporate the PTE theory in the proposal of James-Stein rule R-estimators of $\underset{\sim}{\theta}$, and, in this way, some of the technical difficulties of the James-Stein rule in the nonparametric case have been avoided.

Along with the preliminary notions, the R-estimators, their PTE versions and the proposed estimators are considered in Section 2. The concepts of asymptotic inadmissibility (for fixed and local alternatives) are discussed in Section 3, and in the light of these results, (in-) admissibility results (pertaining to the estimators under consideration) are studied in Section 4. Some general comments are made in the concluding section.

2. The proposed estimators. For every real b, let $R^+_{ij}(b)$ be the rank of $|X_{ij} - b|$ among $|X_{1j} - b|,\ldots,|X_{nj} - b|$, for $i = 1,\ldots,n$, $j = 1,\ldots,p$. Also, for every $n(\geq 1)$ and $j(=1,\ldots,p)$, let $a^+_{nj}(1),\ldots,a^+_{nj}(n)$ be a set of scores, which we shall define more formally later on. Consider the statistics

$$(2.1)\qquad T_{nj}(b) = n^{-1}\sum_{i=1}^{n} \text{sgn}(X_{ij} - b)a^+_{nj}(R^+_{ij}(b)) \ , \ 1 \leq j \leq p.$$

Then, for monotone ($\nearrow$) scores, $T_{nj}(b)$ is $\searrow$ in b [viz, Puri and Sen (1971, Ch.6)], and we set

$$(2.2)\qquad \hat{\theta}_{nj} = \frac{1}{2}\left(\sup\{b: T_{nj}(b) > 0\} + \inf\{b: T_{nj}(b) < 0\}\right), \ 1 \leq j \leq p;$$

$$(2.3)\qquad \hat{\underset{\sim}{\theta}}_n = \left(\hat{\theta}_{n1},\ldots,\hat{\theta}_{np}\right)'.$$

$\hat{\theta}_n$ is termed an R-estimator of θ. It is a robust, translation-invariant and consistent estimator having the (coordinate wise) median-unbiasedness property and other desirable properties too. The performance of $\hat{\theta}_n$ would naturally depend on the underlying F and the set of scores $\{a_{nj}^{+}(k),\ 1 \le k \le n;\ 1 \le j \le p\}$. Towards this study, we set for each $j(=1,\dots,p)$,

$$a_{nj}^{+}(k) = E\phi_j^{+}(U_{nk}) \quad \text{or} \quad \phi_j^{+}\left(\frac{k}{n+1}\right),\ 1 \le k \le n\ , \tag{2.4}$$

where $U_{n1} < \cdots < U_{nn}$ stand for the ordered r.v. of a sample of size n from the uniform (0,1) d.f. and for every $u \in (0,1)$,

$$\phi_j^{+}(u) = \phi_j\left(\frac{1+u}{2}\right);\quad \phi_j(u) + \phi_j(1-u) = 0, \qquad 1 \le j \le p. \tag{2.5}$$

We also assume the ϕ_j to be square integrable. Let then $F_{[j]}$ be the j^{th} marginal d.f. corresponding to the d.f. F, for $j = 1,\dots,p$ and let $F_{[j\ell]}$ be the bivariate marginal d.f. for $j \ne \ell = 1,\dots,p$. We define $\nu = ((\nu_{j\ell}))$ by

$$\nu_{j\ell} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_j\big(F_{[j]}(x)\big)\phi_\ell\big(F_{[\ell]}(y)\big)dF_{[j\ell]}(x,y) \tag{2.6}$$

for $j,\ell=1,\dots,p$. Also, we assume that $F_{[j]}$ possesses an absolutely continuous probability density function (pdf) $f_{[j]}$, and set

$$\psi_j(u) = -f'_{[j]}\big(F_{[j]}^{-1}(u)\big)/f_{[j]}\big(F_{[j]}^{-1}(u)\big),\quad 0 < u < 1; \tag{2.7}$$

$$\xi_j \doteq \int_0^1 \phi_j(u)\psi_j(u)du\ ,\quad 1 \le j \le p; \tag{2.8}$$

$$\xi = \underline{\text{Diag}}\ (\xi_1,\dots,\xi_p); \tag{2.9}$$

$$\Gamma = \xi^{-1} \nu \xi^{-1} = ((\gamma_{j\ell})). \tag{2.10}$$

Then, it is known that

$$n^{\frac{1}{2}}(\hat{\theta}_n - \theta) \xrightarrow[\mathcal{D}]{} \mathcal{N}_p(0, \Gamma). \tag{2.11}$$

We consider next the PTE. This estimator is particularly suitable when θ is suspected to be close to 0, so that a preliminary test for H_0: $\theta = 0$ is made first, and, depending on the outcome of this test, the ultimate estimator is chosen. Towards this, we define $M_n^* = ((m^*_{n,j\ell}))$ by letting

$$m^*_{n,j\ell} = n^{-1}\Sigma_{i=1}^n a_{nj}^+(R_{ij}^+) a_{n\ell}^+(R_{i\ell}^+) \text{sgn } X_{ij} \text{ sgn } X_{i\ell} \tag{2.12}$$

for $j, \ell = 1, \ldots, p$, where $R_{ij}^+ = R_{ij}^+(0)$, for $i = 1, \ldots, n$, $1 \le j \le p$. Then as in Sen and Puri (1967), the test statistic used is

$$L_n = n T_n' \left(M_n^*\right)^- T_n , \tag{2.13}$$

where $T_n = \left(T_{n1}, \ldots, T_{np}\right)'$, $T_{nj} = T_{nj}(0)$, $1 \le j \le p$ and $\left(M_n^*\right)^-$ is a (reflexive) generalized inverse of M_n. For small values of n, a (conditionally) distribution-free test for H_0: $\theta = 0$ may be based on the exact (conditional) distribution of L_n generated by the 2^n equally likely sign-inversions of the vectors X_i, $1 \le i \le n$, while, for large n, L_n has closely the chi square distribution with p degrees of freedom (DF). Thus, if $\chi_p^2(\alpha)$ stands for the upper 100 α% point of the chi square d.f. with p DF and if $\ell_{n,\alpha}$ stands for the (conditional) critical value of L_n at the <u>level of significance</u> $\alpha(0 < \alpha < 1)$, then

(2.14) $$\ell_{n,\alpha} \xrightarrow{p} \chi_p^2(\alpha) \quad \text{as} \quad n \to \infty.$$

The PTE $\hat{\underset{\sim}{\theta}}_n^{PT}$ of $\underset{\sim}{\theta}$ may be defined as

(2.15) $$\hat{\underset{\sim}{\theta}}_n^{PT} = \begin{cases} \underset{\sim}{0}, & L_n < \ell_{n,\alpha} \\ \hat{\underset{\sim}{\theta}}_n, & L_n \geq \ell_{n,\alpha} \end{cases}.$$

Naturally, the performance of $\hat{\underset{\sim}{\theta}}_n^{PT}$ depends on $\alpha (0 < \alpha < 1)$ as well as the true $\underset{\sim}{\theta}$; if $\underset{\sim}{\theta}$ is away from $\underset{\sim}{0}$, then $\hat{\underset{\sim}{\theta}}_n^{PT}$ and $\hat{\underset{\sim}{\theta}}_n$ may behave very similarly, while for $\underset{\sim}{\theta}$ close to $\underset{\sim}{0}$, $\hat{\underset{\sim}{\theta}}_n^{PT}$ may behave better than $\hat{\underset{\sim}{\theta}}_n$.

Following James and Stein (1961), we may consider the estimator $\hat{\underset{\sim}{\theta}}_n^{JS}$ defined by

(2.16) $$\hat{\underset{\sim}{\theta}}_n^{JS} = \left\{1 - a/L_n\right\}\hat{\underset{\sim}{\theta}}_n ,$$

where L_n is defined by (2.13) and a is an appropriate constant, which we may discuss later on. Though the existence and convergence of moments of R-estimates of location have been studied by Sen (1980b), there seems to be a basic problem with the study of the risk of $\hat{\underset{\sim}{\theta}}_n^{JS}$ according to (1.3); this is mainly due to the fact that $P_{\underset{\sim}{\theta}}\{L_n = 0\} > 0$, though it could be very small, and as such, L_n^{-1} may not have a finite moment of any positive order. To avoid this technical problem and to introduce more flexibility in the definition of the shrinkage estimator, we consider the following modifications.

Parallel to (2.4)-(2.5), we let $a_{nj}(k) = E\phi_j(U_{nk})$ or $\phi_j(k/(n+1))$, for $k=1,\dots,n$; $1 \leq j \leq p$, and define $\underset{\sim}{M}_n = ((m_{n,j\ell}))$ by

(2.17) $$m_{n,j\ell} = n^{-1}\sum_{i=1}^{n} a_{nj}\left(R_{ij}\right)a_{n\ell}\left(R_{i\ell}\right), \quad j,\ell=1,\dots,p,$$

where R_{ij} = rank of X_{ij} among $X_{1j},\ldots,X_{nj}$, for $1 \le i \le n$ and $j=1,\ldots,p$. Then $\underset{\sim}{M}_n$ is a translation-invariant, robust and consistent estimator of $\underset{\sim}{\nu}$, defined by (2.6). Also, we define $\hat{\underset{\sim}{\xi}}_n = \underline{\text{Diag}}(\hat{\xi}_{n1},\ldots,\hat{\xi}_{np})$ by letting

$$\hat{\xi}_{nj} = n^{\frac{1}{2}}\left\{T_{nj}(\hat{\theta}_{nj} - a/\sqrt{n}) - T_{nj}(\hat{\theta}_{nj} + a/\sqrt{n})\right\}/(2a),\ 1 \le j \le p, \tag{2.18}$$

where a is some pre-fixed positive number. Let then

$$\hat{\underset{\sim}{\Gamma}}_n = \hat{\underset{\sim}{\xi}}_n^{-1}\ \underset{\sim}{M}_n\ \hat{\underset{\sim}{\xi}}_n^{-1}. \tag{2.19}$$

Finally, let $d_n = ch_p(\underset{\sim}{Q}\hat{\underset{\sim}{\Gamma}}_n)$ be the smallest characteristic root of $\underset{\sim}{Q}\hat{\underset{\sim}{\Gamma}}_n$ and c: $0 < c < 2(p-2)$ be a given number; naturally, we assume that $p \ge 3$. Then our proposed estimator is

$$\hat{\underset{\sim}{\theta}}_n = \begin{cases} \underset{\sim}{0}, & \text{if } L_n < \varepsilon, \\ \left(\underset{\sim}{I} - cd_n L_n^{-1}\underset{\sim}{Q}^{-1}\hat{\underset{\sim}{\Gamma}}_n^{-1}\right)\hat{\underset{\sim}{\theta}}_n, & \text{if } L_n \ge \varepsilon, \end{cases} \tag{2.20}$$

where $\varepsilon(> 0)$ is an <u>arbitrarily small</u> number, L_n is defined by (2.13) and $\hat{\underset{\sim}{\theta}}_n$ by (2.3). Note that the idea of PTE is incorporated in (2.20) through L_n (being $<$ or $\ge \varepsilon$), though we are not adapting (necessarily) $\varepsilon = \ell_{n,\alpha}$ (which would require α to be close to 1). Also, the James-Stein rule is adapted here for $L_n \ge \varepsilon > 0$.

3. Preliminaries on asymptotic admissibility.

Note that according to (1.3), for any $\underset{\sim}{\delta}_n$, if there exists some other estimator $\underset{\sim}{\delta}_n^*$, such that

$$\rho_n(\underset{\sim}{\delta}^*,\underset{\sim}{\theta}) \le \rho_n(\underset{\sim}{\delta},\underset{\sim}{\theta}),\ \forall\,\underset{\sim}{\theta}, \tag{3.1}$$

with strict inequality holding for some $\underset{\sim}{\theta}$, then $\underset{\sim}{\delta}_n$ is termed an inadmissible estimator of $\underset{\sim}{\theta}$. If instead of (3.1), we have

$$(3.2) \qquad \lim_{n\to\infty} \rho_n(\underset{\sim}{\delta}^*,\underset{\sim}{\theta}) \le \lim_{n\to\infty} \rho_n(\underset{\sim}{\delta},\theta),\ \forall\ \underset{\sim}{\theta},$$

with strict inequality holding for some $\underset{\sim}{\theta}$, then $\underset{\sim}{\delta}_n$ is termed asymptotically inadmissible (for fixed alternatives). For the mean of a p-variate normal distribution, it has been observed by Judge and Bock (1981, p.172) that for values of $\underset{\sim}{\theta}$ at or close to the origin, the risk advantage of the James-Stein estimator, relative to that of the maximum likelihood estimator, could be considerable. To exploit this result fully in the context of nonparametric estimation, we conceive of a sequence $\{K_n\}$ of local translation alternatives, where under $K_n, \underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n$ (more precisely, $\underset{\sim}{X}_{n1},\ldots,\underset{\sim}{X}_{nn}$) are i.i.d. r.v. with the d.f. $F(\underset{\sim}{x} - n^{-\frac{1}{2}}\underset{\sim}{\lambda})$, for some given $\underset{\sim}{\lambda} \in E^p$. Then, in (1.2)-(1.4), we replace $\underset{\sim}{\theta}$ by $n^{-\frac{1}{2}}\underset{\sim}{\lambda}$ and denote the corresponding risk by $\rho_n^*(\underset{\sim}{\delta}^*,\underset{\sim}{\lambda})$. A similar notation is used for $\rho_n^*(\underset{\sim}{\delta}^*,\underset{\sim}{\lambda})$. If

$$(3.3) \qquad \lim_{n\to\infty} \rho_n^*(\underset{\sim}{\delta}^*,\underset{\sim}{\lambda}) \le \lim_{n\to\infty} \rho_n^*(\underset{\sim}{\delta},\underset{\sim}{\lambda}),\forall\ \underset{\sim}{\lambda} \in E^p,$$

with the strict inequality holding for some $\underset{\sim}{\lambda}$, then $\underset{\sim}{\delta}_n$ is termed asymptotically inadmissible for local translation alternatives.

The main objectives of this study are two-fold: (i) To establish the asymptotic admissibility of $\hat{\underset{\sim}{\theta}}_n$ for fixed alternatives (according to (3.2)), and (ii) to establish the asymptotic inadmissibility of $\hat{\underset{\sim}{\theta}}_n$ for local translation alternatives, according to (3.3). In either case, for the sake of compatibility, we confine ourselves to the class of

R-estimators. Note that for (3.3), we have to deal with a triangular array of (row wise) i.i.d.r.v. (whose d.f. may depend on n), and this may cause some problems with manipulations. However, $\hat{\underset{\sim}{\theta}}_n$ is a translation-invariant estimator, and hence, most of these problems can be avoided by replacing $\underset{\sim}{X}_{ni}$ by $\underset{\sim}{X}_{ni} - n^{-\frac{1}{2}}\lambda$, $1 \le i \le n$, which are i.i.d.r.v. with location $\underset{\sim}{0}$ (under K_n).

4. <u>Asymptotic admissibility results</u>. We shall consider first the fixed alternative asymptotic admissibility results. In this context, we need some moment convergence results on $\hat{\underset{\sim}{\theta}}_n$, and these we introduce first. We assume that there exist a positive number a (not necessarily greater than or equal to 1), such that

$$E_F|X_{1j}|^a < \infty, \quad \text{for} \quad j = 1,\dots,p. \tag{4.1}$$

It follows from Theorem 2.1 of Sen (1980b) that for monotone ϕ_j with $\nu_{jj} > 0$, for every $k(> 0)$, there exists a sample size $n_0(=n_0(k,a))$, such that $E_F|\hat{\theta}_{nj}|^k < \infty, \forall\, n \ge n_0$, and further

$$\limsup_{n\to\infty} E_F|\hat{\theta}_{nj} - \theta_j|^k < \infty, \ \forall\ k\colon 0 < k < \infty. \tag{4.2}$$

We need, however, results stronger than (4.2). Hence, we introduce some additional regularity conditions. As in Sen (1980b), we assume that for each $j(=1,\dots,p)$, $\phi_j^{(r)}(u) = \left(d^r/du^r\right)\phi_j(u)$, $r = 0,1,2,$ exist almost everywhere $(0 < u < 1)$, and there exist positive constants K_0 $(< \infty)$ and $\delta(< \frac{1}{2})$, such that

(4.3) $$|\phi_j^{(r)}(u)| \le K_0[u(1-u)]^{-\delta-r},\ 0<u<1,\quad r=0,1,2.$$

As in (2.7), we assume the existence of $f'_{[j]}$ almost everywhere and further that for some $\eta > 0$,

(4.4) $$\sup_x f_{[j]}(x)\left\{F_{[j]}(x)[1-F_{[j]}(x)]\right\}^{-\delta-\eta} < \infty,\ \forall\ 1 \le j \le p.$$

Then, from Theorem 2.2 and (2.49)-(2.50) of Sen (1980b), it follows that for each $j(=1,\ldots,p)$,

(4.5) $$n(\hat{\theta}_{nj} - \theta_j) = \xi_j^{-1} n T_{nj}(\theta_j) + \omega_{nj}\ ,$$

where $n^{-\frac{1}{2}}|\omega_{nj}| \to 0$ almost surely (a.s.) as $n \to \infty$, and

(4.6) $$E|n^{-\frac{1}{2}}\omega_{nj}|^k \to 0\ ,\ \forall\ k < (1-2\delta)/\delta = \delta^*,\ \text{say}.$$

In the sequel, we assume that (4.3) holds for some $\delta < \frac{1}{4}$, so that $\delta^* > 2$. In passing, we may remark that (4.3) holds for the Wilcoxon scores ($\delta = 0$), Normal scores (δ arbitrarily close to 0) and all the other commonly used score functions.

Note that by (1.2), (1.3), (2.3), (4.5) and (4.6), under (4.1), (4.3) and (4.4) (for some $\delta < \frac{1}{4}$),

(4.7) $$\lim_{n\to\infty} \rho_n(\hat{\underset{\sim}{\theta}},\underset{\sim}{\theta}) = \text{Tr}\left(\underset{\sim}{Q} \lim_{n\to\infty} nE(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta})'\right)$$
$$= \text{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma})\ ,$$

where $\underset{\sim}{\Gamma}$ is defined by (2.6)-(2.10) and under (4.3), $\text{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma}) < \infty$. Next, we note that by (2.15),

(4.8) $$n(\hat{\underset{\sim}{\theta}}{}_n^{PT} - \hat{\underset{\sim}{\theta}}_n)'\underset{\sim}{Q}(\hat{\underset{\sim}{\theta}}{}_n^{PT} - \hat{\underset{\sim}{\theta}}_n) = n(\hat{\underset{\sim}{\theta}}{}_n'\underset{\sim}{Q}\hat{\underset{\sim}{\theta}}_n)I(L_n < \ell_{n,\alpha}),$$

so that by the Hölder inequality, we obtain that

$$(4.9)\qquad E\left\{n\left(\hat{\theta}_n^{PT}-\hat{\theta}_n\right)'Q\left(\hat{\theta}_n^{PT}-\hat{\theta}_n\right)\right\} \le n\left[E\left\{\left(\hat{\theta}_n'Q\hat{\theta}_n\right)^q\right\}\right]^{\frac{1}{q}}\left[P\left\{L_n < \ell_{n,\alpha}\right\}\right]^{1-\frac{1}{q}},$$

where $q(>1)$ is some positive number. Note that by (4.2), under (4.1),

$$(4.10)\qquad E\left\{\left(\hat{\theta}_n'Q\hat{\theta}_n\right)^q\right\} \le \left[ch_1(Q)\right]^q E\left\{\left(\hat{\theta}_n'\hat{\theta}_n\right)^q\right\}$$

$$\le \left[Tr(Q)\right]^q p^{q-1}\sum_{j=1}^{p} E\left|\hat{\theta}_{nj}\right|^{2q} < \infty, \quad \forall\ n \ge n_0.$$

On the other hand, by (2.13), $L_n \ge \max_{1\le j\le p}\left(nT_{nj}^2/m_{n,ij}^*\right)$, so that

$$P\left\{L_n < \ell_{n,\alpha}\right\} \le \min_{1\le j\le p} P\left\{T_{nj}^2 < n^{-1}\ell_{n,\alpha}m_{n,jj}^*\right\}$$

$$(4.11)\qquad = \min_{1\le j\le p} P\left\{\left|T_{nj}\right| < n^{-\frac{1}{2}}\left(\ell_{n,\alpha}m_{n,jj}^*\right)^{\frac{1}{2}}\right\}.$$

Now, incorporating Theorem 3 of Sen (1970) into the proof of his Theorem 1, we conclude that under (4.3) with $\delta < \frac{1}{2}$, whenever $\theta_j \neq 0$, $P\{|T_{nj}| \le n^{-\frac{1}{2}}(\ell_{n,\alpha}m_{n,jj}^*)^{\frac{1}{2}}|\theta_j\} = O(n^{-2})$, as $n \to \infty$. Thus, if in (4.9), we choose $q > 2$, then by (4.10) and (4.11), we conclude that (4.9) converges to 0 (as $n \to \infty$) whenever $\theta \neq 0$ (i.e., $\theta_j \neq 0$ for some $j=1,\ldots,p$). Hence, under (4.1), (4.3), $(\delta < \frac{1}{2})$ and (4.4), for every (fixed) $\theta \neq 0$, $\hat{\theta}^{PT}$ and $\hat{\theta}_n$ are asymptotically risk equivalent. We shall see later on that this feature does not hold for $\theta = 0$ or close to 0 (in a local sense). After (2.16), we have noticed that because of the positive probability mass (however small) of L_n at 0, $\hat{\theta}_n^{JS}$ may not have a finite risk even asymptotically as $n \to \infty$. Hence, we proceed to study the case of $\hat{\theta}_n^{S}$ in (2.20). Note that

$$(4.12)\qquad n(\hat{\theta}_n^S-\hat{\theta}_n)'Q(\hat{\theta}_n^S-\hat{\theta}_n)$$

$$= I(L_n < \varepsilon)n\hat{\theta}_n'Q\hat{\theta}_n + I(L_n \ge \varepsilon)c^2d_n^2L_n^{-2}n\hat{\theta}_n'\hat{\Gamma}_n^{-1}Q^{-1}\hat{\Gamma}_n^{-1}\hat{\theta}_n.$$

Now, proceeding as in (4.9) through (4.11), it follows that

$$\limsup_{n\to\infty} E\left\{I(L_n < \varepsilon)n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n \,|\, \underset{\sim}{\theta} \neq \underset{\sim}{0}\right\} = 0. \tag{4.13}$$

Also, note that $d_n = ch_p(\underset{\approx}{Q}\hat{\underset{\approx}{\Gamma}}_n)$, so that

$$\begin{aligned}(4.14)\qquad & nd_n^2\hat{\underset{\sim}{\theta}}_n'\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1}\hat{\underset{\sim}{\Gamma}}_n^{-1}\hat{\underset{\sim}{\theta}}_n \\ &= (n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n)\left\{d_n^2\left[\hat{\underset{\sim}{\theta}}_n'\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1}\hat{\underset{\sim}{\Gamma}}_n^{-1}\hat{\underset{\sim}{\theta}}_n/\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n\right]\right\} \\ &\le (n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n)d_n^2 ch_1(\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1}\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1}) \\ &= (n\hat{\underset{\sim}{\theta}}_n\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n)d_n^2/\left[ch_p(\hat{\underset{\sim}{\Gamma}}_n\underset{\approx}{Q})\right]^2 = n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n \; .\end{aligned}$$

Therefore choosing some ε' (> 0) arbitrarily small, we have

$$\begin{aligned}(4.15)\qquad & I(L_n \ge \varepsilon)c^2 d_n^2 L_n^{-2} n\hat{\underset{\sim}{\theta}}_n'\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1}\hat{\underset{\sim}{\Gamma}}_n^{-1}\hat{\underset{\sim}{\theta}}_n \\ &\le I(L_n \ge \varepsilon)c^2 L_n^{-2} n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n \\ &= I(\varepsilon \le L_n < n\varepsilon')c^2 L_n^{-2} n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n + I(L_n \ge n\varepsilon')c^2 L_n^{-2} n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n \\ &\le I(L_n < n\varepsilon')c^2\varepsilon^{-2} n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n + I(L_n \ge n\varepsilon')c^2(\varepsilon')^{-2}\left\{\frac{1}{n}\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n\right\} .\end{aligned}$$

Now proceeding as in (4.9)-(4.11), the first term on the right hand side of (4.15) converges in the first mean to 0 (as $n \to \infty$), when $\underset{\sim}{\theta} \neq \underset{\sim}{0}$, for the second term we make use of (4.2) and conclude (by using the Hölder inequality) that this also converges in the first moment to 0 (as $n \to \infty$) for every fixed $\underset{\sim}{\theta}$. Hence, we obtain that for every (fixed) $\underset{\sim}{\theta} \neq \underset{\sim}{0}$, under (4.1), (4.3) ($\delta < \frac{1}{4}$) and (4.4), as $n \to \infty$,

$$\limsup_{n\to\infty} E\left\{n\left(\hat{\underset{\sim}{\theta}}_n^S - \hat{\underset{\sim}{\theta}}_n\right)'\underset{\approx}{Q}\left(\hat{\underset{\sim}{\theta}}_n^S - \hat{\underset{\sim}{\theta}}_n\right) \,|\, \underset{\sim}{\theta} \neq \underset{\sim}{0}\right\} = 0, \tag{4.16}$$

so that for every (fixed) $\underset{\sim}{\theta} \neq \underset{\sim}{0}$, $\underset{\sim}{\theta}_n^S$ and $\hat{\underset{\sim}{\theta}}_n$ are asymptotically risk equivalent. Again, we shall see later on that the situation is different for local translation alternatives.

We proceed on next to study the asymptotic inadmissibility of the classical R-estimator $\hat{\theta}_n$ for local translation alternatives, as considered before (3.3). We conceive of a triangular array $\{X_{ni},\ 1 \le i \le n;\ n \ge 1\}$ of row-wise i.i.d.r.v. with the d.f. $\{F_{(n)}\}$, and, as before, let K_n stand for the hypothesis that for the given n, $\underset{\sim}{\theta} = \underset{\sim}{\theta}_n = n^{-\frac{1}{2}}\underset{\sim}{\lambda}$, i.e.,

$$\text{(4.17)} \qquad F_{(n)}(\underset{\sim}{x}) = F(\underset{\sim}{x} - n^{-\frac{1}{2}}\underset{\sim}{\lambda}),\ \underset{\sim}{x} \in E^p ,$$

where $\underset{\sim}{\lambda}$ is finite $(\in E^p)$ and fixed, and the d.f. F satisfies all the regularity conditions stated in earlier sections.

For the PTE $\hat{\underset{\sim}{\theta}}{}_n^{PT}$ in (2.15), we have, under K_n in (4.17),

$$\text{(4.18)} \qquad n\left(\hat{\underset{\sim}{\theta}}{}_n^{PT} - \underset{\sim}{\theta}_n\right)' \underset{\sim}{Q}\left(\hat{\underset{\sim}{\theta}}{}_n^{PT} - \underset{\sim}{\theta}_n\right)$$

$$= (\underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda}) I(L_n < \ell_{n,\alpha}) + I(L_n \ge \ell_{n,\alpha})\left[n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\underset{\sim}{Q}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)\right],$$

so that by (2.14) and (4.17),

$$\text{(4.19)} \qquad \lim_{n\to\infty} \rho_n^*\left(\hat{\underset{\sim}{\theta}}{}^{PT},\underset{\sim}{\lambda}\right) = \lim_{n\to\infty} nE\left\{\left(\hat{\underset{\sim}{\theta}}{}_n^{PT} - \underset{\sim}{\theta}_n\right)'\underset{\sim}{Q}\left(\hat{\underset{\sim}{\theta}}{}_n^{PT} - \underset{\sim}{\theta}_n\right) \middle| K_n\right\}$$

$$= (\underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda}) \lim_{n\to\infty} P\left\{L_n < \chi_p^2(\alpha) \middle| K_n\right\} +$$

$$+ \lim_{n\to\infty} E\left\{I\left(L_n \ge \chi_p^2(\alpha)\right) n\left(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n\right)'\underset{\sim}{Q}\left(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n\right) \middle| K_n\right\} .$$

We denote by $H_p(\cdot;\Delta)$ the non-central chi square d.f. with p degrees of freedom (DF) and non-centrality parameter $\Delta(\geq 0)$. Then, we have [see Sen and Puri (1967)]

$$(4.20) \qquad \lim_{n \to \infty} P\left\{L_n \leq \chi_p^2(\alpha) \,|\, K_n\right\} = H_p\left(\chi_p^2(\alpha); \underset{\sim}{\lambda}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{\lambda}\right) .$$

For the second term on the right hand side of (4.19), we make use of (i) the incomplete moments of multinormal distributions [c.f. Section 4 of Sen and Saleh (1979)], (ii) (2.11), (4.5), (4.6) and (iii) the integrability of the $|T_{nj}(\theta_j)|^4$ (under (4.3) for $\delta < \frac{1}{4}$), and obtain that

$$(4.21) \qquad \lim_{n\to\infty} E\left\{I\left(L_n \geq \chi_p^2(\alpha)\right) n\left(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n\right)' \underset{\sim}{Q}\left(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n\right) | K_n\right\}$$

$$= \mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma})\left[1 - H_{p+2}\left(\chi_p^2(\alpha); \underset{\sim}{\lambda}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{\lambda}\right)\right] - (\underset{\sim}{\lambda}' \underset{\sim}{Q} \underset{\sim}{\lambda})\left[H_p\left(\chi_p^2(\alpha); \underset{\sim}{\lambda}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{\lambda}\right)\right.$$

$$\left. - 2H_{p+2}\left(\chi_p^2(\alpha); \underset{\sim}{\lambda}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{\lambda}\right) + H_{p+4}\left(\chi_p^2(\alpha); \underset{\sim}{\lambda}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{\lambda}\right)\right].$$

From (4.19) through (4.21), we have

$$(4.22) \qquad \lim_{n\to\infty} \rho_n^*(\hat{\underset{\sim}{\theta}}^{PT}, \underset{\sim}{\lambda}) = \left[1 - H_{p+2}\left(\chi_p^2(\alpha); \underset{\sim}{\lambda}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{\lambda}\right)\right] \cdot \mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma})$$

$$+ (\underset{\sim}{\lambda}' \underset{\sim}{Q} \underset{\sim}{\lambda})\left[2H_{p+2}\left(\chi_p^2(\alpha); \underset{\sim}{\lambda}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{\lambda}\right) - H_{p+4}\left(\chi_p^2(\alpha); \underset{\sim}{\lambda}' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{\lambda}\right)\right] .$$

We may note that (4.7) holds as well for $\{K_n\}$ in (4.17), so that for the classical R-estimator $\hat{\underset{\sim}{\theta}}_n$, we have

$$(4.23) \qquad \lim_{n\to\infty} \rho_n^*(\hat{\underset{\sim}{\theta}}, \underset{\sim}{\lambda}) = \mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma}), \forall \underset{\sim}{\lambda} \in E^p.$$

Since $H_q(x;\delta) \geq H_{q+i}(x;\delta)$, $\forall\, i \geq 1$, $x \geq 0$, $\delta \geq 0$, the right hand side of (4.22) is bounded from below by

(4.24) $$\mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma})\left\{1 - H_{p+2}\left(\chi_p^2(\alpha);\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda}\right)\right\} + (\underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda})H_{p+2}\left(\chi_p^2(\alpha);\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda}\right).$$

Thus, for all $\underset{\sim}{\lambda}$ for which $\underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda} \geq \mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma})$, (4.24) exceeds (4.23), though the difference may not be large for any $\underset{\sim}{\lambda}$ and it converges to 0 as $\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda} \to +\infty$. On the other hand, for all $\underset{\sim}{\lambda}:\underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda} \leq \frac{1}{2}\mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma})$, the right hand side of (4.22) is bounded from above by

(4.25) $$\left\{1 - \tfrac{1}{2}H_{p+4}\left(\chi_p^2(\alpha);\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda}\right)\right\}\mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma}) < \mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma}),$$

wo that the asymptotic risk of the PTE is smaller than that of $\hat{\underset{\sim}{\theta}}_n$ for $\underset{\sim}{\lambda}:\underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda} \leq \frac{1}{2}\mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma})$. For $\underset{\sim}{\lambda}:\frac{1}{2}\mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma}) < \underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda} < \mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma})$, towards the lower boundary (4.23) is greater than (4.22) and conversely towards the upper boundary, though the actual difference may be quite small. Thus, the PTE $\hat{\underset{\sim}{\theta}}_n^{PT}$ turns out to be a robust competitor, with better performance for $\underset{\sim}{\lambda}$ close to $\underset{\sim}{0}$ and not so poor for $\underset{\sim}{\lambda}$ away from $\underset{\sim}{0}$. However, in accordance with (3.3), we can not conclude that either $\hat{\underset{\sim}{\theta}}_n$ or $\hat{\underset{\sim}{\theta}}^{PT}$ is inadmissible with respect to the other for all $\underset{\sim}{\lambda} \in E^p$.

The picture is different with respect to $\hat{\underset{\sim}{\theta}}_n^S$. By (2.20), we have

(4.26) $$\begin{aligned} & n\left(\hat{\underset{\sim}{\theta}}_n^S - \underset{\sim}{\theta}_n\right)'\underset{\sim}{Q}\left(\hat{\underset{\sim}{\theta}}_n^S - \underset{\sim}{\theta}_n\right) \\ &= I(L_n < \varepsilon)(\underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda}) + I(L_n \geq \varepsilon)\Big\{n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\underset{\approx}{Q}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n) \\ &\quad - 2cd_nL_n^{-1}n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\hat{\underset{\sim}{\Gamma}}_n^{-1}\hat{\underset{\sim}{\theta}}_n + c^2d_n^2L_n^{-2}n\hat{\underset{\sim}{\theta}}_n'\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1}\hat{\underset{\sim}{\Gamma}}_n^{-1}\hat{\underset{\sim}{\theta}}_n\Big\}. \end{aligned}$$

Now, by (4.20), as $n \to \infty$,

(4.27) $$E\left\{I(L_n < \varepsilon)\underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda}\,\middle|\,K_n\right\} \to \underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda}H_p(\varepsilon;\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda}),$$

while proceeding as in (4.21)-(4.22),

$$\text{(4.28)}\qquad \lim_{n\to\infty} E\left\{I(L_n \geq \varepsilon)n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\underset{\approx}{Q}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)|K_n\right\}$$

$$= \left\{1 - H_{p+2}(\varepsilon;\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda})\right\}\mathrm{Tr}(\underset{\approx}{Q}\Gamma) - (\underset{\sim}{\lambda}'\underset{\approx}{Q}\underset{\sim}{\lambda})\left\{H_p(\varepsilon;\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda})\right.$$

$$\left. - 2H_{p+2}(\varepsilon;\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda}) + H_{p+4}(\varepsilon;\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda})\right\}.$$

Also we may note that

$$\text{(4.29)}\qquad d_n^2 n\hat{\underset{\sim}{\theta}}_n'\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1}\hat{\underset{\sim}{\Gamma}}_n^{-1}\hat{\underset{\sim}{\theta}}_n = (n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n)\left\{d_n^2 n\hat{\underset{\sim}{\theta}}'\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1}\underset{\sim}{\Gamma}_n^{-1}\hat{\underset{\sim}{\theta}}_n/n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n\right\}$$

$$\leq (n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n)d_n^2 ch_1(\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1}\hat{\underset{\sim}{\Gamma}}_n^{-1}\underset{\approx}{Q}^{-1})$$

$$= (n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n)d_n^2/ch_p(\hat{\underset{\sim}{\Gamma}}_n\underset{\approx}{Q}\hat{\underset{\sim}{\Gamma}}_n\underset{\approx}{Q}) = n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n \;;$$

$$\text{(4.30)}\qquad |n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\hat{\underset{\sim}{\Gamma}}_n^{-1}\hat{\underset{\sim}{\theta}}_n\cdot d_n|$$

$$= |n^{\frac12}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\hat{\underset{\sim}{\Gamma}}_n^{-\frac12}\cdot\hat{\underset{\sim}{\Gamma}}_n^{-\frac12}n^{\frac12}\hat{\underset{\sim}{\theta}}_n\cdot d_n|$$

$$\leq \left\{\left[n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\hat{\underset{\sim}{\Gamma}}_n^{-1}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)\cdot d_n\right]\left[d_n n\hat{\underset{\sim}{\theta}}_n'\hat{\underset{\sim}{\Gamma}}_n^{-1}\hat{\underset{\sim}{\theta}}_n\right]\right\}^{\frac12}$$

$$\leq \left\{n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\underset{\approx}{Q}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)\cdot n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n\right\}^{\frac12}$$

$$\leq \tfrac12\left[n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\underset{\approx}{Q}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n) + n\hat{\underset{\sim}{\theta}}_n\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n\right].$$

Notice that because of the translation invariance of $\hat{\underset{\sim}{\theta}}_n$, (4.5)-(4.6) hold under $\{K_n\}$ as well (with $\underset{\sim}{\theta}$ being replaced by $\underset{\sim}{\theta}_n$), so that for every k (> 0), under K_n, with n adequately large, $(n\hat{\underset{\sim}{\theta}}_n'\underset{\approx}{Q}\hat{\underset{\sim}{\theta}}_n)^k$ and $(n(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n)'\underset{\approx}{Q}(\hat{\underset{\sim}{\theta}}_n - \underset{\sim}{\theta}_n))^k$ are integrable. Further, making use of the Jurečková-linearity of rank statistics in the multivariate case [viz., Sen and Puri (1977)], it follows that under $\{K_n\}$, as $n \to \infty$, (4.31) $$L_n = n\hat{\theta}_n' \Gamma^{-1} \hat{\theta}_n + \eta_n ,$$

where

(4.32) $$\eta_n \xrightarrow{P} 0 , \quad \text{as} \quad n \to \infty .$$

Therefore, on the set $I(L_n > \varepsilon)$, $\varepsilon \quad 0$, under $\{K_n\}$,

(4.33) $$L_n^{-1} - \left(n\hat{\theta}_n' \Gamma^{-1} \hat{\theta}_n\right)^{-1} \xrightarrow{P} 0 , \quad \text{as} \quad n \to \infty .$$

Finally, under $\{K_n\}$, by (4.5)-(4.6),

(4.34) $$n^{\frac{1}{2}} \Gamma^{-\frac{1}{2}} \hat{\theta}_n \xrightarrow[\mathcal{D}]{} N\left(\Gamma^{-\frac{1}{2}}\lambda, I_p\right).$$

Let us now denote by ω a p-vector having the multinormal distribution with mean vector $\omega = \Gamma^{-\frac{1}{2}}\lambda$ and dispersion matrix I_p. Also, let $\chi^2_{q,\Delta}$ be a random variable having the noncentral chi square d.f. $H_q(\cdot;\Delta)$. Then, by using (4.29)-(4.34) along with the desired integrability conditions, we conclude that for every $\varepsilon > 0$,

(4.35) $$\lim_{n\to\infty} E\left\{I(L_n \geq \varepsilon) d_n L_n^{-1} n(\hat{\theta}_n - \theta_n)' \Gamma_n^{-1} \hat{\theta}_n \middle| K_n\right\}$$
$$= ch_p(Q\Gamma)\left\{E\left(\chi^{-2}_{p+2,\Delta}\right) - E\left[I(\omega' W < \varepsilon)(\omega' W)^{-1} \omega' \Gamma^{-\frac{1}{2}}\lambda\right]\right\}$$

where $\Delta = \lambda'\Gamma^{-1}\lambda$, and

(4.36) $$\lim_{n\to\infty} E\left\{I(L_n \geq \varepsilon) d_n^2 L_n^{-2} n\hat{\theta}_n' \hat{\Gamma}_n^{-1} Q^{-1} \hat{\Gamma}_n^{-1} \hat{\theta}_n \middle| K_n\right\}$$
$$= \left[ch_p(Q\Gamma)\right]^2 \left\{\mathrm{Tr}\left(Q^{-1}\Gamma^{-1}\right) E\left(\chi^{-4}_{p+2,\Delta}\right) + \Delta^* E\left(\chi^{-4}_{p+4,\Delta}\right)\right.$$
$$\left. - E\left(I(\omega' W < \varepsilon)(\omega' W)^{-2} \omega' A W\right)\right\}$$

where $\Delta^* = \lambda'\Gamma^{-1}Q^{-1}\Gamma^{-1}\lambda$ and $A = \Gamma^{-\frac{1}{2}}Q^{-1}\Gamma^{-\frac{1}{2}}$. Now

$$(4.37)\qquad \left|E\left[I(\underset{\sim}{\omega}'\underset{\sim}{W} < \varepsilon)(\underset{\sim}{\omega}'\underset{\sim}{W})^{-1}\underset{\sim}{\omega}'\underset{\sim}{\Gamma}^{-\frac{1}{2}}\underset{\sim}{\lambda}\right]\right|$$

$$\le E\left[I(\underset{\sim}{\omega}'\underset{\sim}{W} < \varepsilon)(\underset{\sim}{\omega}'\underset{\sim}{W})^{-\frac{1}{2}}(\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda})^{\frac{1}{2}}\right]$$

$$= (\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda})^{\frac{1}{2}} \int_0^{\varepsilon} x^{-\frac{1}{2}}\, dH_p(x;\Delta)$$

$$= \Delta^{\frac{1}{2}} e^{-\frac{1}{2}\Delta} \sum_{r=0}^{\infty} \frac{1}{r!}(\tfrac{1}{2}\Delta)^r \frac{1}{\Gamma(\frac{p}{2}+r)\, 2^{(p/2+r)}} \int_0^{\varepsilon} x^{r+(p-3)/2} e^{-\frac{1}{2}x}\, dx$$

$$\le \Delta^{\frac{1}{2}} e^{-\frac{1}{2}\Delta} \sum_{r=0}^{\infty} \frac{1}{r!} (\tfrac{1}{2}\Delta)^r \left(\Gamma(p/2)\, 2^{p/2}\right)^{-1} \varepsilon^{r+(p-1)/2}\big/(r+(p-1)/2)$$

$$\le \left(\Gamma(p/2)\, 2^{p/2}\right)^{-1} \Delta^{\frac{1}{2}} \varepsilon^{(p-1)/2} e^{-\frac{1}{2}\Delta(1-\varepsilon)}, \ \forall\ p \ge 2,\ 0 < \varepsilon \le 1.$$

Also,

$$(4.38)\qquad E\left[I(\underset{\sim}{\omega}'\underset{\sim}{W} < \varepsilon)(\underset{\sim}{\omega}'\underset{\sim}{W})^{-2}\underset{\sim}{\omega}'\underset{\sim}{A}\underset{\sim}{W}\right]$$

$$\le E\left[I(\underset{\sim}{\omega}'\underset{\sim}{W} < \varepsilon)(\underset{\sim}{\omega}'\underset{\sim}{W})^{-1}\mathrm{Tr}(\underset{\sim}{A})\right]$$

$$= \mathrm{Tr}(\underset{\sim}{A})E\left[I(\underset{\sim}{\omega}'\underset{\sim}{W} < \varepsilon)(\underset{\sim}{\omega}'\underset{\sim}{W})^{-1}\right]$$

$$\le \mathrm{Tr}(\underset{\sim}{A})\left(\Gamma(p/2)\, 2^{p/2}\right)^{-1} \varepsilon^{(p-2)} e^{-\frac{1}{2}\Delta(1-\varepsilon)}, \ \forall\ p \ge 2,\ 0 < \varepsilon \le 1.$$

Therefore, whenever $p \ge 3$, from (4.26) through (4.38), we obtain that

$$(4.39)\qquad \lim_{n\to\infty} \rho_n^*(\underset{\sim}{\hat{\theta}}^S, \underset{\sim}{\lambda}) = \lim_{n\to\infty} E\left\{n(\underset{\sim}{\hat{\theta}}_n^S - \underset{\sim}{\theta}_n)'\underset{\sim}{Q}(\underset{\sim}{\hat{\theta}}_n^S - \underset{\sim}{\theta}_n)\,|K_n\right\}$$

$$= \mathrm{Tr}(\underset{\sim}{Q}\underset{\sim}{\Gamma})\{1 - H_{p+2}(\varepsilon;\Delta)\} + (\underset{\sim}{\lambda}'\underset{\sim}{Q}\underset{\sim}{\lambda})\left[2H_{p+2}(\varepsilon;\Delta) - H_{p+4}(\varepsilon;\Delta)\right]$$

$$- 2c\ \mathrm{ch}_p(\underset{\sim}{Q}\underset{\sim}{\Gamma})E\left(\chi^{-2}_{p+2,\Delta}\right) + c^2\left[\mathrm{ch}_p(\underset{\sim}{Q}\underset{\sim}{\Gamma})\right]^2\left\{\mathrm{Tr}(\underset{\sim}{Q}^{-1}\underset{\sim}{\Gamma}^{-1})E\left(\chi^{-4}_{p+2,\Delta}\right)\right.$$

$$\left. + \Delta^* E\left(\chi^{-4}_{p+4,\Delta}\right)\right\} + 2c\ \mathrm{ch}_p(\underset{\sim}{Q}\underset{\sim}{\Gamma})E(I(\underset{\sim}{\omega}'\underset{\sim}{W} < \varepsilon)(\underset{\sim}{\omega}'\underset{\sim}{W})^{-1}\underset{\sim}{\omega}'\underset{\sim}{\omega})$$

$$- c^2\left[\mathrm{ch}_p(\underset{\sim}{Q}\underset{\sim}{\Gamma})\right]^2 E(I(\underset{\sim}{\omega}'\underset{\sim}{W}) < \varepsilon)(\underset{\sim}{\omega}'\underset{\sim}{W})^{-2}\underset{\sim}{\omega}'\underset{\sim}{A}\underset{\sim}{W}),$$

where the last term is non-negative, while the term before last is bounded from above by $2c\ ch_p(\underset{\sim\sim}{Q\Gamma})\varepsilon^{(p-1)/2}/(2^{p/2}\sqrt{p/2})$, uniformly in $\underset{\sim}{\lambda}$. Hence, we have for every $\varepsilon: 0 < \varepsilon \le 1$, $\underset{\sim}{\lambda} \in E^p$,

$$(4.40) \quad \lim_{n\to\infty} \rho_n^*(\hat{\underset{\sim}{\theta}}^S, \underset{\sim}{\lambda}) \le \mathrm{Tr}(\underset{\sim\sim}{Q\Gamma})\left[1 - H_{p+2}(\varepsilon;\Delta)\right] + (\underset{\sim}{\lambda}'\underset{\sim\sim}{Q\lambda})\left[2H_{p+2}(\varepsilon;\Delta) - H_{p+4}(\varepsilon;\Delta)\right] - 2c\cdot ch_p(\underset{\sim\sim}{Q\Gamma})E\left(\chi_{p+2,\Delta}^{-2}\right) + c^2ch_p^2(\underset{\sim\sim}{Q\Gamma})\left[\mathrm{Tr}(\underset{\sim}{Q}^{-1}\underset{\sim}{\Gamma}^{-1})E\left(\chi_{p+2,\Delta}^{-4}\right) + \Delta^*E\left(\chi_{p+4,\Delta}^{-4}\right)\right] + O\left(\varepsilon^{(p-1)/2}\right)$$

$$= \left\{\mathrm{Tr}(\underset{\sim\sim}{Q\Gamma}) - 2c\cdot ch_p(\underset{\sim\sim}{Q\Gamma})E\left(\chi_{p+2,\Delta}^{-2}\right) + c^2ch_p^2(\underset{\sim\sim}{Q\Gamma})\left[\mathrm{Tr}(\underset{\sim}{Q}^{-1}\underset{\sim}{\Gamma}^{-1})E\left(\chi_{p+2,\Delta}^{-4}\right) + \Delta^*E\left(\chi_{p+4,\Delta}^{-4}\right)\right]\right\} - H_{p+2}(\varepsilon;\Delta)\mathrm{Tr}(\underset{\sim\sim}{Q\Gamma}) + (\underset{\sim}{\lambda}'\underset{\sim\sim}{Q\lambda})\left[2H_{p+2}(\varepsilon;\Delta) - H_{p+4}(\varepsilon;\Delta)\right] + O\left(\varepsilon^{(p-1)/2}\right)$$

$$= \left\{\mathrm{Tr}(\underset{\sim\sim}{Q\Gamma}) - 2c\cdot ch_p(\underset{\sim\sim}{Q\Gamma})E\left(\chi_{p+2,\Delta}^{-2}\right) + c^2ch_p^2(\underset{\sim\sim}{Q\Gamma})\left[\mathrm{Tr}(\underset{\sim}{Q}^{-1}\underset{\sim}{\Gamma}^{-1})E\left(\chi_{p+2,\Delta}^{-4}\right) + \Delta^*E\left(\chi_{p+4,\Delta}^{-4}\right)\right]\right\} + O\left(\varepsilon^{(p-1)/2}\right) + O\left(\varepsilon^{p/2}\right), \text{ uniformly in } \underset{\sim}{\lambda}.$$

Now, by the results of Section 2 of Sclove, Morris and Radhakrishnan (1972), the leading term on the right hand side of (4.40) is $< \mathrm{Tr}(\underset{\sim\sim}{Q\Gamma})$, $\forall c: 0 < c < 2(p-2)$, and $\underset{\sim}{\lambda} \in E^p$. Hence,

$$(4.41) \quad \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} \rho_n^*\left(\hat{\underset{\sim}{\theta}}^S, \underset{\sim}{\lambda}\right) \le \mathrm{Tr}(\underset{\sim\sim}{Q\Gamma}) = \lim_{n\to\infty} \rho_n^*(\hat{\underset{\sim}{\theta}}, \underset{\sim}{\lambda}),$$

so that for $\varepsilon(> 0)$ chosen sufficiently small and for $0 < c < 2(p-2)$, $p \ge 3$, $\hat{\underset{\sim}{\theta}}_n$ is asymptotically inadmissable with respect to $\hat{\underset{\sim}{\theta}}_n^S$. This also exhibits the asymptotic inadmissibility of $\hat{\underset{\sim}{\theta}}_n$ for local translation alternatives.

5. Some General Comments. As has been mentioned after (2.16) for the James-Stein type estimator in (2.16), one faces a technical problem in evaluating the risk, as $L_n = 0$, with a positive probability. The proposed estimator in (2.20) eliminates this problem. It may be noted that rank statistics are not invariant under linear transformation, hence general loss function with matrix $\underset{\sim}{Q}$ and $\underset{\sim}{\Gamma}$ is considered. However, in this context, the choice of $\varepsilon(> 0)$ and $c(> 0)$ remains to be worked out. Because of (4.37)-(4.38), we need to restrict ε: $0 < \varepsilon \le 1$, while in (4.40), to derive the inequality in (4.41) we need to restrict c to $0 < c < 2(p-2)$. In the simple James-Stein (1961) estimator, c has been restricted to $c = (p-2)$. As a general rule, the same choice of c may be recommended in (2.20). However, for the normal mean case, the study of Berger, Bock, Brown, Cassella and Gleser (1977) shows that for small values of n, $c(\le c_{n,p})$ has a range depending on n and p and $c_{n,p}$ may be quite small compared to $p-2$ when n is small. Hence, from a practical point of view, for n not very large, we would be tempted to use a smaller value of c. Ideally, $\varepsilon(> 0)$ should be as small as possible, so that in (4.40), these terms are really negligible. However, as ε becomes small, (4.35) and (4.36) demand more larger sample sizes for a valid justification. Hence, the choice of ε also needs to be made with relation to the sample size. Some numerical studies may be more appropriate to study the elegancy of the asymptotic theory for moderate sample sizes. Finally, we may remark that though for local translation alternatives, $\hat{\underset{\sim}{\theta}}_n$ is asymptotically inadmissible with respect to $\hat{\underset{\sim}{\theta}}_n^S$, $\hat{\underset{\sim}{\theta}}_n^{PT}$ is not so. Actually, for $\underset{\sim}{\lambda}$ close to $\underset{\sim}{0}$, the

PTE performs better than $\hat{\underset{\sim}{\theta}}_n^S$, though for some $\Delta_0(>0)$, for all $\underset{\sim}{\lambda}$: $\underset{\sim}{\lambda}'\underset{\sim}{\Gamma}^{-1}\underset{\sim}{\lambda} \geq \Delta_0$, $\hat{\underset{\sim}{\theta}}_n^S$ may perform better than $\hat{\underset{\sim}{\theta}}_n^{PT}$. Hence, the proposed shrinkage estimator $\hat{\underset{\sim}{\theta}}_n^S$ does not dominate over $\hat{\underset{\sim}{\theta}}_n$ everywhere.

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