# EVALUATION OF NORMING CONSTANTS FOR SOME EXPONENTIAL MODELS ON THE SPHERE

by

Robert M. Hoekstra and Donald St. P. Richards

University of North Carolina
Chapel Hill, NC 27514

AMS 1980 subject classification: 62E15, 62H10 .

Key Words and Phrases: directional data, Fisher-Bingham distribution, exponential model, modified Bessel function, trivariate normal distribution, computable series expansion, norming constant.

ABSTRACT

Computable expressions are obtained for the norming constants for certain exponential distributional models on the sphere. The method used evaluates the norming constant for any member of the general exponential family treated by Beran (Ann. Statist., 7 (1979), 1162-1178), and we illustrate by providing explicit results in the case of the 8-parameter Fisher-Bingham distribution considered by Kent (J.R.S.S. B, 44 (1982), 71-80).

## 1. INTRODUCTION

Recently, several exponential models have been proposed in statistical analysis on the sphere. Beran (1979) introduced a general family of these distributions and discussed certain aspects of their behaviour; Watson (1982) encountered a particular instance of Beran's models while treating some directional data problems arising from geophysics.

As Beran notes, the evaluation of the norming constants for these models is of more than academic importance; if these constants are not known, certain important likelihood ratio procedures cannot be performed. Similar remarks have also been made by Kent (1982).

In this note, we use a simple, direct method to evaluate the norming constants for certain exponential models on the sphere. The method produces computable expressions, and is applicable to any distribution whose density is proportional to an exponentiated polynomial. In particular, it covers the general family of exponential models considered by Beran (1979), and we illustrate by providing explicit results for the 8-parameter Fisher-Bingham ($FB_8$) distribution treated earlier by Kent (1982) and others.

## 2. THE NORMING CONSTANT FOR THE $FB_8$ DISTRIBUTION

Throughout, $\Omega_3 = \{x\varepsilon\ \mathbb{R}^3 : \|x\| = 1\}$ denotes the unit sphere in $\mathbb{R}^3$. Following Kent (1982), we say that a random vector $x\varepsilon\Omega_3$ has the 8-parameter Fisher-Bingham ($FB_8$) distribution if its density is proportional to

$$\exp(k\nu'x + \sum_{j=2}^{3} \beta_j (\gamma'_{(j)}x)^2)\ ,\quad x\varepsilon\Omega_3\ , \tag{1}$$

with respect to Lebesgue measure $d\sigma(x)$ on $\Omega_3$. The eight independent parameters in (1) are given by $k \geq 0$, real-valued $\beta_2 \geq \beta_3$, $\nu\varepsilon\Omega_3$, and $\gamma_{(2)}$ and $\gamma_{(3)}$ are two columns of an orthogonal 3×3 matrix $\Gamma$. Kent (1982) has considered various testing problems involving the $FB_8$ distribution and gives a number of references to the literature.

To compute the norming constant C for the $FB_8$ distribution, we first rotate to the frame of reference determined by the columns of $\Gamma$, i.e. we transform from x to $x^* = \Gamma' x$. It follows that

$$C = \int_{\Omega_3} \exp(\lambda' x + \lambda_4 x_2^2 + \lambda_5 x_3^2) d\sigma(x) \quad , \tag{2}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)' = k\Gamma\nu$, $\lambda_4 = \beta_2$ and $\lambda_5 \equiv \beta_3$. On expanding the exponential factor in (2) and integrating term-by-term, we see that

$$C = \sum_{i=0}^{\infty} \sum_{\underset{\sim}{j}} \left( \prod_{k=1}^{5} \frac{\lambda_k^{j_k}}{j_k!} \right) \int_{\Omega_3} x_1^{j_1} x_2^{j_2+2j_4} x_3^{j_3+2j_5} d\sigma(x) \quad , \tag{3}$$

where the second summation is over all quintuples $\underset{\sim}{j} = (j_1, \ldots, j_5)$ of non-negative integers such that $j_1 + \ldots + j_5 = i$. Denote the integral in (3) by $I(\underset{\sim}{j})$. It is easy to see that $I(j) = 0$ if any one of $j_1, j_2$ or $j_3$ is odd. Therefore, we may assume that $j_1, j_2$ and $j_3$ are all even, in which case

$$I(\underset{\sim}{j}) = \int_{\Omega_3} x_1^{2m_1} x_2^{2m_2} x_3^{2m_3} d\sigma(x) \quad , \tag{4}$$

where $2m_1 = j_1$, $2m_2 = j_2 + 2j_4$ and $2m_3 = j_3 + 2j_5$. To compute (4), we transform to polar coordinates: $x_1 = \cos\theta$, $x_2 = \sin\theta\cos\phi$, $x_3 = \sin\theta\sin\phi$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$. The Jacobian of the transformation is $\sin\theta$. Using the result

$$\int_0^{\pi/2} (\sin\theta)^\alpha (\cos\theta)^\beta d\theta = \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{\beta+1}{2})}{2\Gamma(\frac{\alpha+\beta}{2} + 1)} \quad , \tag{5}$$

valid for $\alpha > -1$, $\beta > -1$, we obtain

$$I(\underset{\sim}{j}) = \frac{2\Gamma(m_1+\frac{1}{2})\Gamma(m_2+\frac{1}{2})\Gamma(m_3+\frac{1}{2})}{\Gamma(m_1+m_2+m_3+\frac{3}{2})} \quad ,$$

from which it follows that

$$C = 2\sum_{i=0}^{\infty} \sum_{\underset{\sim}{j}}{}' \frac{\Gamma(\frac{j_1+1}{2})\Gamma(\frac{j_2+2j_4+1)}{2})\Gamma(\frac{j_3+2j_5+1}{2})}{\Gamma(\frac{j_1+j_2+j_3+2j_4+2j_5+3}{2}} \prod_{k=1}^{5} \frac{\lambda_k^{j_k}}{j_k!} , \tag{6}$$

where the prime on the second summation means that we sum over all $\underset{\sim}{j}$ with $j_1, j_2$ and $j_3$ even.

Some remarks are in order. It is evident that this method is applicable to any model whose density is proportional to an exponentiated polynomial on $\Omega_3$, or more generally on $\Omega_p$, the unit sphere in $\mathbb{R}^p$ , $p \geq 3$. Therefore, the norming constants for all the models introduced by Beran (1979) may be evaluated using the method given above.

For a given model, it may even be possible to modify our method and derive alternative expressions for the norming constant. In the case of the $FB_8$ distribution, we can reduce the number of multinomial indices in (6), at the expense of introducing Bessel functions, by proceeding as follows:

$$C = \int_{\Omega_3} \exp(\lambda_1 x_1) \ \exp(\lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_2^2 + \lambda_5 x_3^2) d\sigma(x)$$

$$= \sum_{i=0}^{\infty} \sum_{\underset{\sim}{j}} \prod_{k=2}^{5} \left( \frac{\lambda_k^{j_k}}{j_k!} \right) \int_{\Omega_3} \exp(\lambda_1 x_1) x_2^{j_2+2j_4} x_3^{j_3 \ +2j_5} d\sigma(x) , \tag{7}$$

where the second summation is now over all quartets $\underset{\sim}{j} = (j_2, j_3, j_4, j_5)$ of non-negative integers such that $j_2 + \ldots + j_5 = i$. Again transforming to polar coordinates, the integrals in (7) are either of the form (5), or of the form

$$\int_0^{\pi} \exp(\lambda_1 \cos\theta)(\sin\theta)^{2\nu} d\theta = 2^{\nu}\pi^{\frac{1}{2}}\lambda_1^{-\nu}\Gamma(\nu+\tfrac{1}{2})I_{\nu}(\lambda_1) \quad , \tag{8}$$

where $I_{\nu}(\cdot)$ is the modified Bessel function of order $\nu$. (cf. Mardia et al (1979), p. 430). Collecting terms, we obtain an alternative expression

$$C = \left(\frac{8\pi}{\lambda_1}\right)^{\frac{1}{2}} \sum_{i=0}^{\infty} \sum_{\underset{\sim}{j}}{}' \left(\frac{2}{\lambda_1}\right)^{\frac{1}{2}(j_2+j_3)+j_4+j_5} \Gamma\left(\frac{j_2+1}{2} + j_4\right)\Gamma\left(\frac{j_3+1}{2} + j_5\right)$$

$$\cdot I_{\frac{1}{2}(j_2+j_3)+j_4+j_5+\frac{1}{2}}(\lambda_1) \prod_{k=2}^{5} \frac{\lambda_k^{j_k}}{j_k!} \quad ,$$

where $\Sigma'$ denotes summation over all $\underset{\sim}{j}$ with $j_2$ and $j_3$ even.

To conclude, we consider another approach which is not as general as the previous methods, but which has considerable computational advantages. In a study of palaeomagnetic pole positions, Watson (1982) encounters a density proportional to

$$\exp(\mu'x-\tfrac{1}{2}x'\Sigma^{-1}x), \quad x \in \Omega_3 \ , \tag{9}$$

where $\mu \in \mathbf{R}^3$, and $\Sigma$ is a 3×3 positive-definite symmetric matrix. When $\mu = 0$, (9) is the Bingham distribution, and the norming constant has been evaluated by Bingham (1976). For arbitrary $\mu$, expressions for the norming constant can be derived from DeWaal (1979). In addition, (9) is a particular instance of the general models of Beran (1979), and hence the norming constant can be computed using the methods proposed here. Using the positive-definiteness of $\Sigma$, we can obtain alternative expressions of the constant by noting that

$$\int_{\Omega_3} \exp(\mu'x-\tfrac{1}{2}x'\Sigma^{-1}x)d\sigma(x) = \frac{\partial}{\partial t}\int_{\|x\|<t} \exp(\mu'x-\tfrac{1}{2}x'\Sigma^{-1}x)dx\Big|_{t=1}$$

$$= \exp(\tfrac{1}{2}\mu'\Sigma\mu)\frac{\partial}{\partial t}\int_{\|x\|<t} \exp(-\tfrac{1}{2}(x-\Sigma\mu)'\Sigma^{-1}(x-\Sigma\mu))dx\Big|_{t=1}$$

$$= |2\pi\Sigma|^{3/2}\exp(\tfrac{1}{2}\mu'\Sigma\mu)f(1) \ ,$$

where $f(\cdot)$ is the density function of $\|X\|$, $X$ having a trivariate normal distribution with mean vector $\Sigma\mu$ and covariance matrix $\Sigma$. To compute $f(1)$, we can apply the results of Kotz et al (1967), since $\|X\|^2$ has a distribution identical with that of a positive definite non-central quadratic form in normal variables. This approach leads to highly computable expressions for the norming constant.

Finally, we remark that our methods may be used to compute all moments for the distributions discussed here.

ACKNOWLEDGEMENT

We wish to thank Professor C. Bingham for bringing the problem to our attention, and for providing us with a copy of his unpublished manuscript.

## REFERENCES

BERAN, R. (1979). Exponential models for directional data. Ann. Statist., 7, 1162-1178.

BINGHAM, C. (1976). Expansions related to a hypergeometric function arising in an antipodally symmetric distribution on the sphere. Research report No. 11, Directional Data Analysis Project, Univ. of Leeds.

DEWAAL, D.J. (1979). On the normalizing constant for the Bingham-von Mises-Fisher matrix distribution. S. Afr. Statist. J., 13, 103-112.

KENT, J.T. (1982). The Fisher-Bingham distribution on the sphere. J. Roy. Statist. Soc. B, 44, 71-80.

KOTZ, S. JOHNSON, N.L., and BOYD, D.W. (1967). Series representations of distributions of quadratic forms in normal variables. II. Noncentral case. Ann. Math. Statist., 38, 838-848.

MARDIA, K.V., KENT, J.T. and BIBBY, J.M. (1979). Multivariate Analysis. London: Academic Press.

WATSON, G.S. (1982). The estimation of palaeomagnetic pole positions. In, Statistics and Probability: Essays in Honour of C.R. Rao (eds. Kallianpur, Krishnaiah and Ghosh), Amsterdam: North Holland, 703-712.