SOME REPLACEMENT - TIMES DISTRIBUTIONS IN TWO-COMPONENT SYSTEMS

by

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ABSTRACT

We consider a system consisting of two modules, one of which cannot be easily inspected, while the other is monitored continuously. A fault developing in the first model is called "unrevealed" (U) while the one in the second module (which is assumed to be detected immediately) is called "revealed" (R). It is supposed that repairs are initiated as soon as an R is observed, but not otherwise. Phillips (Microelectron. Reliab. (1979) **18**, 495-503; Reliab. Engineer. (1981), **2**, 221-231) assumed that both modules are always replaced. In this paper we allow for replacement of the first module only if a U fault is found on special inspection. Expressions are derived for the joint distribution of time to first repair and time between first and second repairs. Special cases are considered in which fairly simple results are obtained by general arguments without recourse to analysis. Detailed formulas are developed for a particular parametric model. Relationships with relevation theory are indicated.

1. Introduction

Phillips [1,2] has considered a system consisting of two modules, one of which cannot be easily inspected, while the other is monitored continuously. A fault developing in the first module is called "unrevealved (U) - that is, until a special inspection is carried out - while a fault in the second module, which it is assumed will be detected immediately, is called "revealed" (R). It is supposed that repairs are initiated as soon as an R is observed but not otherwise. As in [2] we will further assume that repairs are effected instantaneously and, if carried out, result in the repaired module being "as good as new".

We will be especially concerned with the distribution of the time at which a second (or later) repair is needed, on similar assumptions to those of Phillips, except that we will allow for the possibility that repairs are made *only* to modules *with faults*. This would mean that if an R occurs, the first module is specially inspected, and replaced only if it has a U fault.

Phillips' model is based on three random variables: X: time from repair of R to next occurrence of R, assuming no U present. Y: time from repair of U to next occurrence of U, assuming no R present. Z: time from a U fault to an R fault. X and Y are assumed independent. Z is independent of X in both [1] and [2], but may depend on Y in [1], though not in [2]. We will retain this possible dependence.

Using the notation $f_W(w)$ to denote the density function of a random variable W, and

$$S_W(w) = \int_w^\infty f_W(t)\,dt$$

to denote its survival function, the density function of the first replacement time, T_1, is ([1], equation (1))

$$f_{T_1}(t_1) = f_X(t_1)S_Y(t_1) + \int_0^{t_1} S_X(t_1-z)f_{Y,Z}(t_1-z,z)dz \qquad (1)$$

(We use subscripts to denote the order of the replacement period.)

2. Distribution of time between first and second replacements

If "repair" consists of complete replacement, as in [1] and [2], then the time T_2 from first to second replacement has the same distribution as T_1, and T_1 and T_2 are mutually independent. Hence the time of second replacement (T_1+T_2) is distributed simply as the convolution of two T_1's. Extension to later replacement times is straightforward.

The situation is different, however, if the first module is replaced only if a U fault is found to be present on inspection after an R fault occurs. When such a fault is not present we start the second replacement period with a first module already aged T_1, rather than a new one. We note in passing that the same situation arises when replacement of *both* modules is automatic, when an R fault occurs, if the first module is replaced from aging stock, with aging occurring at the same rate in storage as in service. Relevation theory, which we have discussed in [3] and elsewhere, is applicable to this situation.

If a U fault *is* present, then X_2, Y_2 and Z_2 will have the same joint distribution as X_1, Y_1 and Z_1; and the sets of variables (X_1,Y_1,Z_1) and (X_2,Y_2,Z_2) will be mutually independent. If a U fault *is not* present, then given $T_1 = t_1$, we will have

X_2 distributed as X_1: - density function $f_X(x_2)$;

Y_2 distributed as (Y_1-t_1), conditional on $Y_1>t_1$: - density function $f_Y(y_2+t_1)/S_y(t_1)$ and survival function $S_y(y_2+t_1)/S_y(t_1)$;

Z_2, given Y_2, distributed as Z_1 given $Y_1 = Y_2$, so that the joint density function of Y_2 and Z_2 is

$$f_{Z|Y}(z_2|y_2)f_Y(y_2|Y_2>t_1) = \frac{f_{Y,Z}(y_2,z_2)}{f_Y(y_2)} \cdot \frac{f_Y(y_2+t_1)}{S_Y(t_1)} \tag{2}$$

Also we have

$$P_U(t_1) = \Pr[U \text{ fault present}|T_1=t_1] = 1-f_X(t_1)S_Y(t_1)/f_{T_1}(t_1) \tag{3}$$

The conditional density function of T_2, given $T_1=t_1$, is therefore

$$f_{T_2|T_1}(t_2|t_1) = P_U(t_1)\{f_X(t_2)S_Y(t_2)+\int_0^{t_2} S_X(t_2-z)f_{Y,Z}(t_2-z,z)dz\}$$
$$+ \frac{1-P_U(t_1)}{S_Y(t_1)}\{f_X(t_2)S_Y(t_1+t_2)+\int_0^{t_2} S_X(t_2-z)f_{Y,Z}(t_2-z,z)\frac{f_Y(t_1+t_2-z)}{f_Y(t_2-z)}dz\} \quad (0<t_2) \tag{4}$$

The joint density function of T_1 and T_2 is

$$f_{T_1,T_2}(t_1,t_2) = f_{T_2|T_1}(t_2|t_1)\ f_{T_1}(t_1)$$
$$= \{f_X(t_2)S_Y(t_2)+\int_0^{t_2} S_X(t_2-z)f_{Y,Z}(t_2-z,z)dz\}$$
$$\cdot \int_0^{t_1} S_X(t_1-z)f_{Y,Z}(t_1-z,z)dz$$
$$+ \{f_X(t_2)S_Y(t_1+t_2)+\int_0^{t_2} S_X(t_2-z)f_{Y,Z}(t_2-z,z)\frac{f_Y(t_1+t_2-z)}{f_Y(t_2-z)}dz\}f_X(t_1) \tag{5}$$
$$= \{f_X(t_2)S_Y(t_2)+H(t_2,0)\}H(t_1,0)$$
$$+ \{f_X(t_2)S_Y(t_1+t_2) + H(t_2,t_1)\}f_X(t_1) \quad (0<t_1,t_2) \tag{5a}$$

where
$$H(a,b) = \int_0^a S_X(a-z)f_{Y,Z}(a-z,z)\{f_Y(a+b-z)/f_Y(a-z)\}dz \tag{6}$$

The overall density of T_2 is

$$f_{T_2}(t_2) = \{f_X(t_2)S_Y(t_2)+\int_0^{t_2} S_X(t_2-z)f_{Y,Z}(t_2-z,z)dz\}$$
$$\cdot \int_0^{\infty}\int_0^{t_1} S_X(t_1-z)f_{Y,Z}(t_1-z,z)dzdt_1$$

$$+ f_X(t_2)\cdot\int_0^\infty f_X(t_1)S_Y(t_1+t_2)dt_1+\int_0^\infty f_X(t_1)\int_0^{t_2} S_X(t_2-z)f_{Y,Z}(t_2-z,z)\frac{f_Y(t_1+t_2-z)}{f_Y(t_2-z)}dz \quad (7)$$

We note that $\int_0^\infty \int_0^{t_1} S_X(t_1-z)f_{Y,Z}(t_1-z,z)dzdt_1$ is simply the overall probability that a U fault is present at the first replacement time, that is, $Pr[X>Y]$, while $\int_0^\infty f_X(t_1)S_Y(t_1+t_2) = Pr[Y>X+t_2]$.

3. Some special cases

Before proceeding to study cases in which the joint distribution of X, Y and Z is completely specified, we first take note of a few results of broader character which can be derived by general reasoning, without recourse to analysis.

Intuitively one might feel that $1 - P_U(t_1) = S_Y(t_1)$, since the probability of no U fault occurring up to time t, is $S_Y(t_1)$, but this does not take into account the differential effect of the times of occurrence of R faults according as they are, or are not preceded by U faults. *If* the time to an R fault (T_1=X if X≤Y; = Y+Z if Y<X) were independent of Y, then $f_{T_1}(t_1) = f_X(t_1)$ and so, we *would* have $1 - P_U(t_1) = S_Y(t_1)$. It is, however, clear that this is not usually the case; indeed independence of Z and Y does not imply independence of T_1 and Y; in fact it usually implies dependence.

If replacements of first modules are from aging stocks, with aging being the same whether in storage or in service, ("relevated" using the terminology of [37]), then it makes no difference (so far as distributions of T_j's are concerned) whether or not a first module is replaced when there is no U fault. This is because after the repair we have, in either case, a first module of age t_1; and this result is valid whether replacement of second modules is from aging stocks or not.

If the lifetime distribution of the first module (that is, of Y) is exponential, then (X_1,Y_1,Z_1) and (X_2,Y_2,Z_2) and so T_1 and T_2 will be mutually independent, whether replacement of the first module is from aging stocks or not (or is, indeed, performed or not when no U is present). This follows from the lack of memory (or old-as-good-as-new) property (see e.g. [4]) of the exponential distribution. If replacement of the *second* module is from aging stock, this may no longer be the case, though it is so if the distributions of X and Y are *both* exponential, *whatever the distribution of* Z, provided this depends only on time (Y) since replacement, and not on actual age.

If the joint distribution of Y and Z is of the Farlie-Gumbel-Morgenstern form, with

$$f_{Y,Z}(y,z) = f_Y(y)f_Z(z)[1+\alpha\{2S_Y(y)-1\}\{2S_Z(z)-1\}] \quad (|\alpha|<1) \tag{8}$$

(see, e.g., [5]), then in (5a)

$$H(a,b) = \int_0^a S_X(a-z)f_Y(a+b-z)f_Z(z)[1+\alpha\{2S_Y(a-z)-1\}\{2S_Z(z)-1\}]dz \tag{9}$$

When we come to consider explicit forms for the distribution functions, numerical evaluation of the distributions of T_1 and T_2 is straightforward, but for elegant analytical results there are technical difficulties, centering mainly on the form of $S_Y(y)$ and in particular of the ratio $f_Y(t_1+t_2-z)/f_Y(t_2-z)$, and also on the form of $S_Z(z)$. Assuming exponential forms does, as we have seen, give simple results, but these are perhaps *too* simple to make good examples. We will take, as an illustrative special case, X, Y and Z to have density functions

$$\theta_i^{-2}t\exp(-t\theta_i^{-1}) \quad (0<\theta_i,t)$$

$i=1,2,3$ respectively, with the joint density function of Y and Z of the Farlie-Gumbel-Morgenstern form (8). We have

$$f_Y(t_1+t_2-z)/f_Y(t_2-z) = (t_1+t_2-z)(t_2-z)^{-1}e^{-t_1/\theta_2};S_X(x) = (1+x\theta_1^{-1})e^{-x\theta_1};$$

$$S_Y(y) = (1+y\theta_2^{-1})e^{-y\theta_2};\ S_Z(z) = (1+z\theta_3^{-1})\ e^{-z/\theta_3}.$$

Now from (5)

$$f_{T_1,T_2}(t_1,t_2) = [\theta_1^{-2}t_2(1+\theta_2^{-1}t_2)\ \exp\{-t_2(\theta_1^{-1}+\theta_2^{-1})+H(t_2,0)]\ H(t_1,0)$$
$$+\ [\theta_1^{-2}t_2\{1+\theta_2^{-1}(t_1+t_2)\}\exp\{-t_2(\theta_1^{-1}+\theta_2^{-1})-t_1\theta_2^{-1}\}+H(t_2,t_1)]\theta_1^{-2}t_1e^{-t_1/\theta_1} \quad (10)$$

$$\text{with } H(a,b) = (\theta_2\theta_3)^{-2}\ \exp\{-a(\theta_1^{-1}+\theta_2^{-1})-b\theta_2^{-1}\}\int_0^a\{1+(a-z)\theta_1^{-1}\}z(a+b-z)e^{z(\theta_1^{-1}+\theta_2^{-1}-\theta_3^{-1})}$$
$$\times\ [1+\alpha-2\alpha\{(1+\frac{a-z}{\theta_2})e^{-(a-z)/\theta_2}+(z/\theta_3)\,e^{-z/\theta_3}\}$$
$$+\ 4\alpha(1+\frac{a-z}{\theta_2})(z/\theta_3)\,e^{-(a-z)\theta_2^{-1}-z\theta_3^{-1}}\,dz \quad (11)$$

Although (11) is complicated, it involves only integrals of form $\int_0^a z^{\alpha}\exp(\beta z)dz = \beta^{-\alpha-1}\int_0^{a\beta}w^{\alpha}e^{-w}dw$ and so can be expressed in terms of elementary functions.

Similar analyses can be performed for other choices of distribution for X, Y and Z (e.g. Weibull) but they lead to even more complicated expressions than (11). We hope to provide some detailed analyses of such cases, in later work, and also to introduce cost functions, on the basis of which replacement strategies can be compared.

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REFERENCES

[1] Phillips, M. J. (1979) The reliability of a system subject to revealed and unrevealed faults, Microelectron. Reliab. **18**, 495-503.

[2] Phillips, M. J. (1981) A preventive maintenance plan for a system subject to revealed and unrevealed faults, Reliab. Eng., **2**, 221-231.

[3] Kotz, S. and Johnson, N. L. (1981) Dependent relevations: Time-to-failure under dependence, Amer. J. Math. and Mgmt. Sci., **1**, 155-165.

[4] Galambos, J. and Kotz, S. (1978) Characterizations of Probability Distributions, Lecture Notes on Mathematics, **675**, Springer-Verlag: Berlin, Heidelberg and New York.

[5] Johnson, N. L. and Kotz, S. (1972) Distributions in Statistics: Multivariate Continuous Distributions, Wiley: New York.

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