SEQUENTIAL NONPARAMETRIC AGE REPLACEMENT POLICIES[1]

by

Edward W. Frees[2]

and

David Ruppert

Department of Statistics

University of North Carolina

Chapel Hill, North Carolina 27514

Running Heading: Age replacement policies

AMS 1980 Subject Classification: Primary 90B25;
Secondary 62L20, 62L12.

Keywords and phrases: age replacement policy, stochastic approximation, adaptive control, kernel estimation.

---

[1] The research of both authors was supported in part by the National Science Foundation (Grant MCS-8100748).

[2] Present address: Department of Statistics, 1210 West Dayton Street, U. of Wisconsin, Madison, Wisconsin 53706

ABSTRACT

Under an age replacement policy, a stochastically failing unit is replaced at failure or after being in service for t units of time, whichever comes first. An important problem is the estimation of ϕ, the optimal replacement time (optimal in terms of achieving the smallest long-run expected cost), when the form of the failure distribution is unknown. It is shown that substantial cost savings can be effected by estimating ϕ sequentially. The sequential methodology employed here is stochastic approximation (SA). The SA estimator converges more rapidly to ϕ than the sequential methodology introduced by Bather (1977).

§1 Introduction and Summary

Consider a functioning unit with specified life distribution F, and probability of survival to age x $S(x)=1-F(x)$. Suppose F is absolutely continuous with probability density f. Let C_1 and C_2 be fixed, known costs with $C_1>C_2>0$. If the unit fails prior to t units of time after installation, it is replaced at that failure time with cost C_1. Otherwise, the unit is replaced t units of time after its installation with cost C_2. It is assumed that replacement is immediate. Under the age replacement policy, the replacement unit is available from a sequence of such units that fail independently with the same distribution function F. The objective is to minimize the long-run accumulation of costs in some sense. The cost function here is the expected long-run average cost,

$$(1.1) \quad R(t) = \{C_1F(t) + C_2S(t)\}/\int_0^t S(u)du.$$

(cf., Barlow and Proschan, 1965, page 87).

Under some fairly general conditions, there is a unique and finite time, say ϕ^*, where R(t) attains a minimum. An example of such a condition is that the failure rate $f(x)/S(x)$ be strictly increasing to infinity with x. See Bergman (1979) for other sufficient conditions. Call ϕ^* the optimal replacement time. We wish to estimate the parameter

ϕ^*.

Estimation of the optimal replacement time based on a fixed number of i.i.d. units with distribution F has been examined in detail. See Arunkumar (1972) for asymptotic results of an interesting nonparametric approach. Using Monte-Carlo simulation, Ingram and Scheaffer (1976) give finite sample results for several cases where the form of F is known up to one parameter. Bergman (1977) and Barlow (1978) discuss graphical methods for the estimation problem.

Because fixed sample procedures rely on i.i.d. observations, during experimentation units must be left in service until failure. Therefore, the experimenter is constrained from using an estimator to achieve cost savings as the estimation procedure continues. Bather (1977) introduced a procedure that can be used on an ongoing basis which constantly updates the estimators using current observations. Suppose $\{X_n\}$ is an i.i.d. sequence with distribution function F and $\{\phi_n^*\}$ is a sequence of random variables that estimates ϕ^*. Let N(t) be the number of failures by time t, i.e.,

(1.2) $$N(t) = \sum_{i=1}^{\infty} I\{\min(X_1,\phi_1^*)+\ldots+\min(X_i,\phi_i^*) < t\}$$

where $I(.)$ is the indicator function. Bather showed how to construct the estimators $\{\phi_n^*\}$ sequentially so that

(1.3) $$\phi_n^* \to \phi^* \quad \text{a.s. and}$$

$$(1.4)\quad t^{-1} \sum_{i=1}^{N(t)} \{C_1 I(X_i<\phi_i^*) + C_2 I(X_i \geq \phi_i^*)\} \to R(\phi) \quad \text{a.s.}$$

Thus, the estimators are strongly consistent, and further, the actual average cost achieved by the experimenter is the same asymptotically as if the optimal replacement time were known a priori.

This paper gives alternative methods for constructing the sequential estimators $\{\phi_n^*\}$. These estimators are easier to calculate and have more fully understood asymptotic properties than the estimators introduced by Bather. In §2 sufficient conditions are given for (1.3) to imply (1.4). In §3 a sequential procedure of the stochastic approximation (SA) type is given. Not only do the resulting estimators satisfy (1.3) (and hence (1.4)), but using well-known theorems on SA we are able to establish rates of convergence for the estimators. In §4 the choices of several parameters needed in the algorithm is studied using Monte-Carlo techniques. Results of this section show that the algorithm behaves satisfactorily from a cost standpoint even in small samples. §5 contains the proofs of §3 results.

§2 A preliminary result

Let X_i be the lifetime of the i^{th} unit. Suppose that $\{X_i\}$ is an i.i.d. sequence with distribution function F having mean μ and finite variance σ^2. Assume the experimenter has available at each stage a replacement time ϕ_i^*, so that if $X_i<\phi_i^*$, then the cost is C_1. Otherwise, the cost is C_2. Use $\{Z_i\}$ for the truncated observations, that is, $Z_i=\min(X_i,\phi_i^*)$. Thus, the cost for the first n units is

$$(2.1) \quad R_n = \sum_{i=1}^{n}\{C_1 \mathbf{I}(Z_i<\phi_i^*) + C_2\mathbf{I}(Z_i\geq\phi_i^*)\}.$$

Using ideas of Bather (1977, Theorem 3), the following result shows that if ϕ_n^* estimates ϕ^* consistently, the best asymptotic cost is achieved.

Theorem 2.1

Let $\{X_n\}$ and $\{Z_n\}$ be as above and let $\mathbf{G}_n=\sigma(Z_1,\ldots,Z_n)$ be the sigma-field generated by $Z_1,\ldots,Z_n$. Suppose there exists a sequence of random variables $\{\phi_n^*\}$ such that ϕ_n^* is $\mathbf{G}_{n-1}$-measurable for $n\geq2$ and (1.3) holds. Then, with R_n, $N(t)$ and $R(t)$ defined in (2.1), (1.3) and (1.1) respectively, we have

$$\lim_{t\to\infty} R_{N(t)}/t = R(\phi^*) \quad \text{a.s.}$$

i.e., that (1.4) holds.

The proof of Theorem 2.1 follows from the application of martingale convergence theorems and can be found in Frees (1983, Theorem 1.1). It is easy to see that we may drop the assumption that F be absolutely continuous for Theorem 2.1. We only need require that ϕ^* be a continuity point of F. Further, the assumption of a finite variance can easily be weakened. However, these stronger assumptions are sufficient for the results in §3.

§3 The sequential procedure and asymptotic results

We now introduce a recursive estimation procedure. As with other stochastic approximation algorithms, its simple form makes it amenable both to practical implementation and to large sample calculations.

Let g(.) be a known, strictly increasing, smooth function such that $g:\mathbf{R}\to[0,\infty)$ and define

$$(3.1)\quad M(t) = (C_1-C_2)\ f(t) \int_0^t S(u)du - S(t)\{C_1F(t)+C_2S(t)\}.$$

Now $\partial/\partial t\ R(t)=K_t\ M(t)$, where K_t is a positive function of t. Thus, by assuming that R(t) is uniquely minimized at some finite point ϕ^* we have that $M(t)(t-\phi^*)>0$ for each $t\neq\phi^*$. Instead of looking for a minimum of R(t), we wish to find the zero of M(t). Define ϕ by $\phi^*=g(\phi)$. Note that ϕ is the unique minimum of R(g(x)) and thus the unique, finite zero of g'(x)M(g(x)) (where x may vary over the entire real line). Since unconstrained recursive estimation is particularly simple, we have introduced g and will estimate ϕ rather than the strictly positive parameter ϕ^*.

Let $\{X_{i,n}\}$, i=1,2, be two sequences of i.i.d. random variables that are mutually independent, each having distribution function F. Let $\mathbf{E}$ and $\mathbf{P}$ denote expectation and probability with respect to F. Suppose ϕ_1 is a random variable such that $\mathbf{E}\ \phi_1^2 < \infty$, and $\{\phi_n\}$, $\{a_n\}$ and $\{c_n\}$ are sequences of random variables. For $i=1,2$ define the truncated observations $\{Z_{in}\}$ by $Z_{in}=\min\{X_{in},g(\phi_n+c_n)\}$. Let $\mathbf{F}_n = \sigma(\phi_1, Z_{ij}, i=1,2\ j=1,\ldots,n-1)$ and require that a_n and c_n be $\mathbf{F}_n$-measurable. In practice, we take $\{a_n\}$ and $\{c_n\}$ to be sequences such that for known $\gamma\in(0,1)$ we have $a_n n\to A$ and $c_n n^{\gamma}\to C$ (A and C are positive constants).

Let $\mathbf{B}_o$ be the class of all Borel-measurable real-valued functions k(.) where k(.) is bounded and equals zero outside [-1,1]. For some positive integer r define

$$\mathbf{M} = \{k\in\mathbf{B}_o\colon 1/j! \int_{-1}^{1} y^j k(y)dy = \begin{cases} 1 & j=0 \\ 0 & j=1,\ldots,r-1\end{cases}\}.$$

$\mathbf{M}$ is similar to a class of kernel functions used by Singh (1977). See, for example, Wertz (1978), for a broad review on using kernel functions to estimate a probability density function.

For $i=1,2$, let $F_{in}(t)=\mathbf{I}\{Z_{in}<t\}$, $S_{in}(t)=1-F_{in}(t)$ and $fg_n(t)=k[(g^{-1}(Z_{1n})-t)/c_n]/c_n$.

The estimator of $g'(t)M(g(t))$ is $M_{g,n}(t)$, where

$$M_{g,n}(t)=(C_1-C_2)fg_n(t)\int_o^{g(t)} S_{2n}(u)du - g'(t)S_{1n}(g(t))\{C_1F_{2n}(g(t))+C_2S_{2n}(g(t))\}. \tag{3.2}$$

The estimators ϕ_n are constructed by the recursive algorithm,

$$\phi_{n+1} = \phi_n - a_n M_{g,n}(\phi_n). \tag{3.3}$$

We define $(F*g)^{(r)}(x) = \partial^r/\partial t^r\, F(g(t))\big|_{t=x}$. For convenience, a list of the most important assumptions is collected below:

A1. The distribution function F of the i.i.d. observations is absolutely continuous with density f, has support on $[0,\infty)$, finite mean μ and variance σ^2.

A2. Let g be a known, strictly increasing function such that $g:\mathbf{R}\to[0,\infty)$ and the first r+1 derivatives exist and are bounded over the entire real line.

A3. For each $x\in\mathbf{R}$, $(x-\phi)M(g(x)) > 0 \quad \forall\ x \neq \phi$.

A4. $(F*g)^{(1)}(x)$ and $(F*g)^{(r+1)}(x)$ exist for each x, are bounded over the entire real line and are continuous in a neighborhood of ϕ.

A5. Let $\gamma\in(0,1)$ and $\lim c_n = 0$, $\sum_1^\infty a_n = \infty$, $\sum_1^\infty a_n c_n^r < \infty$, and $\sum_1^\infty a_n^2/c_n < \infty$.

A6. There exists p > 2 such that $\int_0^\infty t^p\, dF(t) < \infty$. Let q be defined by 2/p+1/q=1 and assume $1/q\geq 1/(2r+1)$.

A7. Let $\gamma=1/(2r+1)$. For some A,C > 0,
$a_n n\to A \quad c_n n^\gamma \to C\ n^{-\gamma}$ a.s. and $1-\gamma<2\Gamma$
where $\Gamma = A(g'(\phi))^2\, M'(g(\phi))$.

A8. $g^{(i)}(\phi) = 0 \quad i=2,\ldots,r+1$.

Remarks: A5 is a weaker condition than A7. A typical SA assumption, stronger than A3, that,

$$\inf\{|M(g(x))|: \epsilon < |x-\phi| < \epsilon^{-1}\} > 0 \text{ for each } \epsilon>0,$$

is not needed here due to the assumed continuity of M(.) and

$g(.)$. Assumption A8 requires that the transform function $g(.)$ behaves approximately like a line at ϕ. Under A8, the asymptotic distribution of the estimator of ϕ^* has a simple form, but otherwise we do not use this assumption.

Let $T = (F*g)^{(r+1)}(\phi) \int_{-1}^{1} y^r/r!\ k(y)dy$, which is a factor in the asymptotic bias. The asymptotic variance of the ϕ_n will be proportional to $\underline{\Sigma}$, where

$$\underline{\Sigma} = (C_1-C_2)^2\ (F*g)^{(1)}(\phi) \int_o^{g(\phi)} u\ S(u)du.$$

We now state some asymptotic properties of our procedure.

Theorem 3.1

Assume A1-A5. Then, for the procedure defined in (3.3),

(3.4) $\phi_n \rightarrow \phi$ a.s. and thus

(3.5) $\phi_n^* = g(\phi_n) \rightarrow \phi^*$ a.s.

Theorem 3.2

Assume A1-A4 and A6-A7. Then, for the procedure defined in (3.3),

(3.6) $n^{(1-\gamma)/2}(\phi_n-\phi) \rightarrow_D N(\mu_1,\sigma_1^2)$

where

$$\mu_1 = A\ C^r\ (C_1-C_2) \int_o^{g(\phi)} S(u)du\ T/(2\Gamma-1+\gamma)$$

$$\sigma_1^2 = A^2\ C^{-1}\ \underline{\Sigma} \int_{-1}^{1} k^2(y)dy/(2\Gamma-1+\gamma).$$

Corollary 3.3

Under the assumptions of Theorem 3.2 and with ϕ_n^* defined in (3.5),

$$(3.7)\quad n^{(1-\gamma)/2}(\phi_n^* - \phi^*) \to_D N(\mu_2, \sigma_2^2)$$

where $\mu_2 = g'(\phi)\mu_1$ and $\sigma_2^2 = (g'(\phi))^2\sigma_1^2$.

Further, assuming A8, we have (3.7) with

$$(3.8)\quad \mu_2 = A\, C^r\, (C_1 - C_2) \int_o^{\phi^*} S(u)du\ f^{(r)}(\phi^*) \int_{-1}^{1} y^r/r!\ k(y)dy/(2\,\Gamma - 1 + \gamma)$$

$$(3.9)\quad \sigma_2^2 = A^2 C^{-1}(C_1 - C_2)^2 \int_o^{\phi^*} u\ S(u)du\ f(\phi^*) \int_{-1}^{1} k^2(y)dy/(2\,\Gamma - 1 + \gamma).$$

Theorem (3.1) tells us that we may use the estimators constructed in (3.3) to achieve the best long-run cost. With some additional mild assumptions, in Theorem 3.2 we can quantify the speed of the convergence of the estimators of ϕ. Using the well-known "δ-method", Corollary 3.3 gives rates of convergence of the estimators of the optimal replacement time, ϕ^*. Under the additional assumption A8, the asymptotic distribution depends only on the slope of the transformation function at ϕ. The parameters A and C may be chosen to be any positive constants, subject only to the restriction in assumption A7. One criterion for selection of parameters suggested by Abdelhamid (1973) is to choose A and C to minimize the asymptotic mean square error. Unfortunately, the best choice depends on knowledge of F and ϕ which are generally unknown a priori.

§4 Monte-Carlo Results

A Monte-Carlo study was undertaken to demonstrate the usefulness of the procedure proposed in §3 even in small samples. Further, the performance of the estimators at finite stages is improved dramatically by parameters that do not appear in the asymptotic theory.

For the model of the problem, we took $C_1=5$ and $C_2=1$. From (1.1), it can be seen that determining ϕ^* depends only on the ratio of the costs. The Weibull distribution was used with location and scale parameters 2 and 2.2, respectively, which produces a mean of 1.7712 and standard deviation of .8499 for the lifetime of the units. It also ensures a unique, finite ϕ^* (=.99505).

For the algorithm, let r=2 which gives $\gamma=.2$ and use $k(y)=1/2$ for $y\in[-1,1]$. Thus, we used a very simple histogram estimator for the density. (Slightly better kernel estimators for the density when r=2 are avaiable, see Epanechnikov (1969) and Rosenblatt (1971).) Instead of using the simpler $a_n=An^{-1}$ and $c_n=Cn^{-\gamma}$, we used $a_n=A(n+k_A)^{-1}$ and $c_n=C(n+k_C)^{-\gamma}$, where k_A and k_C are nonnegative constants. Taking k_A and k_C to be positive provided dramatic improvements in finite samples over the more traditional

$k_A=k_C=0$. This form was first suggested by Dvoretsky (1956), and it has been employed by Ruppert et al. (1984). We used the transform function $g(x)=\log\{1+\exp(x)\}$. Some easy calculations show that the values of A and C that minimize the asymptotic mean square error ($=\mu_2^2+\sigma_2^2$, given in (3.8) and (3.9)) are A=2.3 and C=1.5. In this study we took these values to be fixed. For more complete tables where A and C are allowed to vary, see Frees (1983).

Table 1 describes the performance of the estimator ϕ_n for various values of k_A, k_C and ϕ_1 (the starting value of the procedure). Stages at n= 10, 50 and 250 were chosen to reflect small, moderate and large sample sizes, respectively. Denote X_{ijk} to be the i^{th} sample (i=1,2) at the j^{th} stage (j=10,50,250) from the k^{th} trial (k=1,...,1000). Let $\phi_{j,k}$ be the resulting estimator, $Z_{ijk}=\min\{X_{ijk}, g(\phi_{j,k}+C(j+k_C)^{-\gamma})\}$ and $\delta_{ijk}=\mathbf{I}\{Z_{ijk}<g(\phi_{j,k}+C(j+k_C)^{-\gamma})\}$. For the bias at the j^{th} stage, use $\text{BIAS}_j = (.001)\sum_{k=1}^{1000}\phi_{j,k} - \phi$. Similarly, for the mean square error, use $\text{MSE}_j = (.001)\sum_{k=1}^{1000}(\phi_{j,k}-\phi)^2$. For the k^{th} trial, the actual sample cost per unit time at the n^{th} stage is

$$SC_{j,k} = \sum_{n=1}^{j}\{C_1(\delta_{1nk}+\delta_{2nk}) + C_2(2-\delta_{1nk}-\delta_{2nk})\} \Big/ \sum_{n=1}^{j}(Z_{1nk}+Z_{2nk}).$$

The mean sample cost per unit time at the j^{th} stage is $\text{MSC}_j = (.001)\sum_{k=1}^{1000} SC_{k,j}$. While the asymptotic theory (Theorem 3.2) indicates $n^{.8}\text{MSE}_n=O_p(1)$, we found that an adjusted standardized mean square error $\text{ASMSE}_n=(n+k_A)^{.8}\text{MSE}_n$

(also $O_p(1)$) was more stable. Heuristically, in replacing An^{-1} with $A(n+k_A)^{-1}$, the procedure believes it is at the $(n+k_A)^{th}$ stage when only n iterations have been performed.

[TABLE 1 INSERTED HERE]

The results of the study indicate that the performance of the algorithm was greatly enhanced by the introduction of the parameter k_A and only somewhat by k_C. By (3.3), it can be seen that ϕ_n could fluctuate wildly for small n as compared to larger n. The introduction of positive k_A inhibits the fluctuation in finite samples without altering the asymptotic properties.

A practical upper bound to the asymptotic cost is a failure replacement policy, i.e., where the unit is never replaced prior to failure. The cost of this policy is easily seen from (1.1) by setting $t=\infty$. For our example, $R(\infty)=C_1/\mu=2.823$. In each trial we achieved a lower expected cost, even by the tenth stage! The reduction was substantial in view of the fact that the best one could hope for is $R(\phi^*)=1.904$.

In this example, since $\phi^*=.99505$ and $g(x)=\log\{1+\exp(x)\}$, simple calculations show $\phi=.53349$. With a standard deviation of .8499, $\phi_1=1$ is not an unreasonable starting value for the algorithm. As is usual in SA schemes, starting far away from the optimal value will affect the bias and mean square error even for large n (=250). One happy note is that this adverse effect does not seem too severe on the expected cost. In fact, we seem to do even better by starting with a low starting value ($\phi_1=-1$), an important practical point (but note that g(-1)=.3133, not so far from $\phi^*=.99505$).

§5 Appendix

In this section, we first prove Theorem 3.1 and then Theorem 3.2. All relationships between random variables are meant to hold almost surely unless stated otherwise. We will use positive constants $K_1, K_2, \ldots$ in the inequalities. All random variables are defined on a fixed probability space $(\Omega, \mathbf{F}, \mathbf{P})$. We begin by stating a martingale convergence result due to Robbins and Siegmund.

Theorem 5.1 (Robbins-Siegmund, 1971, Theorem 1)

Let $\mathbf{G}_n$ be a nondecreasing sequence of sub σ-fields of $\mathbf{F}$. Suppose that X_n, β_n, η_n and ζ_n are nonnegative $\mathbf{G}_n$-measurable random variables such that

$$\mathbf{E}_{\mathbf{G}_n} X_{n+1} \leq X_n(1 + \beta_n) + \eta_n - \zeta_n \quad \text{for } n = 1,2,\ldots$$

Then, $\lim_{n\to\infty} X_n$ exists and is finite and

$$\sum \zeta_n < \infty \text{ on } \{\sum \beta_n < \infty, \ \sum \eta_n < \infty\}.$$

Some additional notation will be useful. Define

$$(5.1)\quad \triangle_n = \mathbf{E}_{\mathbf{F}_n}\{M_{g,n}(\phi_n) - g'(\phi_n)M(g(\phi_n))\}$$
$$= (C_1-C_2)\int_o^{g(\phi_n)} S(u)du\ \{\mathbf{E}_{\mathbf{F}_n} fg_n(\phi_n)-(F*g)^{(1)}(\phi_n)\}$$

$$(5.2)\quad V_n = (c_n)^{\frac{1}{2}}\{M_{g,n}(\phi_n) - g'(\phi_n)M(g(\phi_n)) - \triangle_n\}.$$

A useful lemma which we use repeatedly is

Lemma 5.2

Assume A1, A2 and A4. Then, for $\mathbf{F}_n$-measurable $x\leq g(\phi_n+c_n)$,

$$(5.3)\quad \mathbf{E}_{\mathbf{F}_n} fg_n(x) = (F*g)^{(1)}(x) + c_n^r\int_{-1}^{1} y^r/r!\ k(y)(F*g)^{(r+1)}(\eta_n(y))dy$$

where $|\eta_n(y)-x|\leq c_n$.

Proof:

By a change of variables,

$$\mathbf{E}_{\mathbf{F}_n} fg_n(x) = \int k\{(g^{-1}(s)-x)/c_n\}/c_n\ f(s)ds$$
$$= \int_{-1}^{1} k(y)\ g'(x+c_ny)\ f(g(x+c_ny))dy$$
$$= \int_{-1}^{1} k(y)\ (F*g)^{(1)}(x+c_ny)dy.$$

The result follows from a Taylor-series expansion and since $k\in\mathbf{M}$. □

Proof of Theorem 3.1

Using (5.2) in (3.3) gives,

$$(5.4)\quad \phi_{n+1} = \phi_n - a_n[g'(\phi_n)M(g(\phi_n))+c_n^{-\frac{1}{2}}V_n+\triangle_n].$$

Subtracting ϕ, squaring and taking conditional expectations with respect to $\mathbf{F}_n$ gives,

$$(5.5)\quad \mathbf{E}_{F_n}(\phi_{n+1}-\phi)^2 = (\phi_n-\phi)^2 - 2a_n(\phi_n-\phi)[g'(\phi_n)M(g(\phi_n))+\triangle_n] + a_n^2\,[(g'(\phi_n)M(g(\phi_n)) + \triangle_n)^2 + c_n^{-1}\mathbf{E}_{F_n}V_n^2].$$

Let $h_{n,1}=2\,a_n|\triangle_n|$ and $h_{n,2}=a_n^2\,c_n^{-1}\mathbf{E}_{F_n}V_n^2$. Suppose

$$(5.6)\quad \sum h_{n,1} < \infty \quad \text{a.s.}$$

$$(5.7)\quad \sum h_{n,2} < \infty \quad \text{a.s.}$$

From A1, $\int_0^t S(u)du \leq \int_0^\infty S(u)du = \mu < \infty$. Thus, by A2 and A4, $|g'(t)M(g(t))|$ is bounded, say, by K_1. Using the inequality $x \leq 1+x^2$ and (5.5)-(5.7), we get

$$(5.8)\quad \mathbf{E}_{F_n}(\phi_{n+1}-\phi)^2 \leq (\phi_n-\phi)^2(1+h_{n,1})-2a_n(\phi_n-\phi)g'(\phi_n)M(g(\phi_n)) + h_{n,1} + \tfrac{1}{2}h_{n,1}^2 + 2K_1^2a_n^2 + h_{n,2}.$$

By Theorem 5.1 and A5, we get that $\lim_{n\to\infty} \phi_n-\phi = X$ a.s. and $\sum a_n(\phi_n-\phi)g'(\phi_n)M(g(\phi_n)) < \infty$ a.s. This and A3 give the result. We need only show (5.6) and (5.7). By (5.1), Lemma 5.2 and A4, we have

$$(5.9)\quad \triangle_n = O(c_n^r).$$

This and A5 prove (5.6). From (3.2), for some $K_2, K_3 \geq 0$,

$$(5.10)\quad \mathbf{E}_{F_n}(M_{g,n}(\phi_n))^2 \leq K_2 + K_3\,\mathbf{E}_{F_n}(fg_n(\phi_n))^2.$$

As in Lemma 5.2, we can show $\mathbf{E}_{F_n}(fg_n(\phi_n))^2=O(c_n^{-1})$. This, the boundedness of $g'(t)M(g(t))$ and (5.9) prove (5.7) and hence the result.□

To prove Theorem 3.2, we use a special case of a theorem due to V. Fabian.

Theorem 5.3 (Fabian, 1968, Theorem 2.2)

Suppose $\mathbf{G}_n$ is a nondecreasing sequence of sub σ-fields of $\mathbf{F}$. Suppose U_n, V_n, T_n, Γ_n, and Φ_n are random variables such that Γ_n, Φ_{n-1}, V_{n-1} are $\mathbf{G}_n$-measurable. Let α, β, T', Σ, Γ, and Φ be real constants with $\Gamma > 0$ such that

(5.11) $\Gamma_n \to \Gamma$, $\Phi_n \to \Phi$, $T_n \to T'$ or $\mathbf{E}|T_n - T'| \to 0$, $\mathbf{E}_{\mathbf{G}_n} V_n = 0$

and

(5.12) $C > \left| \mathbf{E}_{\mathbf{G}_n} V_n^2 - \Sigma \right| \to 0.$

Suppose, with $\sigma_{j,r}^2 = \mathbf{E}\ \mathbf{I}[V_j^2 \geq r j^{\alpha}]\ V_j^2$, that

(5.13) $\lim n^{-1} \sum_1^n \sigma_{j,r}^2 = 0\ \forall\ r.$

Let $0 \leq \beta < 2\Gamma$ and

(5.14) $U_{n+1} = U_n[1 - n^{-1}\Gamma_n] - n^{-(1+\beta)/2}\ \Phi_n V_n + n^{-1-\beta/2}\ T_n.$

Then, $n^{\beta/2}\ U_n \to_D N(T'/(\Gamma - \beta/2),\ \Sigma\ \Phi^2/(2\Gamma - \beta)).$

Proof of Theorem 3.2:

By a Taylor-series expansion, $g'(\phi_n)M(g(\phi_n)) = (\phi_n - \phi)\ \{g''(\eta_n)M(g(\eta_n)) + (g'(\eta_n))^2 M'(g(\eta_n))\}$ for some η_n such that $|\eta_n - \phi| \leq |\phi_n - \phi|$. This and (5.4) give

(5.15) $\phi_{n+1} - \phi = (\phi_n - \phi)(1 - n^{-1}\Gamma_n) + n^{-1+\gamma/2}\ \Phi_n V_n + n^{-3/2+\gamma/2}\ T_n$

where $\Gamma_n = a_n n\{g''(\eta_n)M(g(\eta_n)) + (g'(\eta_n))^2 M'(g(\eta_n))\}$

$\Phi_n = a_n c_n^{-\frac{1}{2}} n^{1-\gamma/2}$ $\quad T_n = a_n n^{3/2-\gamma/2}\ \Delta_n.$

By Theorem 3.1 and A7, $\Gamma_n \to \Gamma$. By A7, $\Phi_n \to \Phi = A\ C^{-\frac{1}{2}}$. By Theorem 3.1, Lemma 5.2 and (5.1),

(5.16) $T_n \to T\ \ A\ C^r(C_1 - C_2) \int_0^{g(\phi)} S(u)du = T'.$

Since $\mathbf{E}_{\mathbf{F}_n} V_n = 0$ by the definition of Δ_n, we have (5.11).

To prove (5.12), we first recall the boundedness of $g'(t)M(g(t))$ and (5.9). Thus, we need only show for some K_4,

(5.17) $K_4 > \left| c_n \, \mathbf{E}_{\mathbf{F}_n}\{M_{g,n}(\phi_n)\}^2 - \Sigma \right| \to 0.$

From (3.2), A2 and (5.17), we need only show for some K_5,

(5.18) $$K_5 > \left| (C_1 - C_2)^2 c_n \, \mathbf{E}_{\mathbf{F}_n}\{fg_n(\phi_n) \int_0^{g(\phi_n)} S_{2n}(u)du\}^2 - \Sigma \right| \to 0.$$

By construction, $fg_n(.)$ and $S_{2n}(.)$ are conditionally independent given $\mathbf{F}_n$. From Theorem 3.1, it is easy to show that

(5.19) $$\infty > \mathbf{E}_{\mathbf{F}_n}\left(\int_0^{g(\phi_n)} S_{2n}(u)du\right)^2 \to 2 \int_0^{g(\phi)} u\, S(u)du \text{ a.s.}$$

Further,

$c_n \, \mathbf{E}_{\mathbf{F}_n}(fg_n(\phi_n))^2 = \frac{1}{2} \int_{-1}^{1} k^2(y)(F^*g)^{(1)}(\phi_n + c_n y)dy$ is bounded. This gives the boundedness of $\mathbf{E}_{\mathbf{F}_n} V_n^2$ and with (5.19) proves (5.18) and hence (5.12).

To prove (5.13), and hence the result, we need only show

(5.20) $\sigma^2_{n,r} = \mathbf{E}[V_n^2 \, \mathbf{I}[V_n^2 \geq rn]] \to 0$ for each r.

Suppose that for the p in A6

(5.21) $\mathbf{E}\,(c_n V_n^2)^{p/2} = o(1)$ as $n \to \infty$.

Then, by Holder's and Markov's inequalities, we get

$$\sigma^2_{n,r} \le (\mathbb{P}\{V_n^2 \ge rn\})^{1/q} \, (\mathbb{E}\{(c_n V_n^2)^{p/2}\})^{2/p}/c_n$$

$$\le (\mathbb{E}\{c_n V_n^2\}^{p/2}/\{r\ n\ c_n\}^{p/2})^{1/q} \ o(1)/c_n$$

$$= o(1)\ (r\ n)^{-p/(2q)}\ (c_n)^{-p/(2q)-1}$$

$$= o(1)\ n^{-p/(2q)+\gamma(1+p/(2q))} = o(1).$$

Thus, to show (5.20), we need only prove (5.21). Recall the algebraic inequality for nonnegative constants a, b, c and d $(a+b+c)^d \le 3^d(a^d+b^d+c^d)$. From (5.2),

(5.22) $$(c_n V_n^2)^{p/2} \le 3^p c_n^p \{|M_{g,n}(\phi_n)|^p + |g'(\phi_n) M(g(\phi_n))|^p + |\triangle_n|^p\}.$$

As before, both $g'(\phi_n)M(g(\phi_n))$ and $\triangle_n$ are bounded. From (3.2), so is $c_n\ |M_{g,n}(\phi_n)|$. Further, from (5.22),

(5.23) $$\mathbb{E}_{F_n}(c_n V_n^2)^{p/2} \le \mathbb{E}_{F_n}\left|c_n\ M_{g,n}(\phi_n)\right|^p + o(1).$$

Since $\mathbb{E}_{F_n}|c_n\ fg_n(\phi_n)|^p = o(1)$, from (5.23) and (3.2) we have

(5.24) $$\mathbb{E}_{F_n}(c_n\ V_n^2)^{p/2} = o(1).$$

(5.21) follows immediately from the Bounded Convergence Theorem. □

REFERENCES

Abdelhamid, S.N. (1973). Transformation of observations in stochastic approximation. Ann. Statist. 1, 1158-1174.

Arunkumar, S. (1972). Nonparametric age replacement policy. Sankhya A 34, 251-256.

Barlow, R.E. (1978). Analysis of retrospective failure data using computer graphics. In Proceedings 1978 Annual Reliability and Maintainability Symposium, Philadelphia.

Barlow, R.E. and Proschan, F. (1965). Mathematical Theory of Reliability. Wiley, New York.

Bather, J.A. (1977). On the sequential construction of an optimal age replacement policy. Bull. Int. Statist. Instit. 47, 253-266.

Bergman, B. (1977). Some graphical methods for maintenance planning. In Proceedings 1977 Annual Reliability and Maintainability Symposium, Philadelphia.

Bergman, B. (1979). On age replacement and the total time on test concept. Scand. J. Statist. 6, 161-168.

Dvoretsky, A. (1956). On stochastic approximation. Proc. Third Berkeley Symp. Math. Statist. Probab., 1 (J. Neyman, ed.), 39-55. Univ. California Press.

Epanechnikov, V. A. (1969). Nonparametric estimation of multivariate probability density. Theory Probab. Appl. 14, 153-158.

Fabian, V. (1968). On asymptotic normality in stochastic approximation. Ann. Math. Statist. 39, 1327-1332.

Frees, E. W. (1983). On Construction of Sequential Age Replacement Policies via Stochastic Approximation. Ph.D. Dissertation, Department of Statistics, University of North Carolina. Chapel Hill, North Carolina.

Ingram, C. R. and Scheaffer, R. L. (1976). On consistent estimation of age replacement intervals. Technometrics 18, 213-219.

Robbins, H. and Siegmund, D. (1971). A convergence theorem for nonnegative almost supermartingales and some applications. In: J. S. Rustagi, Ed., Optimizing Methods in Statistics, 233-257. Academic Press, N.Y.

Rosenblatt, M. (1971). Curve estimates. Ann. Math. Statist. 42, 1815-1842.

Ruppert, D., Reish, R.L., Deriso, R.B., and Carroll, R.J. (1984). Optimization using stochastic approximation and Monte Carlo simulation (with application to harvesting of Atlantic menhaden). (To appear in Biometrics).

Singh, R.S. (1977). Improvement on some known nonparametric uniformly consistent estimators of derivatives of a density. Ann. Statist. 5, 394-399.

Wertz, W. (1978). Statistical Density Estimation. A Survey. Vendenhoeck & Ruprecht, Gottingen.

TABLE 1 - PERFORMANCE OF ESTIMATORS

	ϕ_1	k_A	k_C	Stage of Algorithm			
				10	50	250	∞
$BIAS_n$	1.0	50	50	.3544	.1284	.0113	0
MSE_n				.2016	.1091	.0377	0
$ASMSE_n$				5.335	4.344	3.617	.8161
MSC_n				2.268	2.159	2.053	1.904
$BIAS_n$	1.0	0	0	-1.205	-1.248	-1.230	
MSE_n				40.03	39.24	38.68	
$ASMSE_n$				252.6	897.3	3204.	
MSC_n				4.767	13.09	48.47	
$BIAS_n$	1.0	0	50	-1.109	-1.112	-1.069	
MSE_n				34.89	34.16	33.56	
$ASMSE_n$				220.1	781.1	2781.	
MSC_n				4.900	13.18	43.44	
$BIAS_n$	1.0	50	0	.3896	.1659	.0247	
MSE_n				.2030	.1082	.0374	
$ASMSE_n$				5.371	4.307	3.584	
MSC_n				2.472	2.261	2.091	
$BIAS_n$	2.5	50	50	1.582	.7228	.1177	
MSE_n				2.640	.7360	.0649	
$ASMSE_n$				69.84	29.30	6.218	
MSC_n				2.745	2.522	2.210	
$BIAS_n$	-1.0	50	50	-1.434	-1.132	-.4743	
MSE_n				2.058	1.289	.2473	
$ASMSE_n$				54.44	1.289	23.71	
MSC_n				2.256	2.136	1.970	
$BIAS_n$	-2.0	50	50	-2.484	-2.333	-1.928	
MSE_n				6.171	5.443	3.718	
$ASMSE_n$				163.2	216.7	356.4	
MSC_n				4.334	4.169	3.648	