COEFFICIENTS IN EXPANSIONS OF CERTAIN RATIONAL MULTIVARIABLE FUNCTIONS

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ABSTRACT

Motivated by the work of Evans, Ismail and Stanton (Canad. J. Math. 34 (1982), 1011-1024), we obtain the coefficients appearing in the series expansion of certain rational multivariate functions.

1. INTRODUCTION

In a recent article, Evans, Ismail and Stanton [3] considered the problem of evaluating the constant terms of some interesting rational functions. Their work was motivated by a problem of Mallows [8]. One of the main results in [3] is

Theorem 1. *Let* A *and* B *be positive integers. For each pair of non-negative integers* u,v, *the coefficient* K(u,v) *of* $y^u z^v$ *in the power series expansion of*

$$g(y,z) = (1-yz)^{A+B}(1-y)^{-A}(1-z)^{-B} \tag{1.1}$$

is

$$K(u,v) = \binom{A+u-v-1}{u}\binom{B+v-u}{v} - \binom{A+u-v}{u}\binom{B+v-u-1}{v-1} . \tag{1.2}$$

This paper is concerned with the problem of obtaining multivariable analogues of Theorem 1. Along the way, there are at least two obstacles to deriving suitable extensions. One is that the natural extensions of some of the techniques used in [3], for example, Lagrange inversion in several variables, do not seem to be particularly helpful. A second difficulty is that one is not quite certain of what will constitute a "suitable" extension of g(y,z); further remarks are directed at this problem in Section 5.

Despite these problems, it turns out that some interesting results can be obtained for the function

$$f(x) \equiv f(x_1,x_2,\ldots,x_n) := (1-x_1x_2\ldots x_n)^A \prod_{j=1}^{n} (1-x_j)^{-B_j} \tag{1.3}$$

which reduces to (1.1) when n=2. Here and throughout, $A,B_1,B_2,\ldots,B_n$ are positive integers.

Actually, even this choice seems recondite (cf. Section 5). So we shall choose the integers B_j to have a common value B, and consider not f(x) but

$f(x)\ \Delta(x)$, where $\Delta(x) = \prod_{j<k}^{n} (x_j - x_k)$ is the usual Vandermonde determinant. With the above restriction on the B_j, $f(x)$ becomes a symmetric function of the variables $x_1, x_2, \ldots, x_n$, and we are able to use various results from the theory of symmetric functions.

Interestingly, Theorem 2 below shows that the coefficients in the power series expansion of $f(x)\Delta(x)$ can be expressed as linear combinations of determinants of binomial coefficients. This result is already valid when $n = 2$ by virtue of (1.2), since $K(u,v)$ is actually a determinant. However, the difference in the methods used is probably of independent interest. In a sense, we reverse the techniques of Evans et al [3] by using integrals to evaluate the series coefficients.

The layout of the paper goes as follows; Section 2 lists some preliminary material, while Section 3 derives the coefficients of $f(x)\Delta(x)$. Section 4 considers two q-analogues of $f(x)$, and expresses the related coefficients in terms of determinants of q-binomial coefficients. Finally, Section 5 relates the problems of obtaining the coefficients of $f(x)$ itself and evaluating terminating balanced generalised hypergeometric function. Some remarks are also directed towards the problem of developing suitable multivariate analogues of $g(y,z)$, and deriving combinatorial interpretations for some integrals similar to some of those in [3].

2. PRELIMINARY MATERIAL

Throughout, we use the usual notation for rising factorials, viz., for any non-negative integer k,

(2.1) $$(a)_k = \Gamma(a+k)/\Gamma(a) .$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition, i.e. a set of non-negative integers arranged in non-increasing order,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 .$$

The sum $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ of the parts of λ, denoted by $|\lambda|$, is called the weight of λ; for brevity, we write $\lambda \vdash k$. The number of non-zero λ_i, denoted by $\ell(\lambda)$, is called the length of λ. The conjugate of λ is the partition $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_n')$ such that

$$\lambda_i' = \text{Card}\,\{j : \lambda_j \geq i\} \quad , \; i = 1,2,\ldots,n \; .$$

For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\ell(\lambda) \leq n$, the Schur functions s_λ are defined by

$$(2.2) \qquad s_\lambda(x) \equiv s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\det(x_i^{\lambda_i+n-j})_{1\leq i,j\leq n}}{\det(x_i^{n-j})_{1\leq i,\, j\leq n}}$$

It is clear that $s_\lambda(x)$ is symmetric in $x_1, x_2, \ldots, x_n$. Further, since the numerator of (2.2) is divisible by each of the factors $x_i - x_j$, $1 \leq i < j \leq n$, then it is divisible also by the Vandermonde determinant

$$(2.3) \qquad \Delta(x) = \prod_{i<j} (x_i - x_j) = \det(x_i^{n-j}) \; .$$

Thus, the $s_\lambda(x)$ are symmetric polynomials in $x_1, x_2, \ldots, x_n$, and it can be shown that they form a $\mathbb{Z}$-basis for the vector space of homogeneous symmetric polynomials in $x_1, \ldots, x_n$.

Schur functions were first defined as in (2.2) by Jacobi, and later related to the theory of group representations by Schur. A recent treatment of Schur functions is given by Macdonald [7].

Two useful series expansions, which are to be regarded as purely formal, without regard to questions of convergence, are ([7], pp. 33-35):

$$(2.4) \qquad \prod_{i=1}^{n} \prod_{j=1}^{m} (1 - x_i y_j)^{-1} = \sum_{k=0}^{\infty} \sum_{\lambda \vdash k} s_\lambda(x) s_\lambda(y) \quad ,$$

$$(2.5) \qquad \prod_{i=1}^{n} \prod_{j=1}^{m} (1 - x_i y_j) = \sum_{k=0}^{\infty} \sum_{\lambda \vdash k} s_\lambda(x) s_{\lambda'}(y) \; .$$

Setting $y_1 = y_2 = \ldots = y_m = t$ in (2.4), we obtain

$$(2.6)\qquad \prod_{i=1}^{n}(1-x_i t)^{-m} = \sum_{k=0}^{\infty} t^k \sum_{\lambda\vdash k} \binom{m}{\lambda} s_\lambda(x) \quad .$$

In fact, (2.6) holds more generally for any real number m. Only a finite number of the <u>generalised binomial coefficients</u> $\binom{m}{\lambda}$ are non-zero if m is a negative integer. An explicit formula for $\binom{m}{\lambda}$ is ([7], Exercises 3.4 and 4.1)

$$(2.7)\qquad \binom{m}{\lambda} = \det\left(\binom{m}{\lambda_j - j + k}\right)_{1\le j,k\le n} = (-1)^{|\lambda|}\binom{-m}{\lambda'} \quad .$$

To evaluate certain sums whose terms contain determinants, we shall use the <u>basic composition formula</u> ([6],p.17). This result states that if β is a sigma-finite measure on the real line $\mathbb{R}$, and $\{f_j\}_{j=1}^n$ and $\{g_j\}_{j=1}^n$ are subsets of $L^1(\beta)$, then

$$(2.8)\qquad \int\cdots\int_{t_1>t_2>\cdots>t_n} \det(f_j(t_k))\det(g_j(t_k))\,d\beta(t_1)\ldots d\beta(t_n)$$

$$= (n!)^{-1}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \det(f_j(t_k))\det(g_j(t_k))\,d\beta(t_1)\cdots d\beta(t_n)$$

$$= \det\left(\int_{\mathbb{R}} f_j(t) g_k(t)\,d\beta(t)\right) \quad .$$

The basic composition formula is attributed by Muir [10] to Andreief [1]. We shall apply it when β is supported by an interval or by a discrete set.

3. COEFFICIENTS IN THE EXPANSION OF $f(x)\Delta(x)$

<u>Theorem 2. For each n-tuple</u> $(u_1,u_2,\ldots,u_n)$ <u>of non-negative integers, the coefficient</u> $C(u_1,u_2,\ldots,u_n)$ <u>of</u> $x_1^{u_1}x_2^{u_2}\cdots x_n^{u_n}$ <u>in the expansion of</u>

$$(3.1)\qquad g(x) = (1-x_1x_2\ldots x_n)^A \prod_{j=1}^{n}(1-x_j)^{-B} \prod_{j<k}(x_j-x_k)$$

<u>is</u>

$$(3.2)\qquad C(u_1,u_2,\ldots,u_n) = (-1)^{\frac{1}{2}n(n-1)+v} \sum_{\ell=0}^{\infty} \frac{(-1)^{n\ell}(-A)_\ell}{\ell!} \det\left(\binom{B}{j+u_k-\ell-n}\right)$$

where $v = u_1+u_2+\ldots+u_n$.

Proof. Applying the expansion (2.6), we obtain

$$(3.3)\qquad (1-x_1x_2\ldots x_n)^A \prod_{j=1}^{n}(1-x_j)^{-B} = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\lambda\vdash k} \frac{(-1)^k(-A)_\ell}{\ell!}\binom{B}{\lambda}(x_1\ldots x_n)^\ell s_\lambda(x).$$

To compute $C(u_1,\ldots,u_n)$, which is the constant term in the expansion of $x_1^{-u_1}x_2^{-u_2}\ldots x_n^{-u_n}\, g(x)$, we set $x_j = e^{i\theta_j}$, $0 \le \theta_j < 2\pi$, $1 \le j \le n$, and integrate over $[0,2\pi)^n$. Then the integral to be evaluated is

$$(2\pi)^{-n}\int\cdots\int_{[0,2\pi)^n} \left(\prod_{j=1}^{n} e^{i\theta_j(\ell-u_j)}\right) s_\lambda(e^{i\theta_1},\ldots,e^{i\theta_n}) \prod_{j<k}(e^{i\theta_j}-e^{i\theta_k})\,d\theta_1\ldots d\theta_n$$

$$(3.4)\qquad = (2\pi)^{-n}\int\cdots\int_{[0,2\pi)^n} \left(\prod_{j=1}^{n} e^{i\theta_j(\ell-u_j)}\right)\det(e^{i\theta_j\mu_k})\,d\theta_1\ldots d\theta_n ,$$

where $\mu_k = \lambda_k+n-k$, $k=1,2,\ldots,n$. Letting S_n denote the group of permutations on n symbols, we expand the determinant in the integrand of (3.4) and obtain

$$(3.5)\qquad \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)(2\pi)^{-n}\int_0^{2\pi}\cdots\int_0^{2\pi}\left(\prod_{j=1}^{n}\exp(i\theta_j(\ell-u_j+\mu_{\sigma_j}))\right)d\theta_1\ldots d\theta_n .$$

With $\delta(a,b)$ denoting Kronecker's delta symbol, the orthogonality properties of the $\{e^{i\theta_j}\}$ show that (3.4) equals

$$(3.6)\qquad \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{j=1}^{n}\delta(\ell+\mu_{\sigma_j},u_j) = \det(\delta(\ell+\mu_j,u_k)).$$

Hence, along with (2.7) and (3.3), (3.6) shows that

$$(3.7)\quad C(u_1,\ldots,u_n) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\lambda\vdash k}\frac{(-1)^k(-A)_\ell}{\ell!}\det(\binom{B}{\lambda_j-j+k})\det(\delta(\ell+\mu_j,u_k))$$

$$= \sum_{\ell=0}^{\infty}\frac{(-A)_\ell}{\ell!}\sum_{\mu_1>\mu_2>\cdots\mu_n>0}^{\infty}(-1)^{\sum_1^n(\mu_j-n+j)}\det(\binom{B}{\mu_j-n+k})$$

$$\times\ \det(\delta(\ell+\mu_j,u_k))\ ,$$

the second equality following from the substitutions $\mu_j=\lambda_j+n-j$, $1 \le j \le n$. In the second series, notice that we may replace the limits $\mu_1 > \mu_2 >\cdots> \mu_n > 0$ by $\mu_1 \ge \mu_2 \ge\cdots\ge \mu_n \ge 0$, since if $\mu_r = \mu_s$ for some r,s, both determinants in the summand are zero. Then, we apply the discrete version of the basic composition formula (2.8), and obtain (3.2) after some simplifications are made. □

Since A is a positive integer, it follows that the sum in (3.2) terminates, at latest, when $\ell = A$.

We next consider several interesting special cases of Theorem 2.

Corollary 3. *If any two of the* u_j *are equal, then* $C(u_1,u_2,\ldots,u_n) = 0$.

Proof. In this event, it is easy to see that all the determinants in (3.2) are identically zero. □

Corollary 4. $C(1,2,3,\ldots,n) = (-1)^{\frac{1}{2}n(n+1)}\ \{\frac{(B)_n}{n!} + (-1)^{n+1}A\}$.

Proof. Let $D(m,p,r) = \det(\binom{m}{p-i+j})$, $1 \le i,\ j \le r + 1$, where m,p and r are non-negative integers. The evaluation of D(m,p,r) may be located in Muir [11], Section 733, pp. 681-2. By reversing the order in which the rows appear, it follows that when $\ell \le 1$, $\det(\binom{B}{j+k-\ell-n}) = (-1)^{\frac{1}{2}n(n-1)}D(B,1-\ell,n-1)$. Since ([11], Section 732, pp. 680-1)

$$D(m,p,r) = D(m,r+1,p) = \prod_{j=0}^{r} \frac{\binom{m+j}{p}}{\binom{p+j}{p}} , \tag{3.8}$$

with the convention that $D(m,p,r) = 0$ if $r < 0$, then the result follows from (3.8). □

More generally, we can obtain an explicit result if the $\{u_j\}$ form an arithmetic progression, $u_j = u + jd$, $j=1,2,\ldots,n$. The result following can be deduced from Theorem 2 using the formulae in Muir [11], Section 734, pp. 683-5.

Corollary 5. For any positive integer d, $C(u+d,u+2d,\ldots,u+nd) =$

$$(-1)^{nu+\frac{1}{2}n(n+1)d} \sum_{\ell=0}^{\infty} \frac{(-1)^{n\ell}(-A)_\ell}{\ell!} \left[\prod_{i=0}^{n-1} \frac{\binom{B+j}{u-\ell+d}}{\binom{u-\ell+(j+1)d}{jd}}\right] \frac{\prod_{i=0}^{n-2} \prod_{j=0}^{n-2-i} \binom{B-u+\ell-d-(d-1)i+j}{d-1}}{\prod_{j=1}^{n-1} \binom{jd-1}{(j-1)d}^{n-j}} .$$

Although Corollary 3 shows that $C(u_1,u_1,u_3,\ldots,u_n) = 0$, this does not seem to be a general phenomenon for functions similar to $f(x)$. The proof of the following result is similar to that of Theorem 2.

Theorem 6. The coefficient of $(x_1x_2\ldots x_n)^u$ in the power series expansion of $f(x)[\Delta(x)]^2$ is zero if $u < n-1$, and for $u \geq n-1$ is

$$(-1)^{un+\frac{1}{2}n(n-1)}(n!) \sum_{\ell=0}^{u-n+1} \frac{(-1)^{n\ell}(-A)_\ell}{\ell!} \prod_{j=0}^{u-n+1-\ell} \frac{\binom{B+j}{n}}{\binom{n+j}{n}} .$$

In this situation, it seems far more difficult to compute the most general coefficients.

4. TWO q-ANALOGUES

In this section, we use the standard notation [12]

$$(a;q)_k = \prod_{r=0}^{\infty} (1-aq^r)/(1-aq^{k+r}) , \quad k = 0,1,2,\ldots,$$

when working with basic hypergeometric series.

For the case of two variables, the results in [3] pertain to expansions of the functions $(wx;q)_{A+B}/(w;q)_A\ (qx;q)_B$ and $(wx;q)_{A+B}/(w;q)_A(x;q)_B$. This section treats two multivariate analogues,

$$(4.1)\qquad f_q(x_0;x_1,\ldots,x_n) = \frac{\prod_{i=1}^{n} (x_0x_i;q)_{A+B}}{(x_0;q)_A \prod_{i=1}^{n} (qx_i;q)_B}\ ,$$

and

$$(4.2)\qquad g_q(x_0;x_1,\ldots,x_n) = \frac{\prod_{i=1}^{n} (x_0x_i;q)_{A+B}}{(x_0;q)_A \prod_{i=1}^{n} (x_i;q)_B}\ .$$

Since both f_q and g_q are symmetric functions in the n variables $x_1,x_2,\ldots,x_n$, it makes sense to ask not for the coefficient of the monomial $x_0^{u_0}x_1^{u_1}\ldots x_u^{u_n}$, but instead for the coefficient of $x_0^{u_0}s_\lambda(x_1,x_2,\ldots x_n)$, where λ is a partition with $\ell(\lambda) \le n$. It turns out that, analogous to the results of Section 3, the coefficients may be expressed in terms of linear combinations of determinants of the q-<u>binomial coefficients</u>,

$$(4.3)\qquad [{}^{j}_{k}]_q = \frac{(q;q)_j}{(q;q)_k(q;q)_{j-k}}\ .$$

First some notation is required. For any partition λ with $\ell(\lambda) \le m$, we define

$$(4.4)\qquad n(\lambda) = \sum_{i\ge 1} (i-1)\lambda_i\ .$$

The significance of $n(\lambda)$ arises from the formula ([7], Exercise 3.1)

$$(4.5)\qquad s_\lambda(1,q,q^2,\ldots,q^{m-1}) = q^{n(\lambda)}[{}^{m}_{\lambda'}]\ ,$$

where $[{}^{k}_{\lambda}]$, the <u>generalised</u> q-<u>binomial coefficient</u> may be defined by

$$(4.6)\qquad [{}^{k}_{\lambda}] = \det\left(q^{\frac{1}{2}(\lambda_i'-i+j)(\lambda_i'-i+j-1)} [{}^{k}_{\lambda_i'-i+j}]_q\right)_{1\le i,j\le \ell(\lambda')}\ .$$

We also recall the Clebsch-Gordan numbers $m_{\lambda\mu}(\nu)$, defined by the relation

$$(4.7) \qquad s_\lambda(x)s_\mu(x) = \sum_\nu m_{\lambda\mu}(\nu)s_\nu(x) \quad ,$$

where the sum is over all partitions ν with $\ell(\nu) \le n$, and $|\nu| = |\lambda|+|\mu|$. An introductory treatment of Clebsch-Gordan numbers and related topics is given by Humphreys [5].

<u>Theorem 7</u>. <u>For any non-negative integer</u> j, <u>and any partition</u> ν <u>with</u> $\ell(\nu) \le n$, <u>the coefficient of</u> $x_0^j s_\nu(x)$ <u>in the expansion of</u> $f_q(x_0;x)$ <u>is</u>

$$(4.8) \qquad \sum_{\lambda,\mu,\rho} m_{\lambda\mu}(\nu)q^{n(\lambda')+|\mu|+n(\mu)+n(\rho)} \begin{bmatrix} A+B \\ \lambda \end{bmatrix} \begin{bmatrix} B \\ \mu' \end{bmatrix} \begin{bmatrix} n \\ \rho' \end{bmatrix} \binom{n}{\rho}$$

<u>where the sum is over all partitions</u> λ,μ,ρ <u>such that</u> $|\lambda| \le A+B$, $|\lambda'|+|\rho| = j$ and $\max\{\ell(\lambda),\ell(\mu), \ell(\rho)\} \le n$.

<u>Proof</u>. As in [7], set $y_j = x_0q^{j-1}$ for all j and m = A+B in (2.5). After applying (4.5), we obtain

$$(4.9) \qquad \prod_{i=1}^{n} (x_0x_i;q)_{A+B} = \sum_{\substack{\ell(\lambda)\le n \\ |\lambda|\le A+B}} q^{n(\lambda')} \begin{bmatrix} A+B \\ \lambda \end{bmatrix} x_0^{|\lambda'|} s_\lambda(x) \quad .$$

In a similar way, (2.4) and (2.6) imply, respectively,

$$(4.10) \qquad \prod_{i=1}^{n} (qx_i;q)_B^{-1} = \sum_{\ell(\mu)\le n} q^{n(\mu)+|\mu|} \begin{bmatrix} B \\ \mu' \end{bmatrix} s_\mu(x) \quad ,$$

and

$$(4.11) \qquad (x_0;q)_A^{-n} = \sum_{i=0}^{\infty} x_0^i \sum_{\substack{\rho\vdash i \\ \ell(\rho)\le n}} q^{n(\rho)} \binom{n}{\rho} \begin{bmatrix} n \\ \rho' \end{bmatrix} \quad .$$

It is straightforward to verify that (4.11) reduces to Heine's q-binomial theorem [12] when n=1. Now, multiplying the series in (4.9)-(4.11), expanding $s_\lambda(x)s_\mu(x)$ as in (4.7), and collecting terms completes the proof. □

It seems difficult to simplify (4.8), not only because the Clebsch-Gordan numbers are difficult to manipulate, but also because the set of admissible partitions λ,μ,ρ is somewhat complicated. To complete this section, we obtain a similar result for $g_q(x_0;x)$; the proof is similar to that of Theorem 7, and is omitted.

Theorem 8. For any non-negative integer j, and any partition ν, $\ell(\nu) \le n$, the coefficient of $x_0^j s_\nu(x)$ in the expansion of $g_q(x_0;x)$ is

$$(4.12)\qquad \sum_{\lambda,\mu,\rho} m_{\lambda\mu}(\nu) q^{n(\lambda')+n(\mu)+n(\rho)} \left[{A+B \atop \lambda}\right] \left[{B \atop \mu'}\right] \left[{n \atop \rho'}\right] \binom{n}{\rho} \quad ,$$

where the sum is over the same set of partitions listed earlier.

5. CONCLUDING REMARKS

There seems to be some connection between the functions considered here, and balanced, terminating, generalised hypergeometric series. To illustrate, if each factor in (1.3) is expanded and the powers of x_i's are collected, we find that the coefficient of $x_1^{u_1} \ldots x_n^{u_n}$ is the terminating sum

$$(5.1)\qquad \sum_{j=0}^{\infty} \frac{(-A)_j}{j!} \prod_{i=1}^{n} \frac{(B_i)_{u_i-j}}{(u_i-j)!}$$

Without losing generality, let u_1 be the smallest of the u_i. Reversing the summation, we rewrite (5.1) in the form

$$(5.2)\qquad \prod_{i=2}^{n} \left[\frac{(B_i)_{u_i-u_1}}{(u_i-u_1)!}\right] \sum_{j=0}^{u_1} \frac{(-A)_{u_1-j}(B_1)_j}{(u_1-j)!\,j!} \prod_{i=2}^{n} \frac{(B_i+u_i-u_1)_j}{(u_i-u_1+1)_j} \quad .$$

So in the case $B_1 = B_2 = \ldots = B_n$, (5.2) is reminiscent of terminating balanced generalised hypergeometric series. Unfortunately, what is presently known about these series seems to be not adequate for our purposes.

As noted in Section 1, there still remains the questions of what constitutes a suitable multivariable extension of g(y,z) in (1.1). The original motivation for [3] was a problem posed by Mallows [8]. In turn, Mallows' problem was derived from certain properties of stable probability measures [4] on the real line. Therefore, it seemed natural that the theory of multivariate stable measures on $\mathbb{R}^n$ should give rise to generalisations of Mallows' integral and hence to suitable extensions of g(y,z). However, multivariate stable laws are far more complicated than their one-dimensional counterparts; see Miller [9] and Cambanis [2] for references to the literature. In the more tractable situations when the stable laws on $\mathbb{R}^n$ have radial (i.e. orthogonally invariant) characteristic functions, the results of Zolotarev [14] can be used in attempts to find extensions of g(y,z). Disappointingly, they seem to lead precisely to Mallows' integrals.

Finally, we obtain a combinatorial interpretation for some integrals similar to the ones considered earlier in [3]. Let $N, m_1, \ldots, m_A, n_1, \ldots, n_B$ be given positive integers such that $m_j | N$ and $n_k | N$ for all $j=1,\ldots,A$, $k=1,\ldots,B$. Consider the equation

$$N = m_1 p_1 + \ldots + m_A p_A + n_1 q_1 + \ldots + n_B q_B \tag{5.3}$$

where the p_i and q_j are non-negative integers.

Theorem 9. *The number* I(N) *of unordered solutions of* (5.3) *may be represented as*

$$I(N) = \frac{1}{2\pi} \int_0^{2\pi} e^{-itN} \prod_{j=1}^{A} \left[\frac{(1-e^{it(N+m_j)})}{(1-e^{itm_j})} \right] \prod_{k=1}^{B} \left[\frac{(1-e^{it(N+n_k)})}{(1-e^{itn_k})} \right] dt .$$

Proof. Our method of proof is based on Vinogradov's method ([13], p. 167). First, it is clear that

$$I(N) = \sum_{p_1 \geq 0} \cdots \sum_{q_B \geq 0} \frac{1}{2\pi} \int_0^{2\pi} \exp(it(-N+\sum_1^A m_j p_j + \sum_1^B n_j q_j))dt.$$

where for each j,k, the sums over p_j and q_k are truncated at N/m_j and N/n_j , respectively. Since the sum over p_1 is

$$\sum_{p_1=0}^{N/m_1} e^{itm_1 p_1} = \frac{1-e^{it(N+m_1)}}{1-e^{itm_1}} ,$$

with similar expressions for the other sums, the results follows. □

It is hoped that Theorem 9 will eventually lead to a combinatorial interpretation of Corollary 4 in [3].

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