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SELBERG POLYNOMIALS

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ABSTRACT

In the course of proving an important multivariate beta-type integral formula, Selberg (Norsk. Mat. Tidsskr. 26 (1944), 71-78) utilized the following properties of the discriminant polynomial $p(t_1,\ldots,t_n) = \prod_{i<j} (t_i - t_j)^2$ in n real variables $t_1,\ldots,t_n$:

(1) $p(t_1,\ldots,t_n)$ is homogeneous of some (even) degree k

(2) $p(t_1,\ldots,t_n) = (t_1 t_2 \ldots t_n)^{\ell}\, p(t_1^{-1},\ldots,t_n^{-1})$ for some integer ℓ,

(3) $p(t_1,\ldots,t_n)$ is symmetric in $t_1,\ldots,t_n$, and (4) $p(t_1,\ldots,t_n) = p(1-t_1,\ldots,1-t_n)$. An arbitrary polynomial $p(t_1,\ldots,t_n)$ which satisfies (1)-(4) for compatible n, ℓ and k will be called a *Selberg polynomial of type* (n,ℓ,k). In this paper, we obtain a complete description of the Selberg polynomials.

Section 1: Introduction

In [8], A. Selberg evaluated an important multivariate beta-type integral involving the discriminant polynomial

$$\Delta(x_1,\dots,x_n) = \prod_{1\le i<j\le n} (x_i-x_j)^2.$$

He proved the

Theorem (1.1): Let r,s,z be complex numbers with Re $r > 0$, Re $s > 0$ and Re $z > \max\{-\frac{1}{n}, -\mathrm{Re}\,\frac{r}{n-1}, -\mathrm{Re}\,\frac{s}{n-1}\}$. Then

$$\int_0^1\cdots\int_0^1 \Delta(x_1,\dots,x_n)^z \prod_{i=1}^n x_i^{r-1}(1-x_i)^{s-1}dx_i$$

$$= \prod_{j=1}^n \frac{\Gamma(r+(j-1)z)\Gamma(s+(j-1)z)\Gamma(jz+1)}{\Gamma(r+s+(n+j-2)z)\Gamma(z+1)}.$$

This result has been shown to include, as a limiting case, the Mehta-Dyson conjecture [5; p. 42]: if Re $z > -\frac{1}{n}$, then

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \Delta(x_1,\dots,x_n)^z \prod_{i=1}^n \exp(-\tfrac{1}{2}t_i^2)dt_i$$

$$= \prod_{j=1}^n \frac{\Gamma(jz+1)}{\Gamma(z+1)}.$$

Recently, Andrews [1], Askey [2], [3], Macdonald [4], Morris [6] and others have related Theorem (1.1) such topics as basic hypergeometric series, orthogonal polynomials, the Dyson conjecture and the root systems of finite reflection groups.

Selberg's ingenious proof of Theorem (1.1) (see also [6], [7]) utilized the following properties of the polynomial $p(\underset{\sim}{x}) = \Delta(\underset{\sim}{x})$ (we write $\underset{\sim}{x} = (x_1,\ldots,x_n)$):

(1.1) $p(\underset{\sim}{x})$ is *homogeneous*: for some nonnegative integer k,

$$p(t\,\underset{\sim}{x}) = t^k p(\underset{\sim}{x})$$

for any real number t.

(1.2) $p(\underset{\sim}{x})$ is *translational*

$$p(\underset{\sim}{x}+\underset{\sim}{1}) = p(\underset{\sim}{x})$$

where $\underset{\sim}{1} = (1,1,\ldots,1)$.

(1.3) $p(\underset{\sim}{x})$ is *ℓ-reciprocal:* for some nonnegative integer ℓ,

$$p(\underset{\sim}{x}) = (x_1x_2\ldots x_n)^{\ell} p(\underset{\sim}{x}^{-1})$$

where $\underset{\sim}{x}^{-1} = (x_1^{-1},\ldots,x_n^{-1})$.

(1.4) $p(\underset{\sim}{x})$ is *symmetric*: for any permutation σ of $\{1,2,\ldots,n\}$,

$$p(\sigma\underset{\sim}{x}) = p(\underset{\sim}{x})$$

where $\sigma\underset{\sim}{x} = (x_{\sigma(1)},\ldots,x_{\sigma(n)})$.

It may be easily checked that $\Delta(\underset{\sim}{x})$ satisfies these four conditions with $k = n(n+1)/2$ and $\ell = n+1$. As a natural generalization, we shall refer to any nontrivial polynomial satisfying (1.1)-(1.4) for compatible n, ℓ and k as a *Selberg polynomial of type* (n,ℓ,k). It is easy to check that the Selberg polynomials of type (n,ℓ,k) form a vector space; we denote it by $S_k^{\ell}(n)$.

Our aim in this paper is three-fold. We intend to

(i) characterize the Selberg polynomials;

(ii) determine the dimension of the vector space $S_k^{\ell}(n)$ (as well as the dimensions of some larger spaces consisting of polynomials typified by a subset of the properties (1.1)-(1.4);

(iii) indicate how the Selberg polynomials can be constructed.

Before stating the main results, we make several remarks. First, property (1.2) is not the one used by Selberg. Instead he required

(1.2') $\quad p(\underset{\sim}{1} - \underset{\sim}{x}) = p(\underset{\sim}{x})$.

However, (1.2') is more restrictive (in conjunction with (1.1)) than (1.2), as we shall show in Section 2. Further, we shall also consider polynomials which satisfy the more general reciprocal property:

$(1.3)^{\pm}$ $\quad p(\underset{\sim}{x})$ is $\pm\ell$-*reciprocal*, that is,

$$p(\underset{\sim}{x}) = \pm(x_1 x_2 \ldots x_n)^{\ell} p(\underset{\sim}{x}^{-1}).$$

The corresponding vector spaces are denoted by $S_k^{\pm\ell}(n)$.

One of the main results is that, in some sense, (1.2) and (1.3) are not both needed here. Theorem (3.3) proves that, aside from certain obvious necessary conditions, homogeneous polynomials are translational if and only if they are reciprocal. This result requires some detailed machinery; we shall first show that a polynomial $p(\underset{\sim}{x})$ is translational if and only if $\partial p(\underset{\sim}{x}) = 0$, where $\partial = \sum_{1}^{n} \partial/\partial x_i$. Then, Theorem (3.3) follows from a detailed analysis of the action of powers of the operator ∂ on certain vector spaces of polynomials.

Our main result evaluates the dimension of $S_k^\ell(n)$; if par(n,ℓ,k) denotes the number of ordered partitions (Appendix 1) of k into at most n parts with no part exceeding ℓ, then we prove in Section 4 that when $2k = n\ell$,

$$\dim S_k^\ell(n) = \text{par}(n,\ell,k) - \text{par}(n,\ell,k-1)$$

and that $S_k^\ell(n) = (0)$ if $2k \neq n\ell$. To obtain this result we set up certain "tactical decompositions", of $S_k^\ell(n)$, which are partly motivated by the operator ∂.

We should also remark on some implications of the dimension formula. First, we clearly have $\dim S_k^\ell(2) = 1$, so that $p(x_1,x_2) = (x_1-x_2)^2$ is the only linearly independent Selberg polynomial. For $n \geq 4$ however, $\dim S_k^\ell(n) > 1$ and we are naturally led to surmise that for increasingly large n, there exists a plethora of integral formulas similar to Theorem (1.1). This is indeed the case as is shown in [7], and we expect that in time, these results will lead to hypergeometric integrals which are linearly independent of those in [2] and [3].

Finally, a word to the reader. We emphasize that the integers n, ℓ and k are fixed throughout. With this in mind, it is perhaps best if one peruses the appendices before reading Sections 3 and 4.

Section 2: Translational Polynomials and the Differential Operators A(i,j)

In this section we prove the main results concerning translational polynomials and their connection with the differential operators A(i,j). In particular, we give several characterizations of these polynomials. One of these characterizations forms the heart of an algorithm for computing a basis for the space of Selberg polynomials (see Appendix 3). In addition, we characterize the spaces $\mathcal{T}$, $\mathcal{T}_k$, $\mathcal{T}^\ell$, $\mathcal{T}_k^\ell$.

For completeness, we begin with the

Definition (2.1): The polynomial p in $\mathbb{F}[\underset{\sim}{x}]$ is *translational* if

$$(Tp)(\underset{\sim}{x}) = p(\underset{\sim}{x})$$

where T is the translation operator

$$(Tp)(\underset{\sim}{x}) = p(\underset{\sim}{x}+\underset{\sim}{1}).$$

The vector space of translational polynomials is denoted by $\mathcal{T}$.

The main tool that we use to characterize the translational polynomials is the

Lemma (2.2): Suppose $p \in \mathbb{F}[\underset{\sim}{x}]$. Then p is translational if and only if

$$p(\underset{\sim}{x}+t\,\underset{\sim}{1}) = p(\underset{\sim}{x}).$$

The conditional that p be translational is slightly more general (in conjunction with homogeneity) than the condition imposed by Selberg. He required that

$$p(\underset{\sim}{1}-\underset{\sim}{x}) = p(\underset{\sim}{x}).$$

In fact, we have the

Lemma (2.1a): Suppose p is k-homogeneous.

(i) If $p(\underset{\sim}{1}-\underset{\sim}{x}) = p(\underset{\sim}{x})$, then k is even; if $p(\underset{\sim}{1}-\underset{\sim}{x}) = -p(\underset{\sim}{x})$, then k is odd. In either case, p is translational.

(ii) If p is translational, then $p(\underset{\sim}{1}-\underset{\sim}{x}) = (-1)^k p(\underset{\sim}{x})$.

Proof: Let p be k-homogeneous.

(i) If $p(\underset{\sim}{1}-\underset{\sim}{x}) = \pm p(\underset{\sim}{x})$, then

$$p(\underset{\sim}{x}+\underset{\sim}{1}) = p(\underset{\sim}{1}+\underset{\sim}{x}) = \pm p(-\underset{\sim}{x}) = \pm(-1)^k p(\underset{\sim}{x}) .$$

Since the highest order terms of $p(\underset{\sim}{x}+\underset{\sim}{1})$ are the same as those of $p(\underset{\sim}{x})$, the result follows.

(ii) If p is translational, then

$$p(\underset{\sim}{1}-\underset{\sim}{x}) = (-1)^k p(\underset{\sim}{x}-\underset{\sim}{1}) = (-1)^k p(\underset{\sim}{x}).$$ QED

Proof of Lemma (2.2): Setting t=1 shows that the condition is sufficient. To prove it is necessary, assume that p is translational and consider the polynomial

$$q(t) \equiv p(\underset{\sim}{x}+t\,\underset{\sim}{1}) - p(\underset{\sim}{x}) .$$

Since p is translational, q(t)=0 for $t=0,\pm1,\pm2,\ldots$. But the only polynomial with infinitely many zeros is identically zero. Hence,

$$p(\underset{\sim}{x}+t\,\underset{\sim}{1}) = p(\underset{\sim}{x}) .$$ QED

With this Lemma we can prove our main characterization result.

Theorem (2.3): [Characterization of Translational Polynomials]

Suppose $p \in \mathbb{F}[\underset{\sim}{x}]$. Then the following statements are equivalent:

(i) p is translational

(ii) $p(\underset{\sim}{x}+t\,\underset{\sim}{1}) = p(\underset{\sim}{x})$

(iii) There is a polynomial q in $\mathbb{F}[x_1,\ldots,X_{n-1}]$ for which $p(\underset{\sim}{x}) = q(x_2-x_1,\ldots,x_n-x_1)$

(iv) Each p_k in the homogeneous decomposition of p is translational;

(v) $\partial p \equiv (\frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_n})p = 0$.

Proof: Suppose $p \in \mathbb{F}[\underset{\sim}{x}]$. Lemma 2.2 shows that (i)$\Leftrightarrow$(ii). Also, it is easy to see that (iii)$\Rightarrow$(i) and (iv)$\Rightarrow$(i). We will show (ii)$\Rightarrow$(iii)$\Rightarrow$(iv) and (ii)$\Leftrightarrow$(v).

So suppose (ii) is true. Set

$$q(y_1,\ldots, y_{n-1}) \equiv p(0,y_1,\ldots, y_{n-1}) .$$

Then

$$\begin{aligned} p(\underset{\sim}{x}) &= p(\underset{\sim}{x}-x_1 \underset{\sim}{1}) \\ &= p(0,x_2-x_1,\ldots, x_n-x_1) \\ &= q(x_2-x_1,\ldots, x_n-x_1) . \end{aligned}$$

So (ii) implies (iii). Now suppose (iii) is true. Let the homogeneous decomposition of q be $q = \sum_{k \geq 0} q_k$. Then

$$\begin{aligned} p(\underset{\sim}{x}) &= q(x_2-x_1,\ldots, x_n-x_1) \\ &= \sum_{k \geq 0} q_k(x_2-x_1,\ldots, x_n-x_1) \end{aligned}$$

is a homogeneous decomposition of p. Since the decomposition is unique, $p_k(\underset{\sim}{x}) = q_k(x_2-x_1,\ldots,x_n-x_1)$ for each k. Thus, each p_k is translational since it satisfies (iii).

To show the equivalence of (ii) and (v), let $q(t) \equiv p(\underset{\sim}{x}+t\underset{\sim}{1})$ and note that

$$\begin{aligned} q'(t) &= \frac{d}{dt} p(\underset{\sim}{x}+t\underset{\sim}{1}) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} p(\underset{\sim}{x}+t\underset{\sim}{1}) \\ &= \partial p(\underset{\sim}{x}+t\underset{\sim}{1}) . \end{aligned}$$

Thus, if (ii) is true, then

$$\partial p(\underset{\sim}{x}) = q'(0) = 0$$

since q is constant. If (v) is true, then

$$q'(t) = \partial p(\underset{\sim}{x}+t\,\underset{\sim}{1}) = 0;$$

so q is constant, and

$$p(\underset{\sim}{x}+t\,\underset{\sim}{1}) = q(t) = q(0) = p(\underset{\sim}{x}) . \qquad \text{QED}$$

We obtain immediately the

Theorem (2.4): [Characterization of $\mathcal{T}$, $\mathcal{T}_k$]

(i) $\mathcal{T}$ has the direct sum subspace decomposition

$$\mathcal{T} = \bigoplus_{k\geq 0} \mathcal{T}_k .$$

(ii) $\mathcal{T}$ is isomorphic to $\mathbb{F}[x_1,\ldots x_{n-1}]$.

(iii) $\mathcal{T}_k$ is isomorphic to $\mathbb{F}[x_1,\ldots,x_{n-1}]$. In particular dim $\mathcal{T}_k = \binom{n+k-2}{k-1}$.

Moreover, if $\phi: \mathbb{F}[\underset{\sim}{x}] \to \mathbb{F}[x_1,\ldots x_{n-1}]$ and

$$(\phi p)(y_1,\ldots,y_{n-1}) = p(0,y_1,\ldots,y_{n-1})$$

$$(\psi q)(x_1,\ldots,x_n) = q(x_2-x_1,\ldots,x_n-x_1),$$

then the restriction of ϕ to $\mathcal{T}$ (resp. $\mathcal{T}_k$) provides the isomorphism in (ii) (resp. (iii)), and the restriction of ψ to $\mathbb{F}[x_1,\ldots,x_{n-1}]$ (resp. $\mathbb{F}_k[x_1,\ldots,x_{n-1}]$) is its inverse.

Proof: (i) This follows immediately from the fact that p is translational if and only if each p_k in the homogeneous decomposition of p is translational.

(ii) This follows from the fact that p is translational if and only if $p=\psi q$ for some q in $\mathbb{F}[x_1,\ldots,x_{n-1}]$. We need also the fact that ψ is one-to-one. To show this, assume that $\psi q_1=\psi q_2$. Then

$$\begin{aligned} q_1(y_1,\ldots,y_{n-1}) &= (\psi q_1)(0,y_1,\ldots,y_{n-1}) \\ &= (\psi q_2)(0,y_1,\ldots,y_{n-1}) \\ &= q_2(y_1,\ldots,y_{n-1}) . \end{aligned}$$

So, $q_1=q_2$, and ψ is one-to-one. Moreover, the restriction of ϕ to $\mathcal{T}$ is its inverse since for q in $\mathbb{F}[x_1,\ldots,x_{n-1}]$

$$\begin{aligned} (\phi(\psi q))(y_1,\ldots,y_{n-1}) &= (\psi q)(0,y_1,\ldots,y_{n-1}) \\ &= q(y_1,\ldots,y_{n-1}), \end{aligned}$$

and for p in $\mathcal{T}$

$$\begin{aligned} (\psi(\phi p))(x_1,\ldots,x_n) &= (\phi p)(x_2-x_1,\ldots,x_n-x_1) \\ &= p(0,x_2-x_1,\ldots,x_n-x_1) \\ &= p(\underset{\sim}{x}-x_1\,\underset{\sim}{1}) \\ &= p(\underset{\sim}{x}) \end{aligned}$$

since $p\in\mathcal{T}$. QED

The characterizations of $\mathcal{T}^{\ell}$ and $\mathcal{T}_k^{\ell}$ are not quite so simple. If we require that the polynomial p in Theorem (2.3) be ℓ-bounded, then the five conditions of the Theorem are still equivalent. Moreover, the polynomial q in condition (iii) (which is given by $q=\ell(p)$) will also be ℓ-bounded. However, if q is ℓ-bounded, then $p=\psi(q)$ need not be. (Consider the example: $n=3$, $\ell=1$, $q(y_1,y_2)=y_1y_2$, $p(x_1,x_2,x_3)=(x_2-x_1)(x_3-x_1)$.) Therefore, ϕ is a one-to-one map from T^{ℓ} (resp. T_k^{ℓ}) into $\mathbb{F}^{\ell}[x_1,\ldots,x_{n-1}]$ (resp. $\mathbb{F}_k[x_1,\ldots,x_{n-1}]$), but it is not necessarily onto.

We must therefore use a different approach to characterize the spaces T^{ℓ}, T_k^{ℓ}. The approach is provided by condition (v) of Theorem (2.3) from which we see that T^{ℓ} (resp. T_k^{ℓ}) is the null space of the restriction of the operator ∂ to $\mathbb{F}^{\ell}[\underset{\sim}{x}]$ (resp. $\mathbb{F}_k^{\ell}[\underset{\sim}{x}]$). In addition, we see from condition (iv) that the characterization of T^{ℓ} reduces to that of T_k^{ℓ}. Therefore, the restrictions of the operator ∂ (and its powers) to the spaces $\mathbb{F}_k^{\ell}[\underset{\sim}{x}]$ as well as the null spaces of these operators will play a role in our characterization. Consequently, we make the

<u>Definition (2.5)</u>: Let i,j be integers with $i \leq j$.

(i) The map

$$A(i,j)\colon \mathbb{F}_j^{\ell}[\underset{\sim}{x}] \to \mathbb{F}_i^{\ell}[\underset{\sim}{x}]$$

denotes the restriction of the operator $(1/(j-i)!)\partial^{j-i}$ to $\mathbb{F}_j^{\ell}[\underset{\sim}{x}]$.

(ii) $c(i,j)$ denotes the range of $A(i,j)$.

(iii) $n(i,j)$ denotes the null spaces of $A(i,j)$.

A basis for $\mathbb{F}_j^{\ell}[\underset{\sim}{x}]$ is given by the monomials $\underset{\sim}{x}^{\underset{\sim}{u}}$ as $\underset{\sim}{u}$ ranges over P_j^{ℓ}. Therefore, the matrix representation of A(i,j) (which we also denote by A(i,j)) with respect to these bases has rows indexed by P_i^{ℓ} and columns indexed by P_j^{ℓ}. The actual matrix elements of A(i,j) are given in the

<u>Lemma (2.6)</u>: Let i,j be non-negative integers with $i \leq j$. If $\underset{\sim}{v} \in P_i^{\ell}$ and $\underset{\sim}{u} \in P_j^{\ell}$, then the $(\underset{\sim}{v},\underset{\sim}{u})$ elements of A(i,j) is

$$\binom{\underset{\sim}{u}}{\underset{\sim}{v}} \equiv \binom{u_1}{v_1}\cdots\binom{u_n}{v_n}.$$

Proof: We must determine the coefficient of $\underset{\sim}{x}^{\underset{\sim}{v}}$ in $(1/(j-i)!)\partial^{j-i}\underset{\sim}{x}^{\underset{\sim}{u}}$. Now,

$$\partial^s = (\partial_1+\ldots+\partial_n)^s = \sum_{\underset{\sim}{\alpha}\in P_s} \frac{s!}{\underset{\sim}{\alpha}!}\,\partial^{\underset{\sim}{\alpha}}$$

and

$$\partial^{\underset{\sim}{\alpha}}\underset{\sim}{x}^{\underset{\sim}{u}} = \begin{cases} \dfrac{\underset{\sim}{u}!}{(\underset{\sim}{u}-\underset{\sim}{\alpha})!}\,\underset{\sim}{x}^{\underset{\sim}{u}-\underset{\sim}{\alpha}}, & \text{if } \underset{\sim}{\alpha} \leq \underset{\sim}{u}, \\ 0, & \text{if } \alpha_k > u_k \text{ for some } k, \end{cases}$$

so that

$$\begin{aligned}(1/s!)\partial^s\underset{\sim}{x}^{\underset{\sim}{u}} &= \sum_{\substack{\underset{\sim}{\alpha}\in P_s\\ \underset{\sim}{\alpha}<\underset{\sim}{u}}} \frac{1}{\underset{\sim}{\alpha}!}\,\frac{\underset{\sim}{u}!}{(\underset{\sim}{u}-\underset{\sim}{\alpha})!}\,\underset{\sim}{x}^{\underset{\sim}{u}-\underset{\sim}{\alpha}} \\ &= \sum_{\underset{\sim}{\alpha}\in P_s^{\ell}} \binom{\underset{\sim}{u}}{\underset{\sim}{u}-\underset{\sim}{\alpha}}\,\underset{\sim}{x}^{\underset{\sim}{u}-\underset{\sim}{\alpha}} \\ &= \sum_{\underset{\sim}{\beta}\in P_{j-s}^{\ell}} \binom{\underset{\sim}{u}}{\underset{\sim}{\beta}}\,\underset{\sim}{x}^{\underset{\sim}{\beta}} \qquad (\underset{\sim}{\beta}=\underset{\sim}{u}-\underset{\sim}{\alpha}).\end{aligned}$$

Taking s=j-i yields the result. QED

Theorem (2.7): [Characterization of ℓ-bounded, k-homogeneous, translational polynomials] Let k be an integer satisfying $0 \leq k \leq n\ell$, and suppose $p \in \mathbb{F}_k^{\ell}[\underset{\sim}{x}]$, say $p = \sum_{\underset{\sim}{u}\in P_k^{\ell}} a_{\underset{\sim}{u}}\underset{\sim}{x}^{\underset{\sim}{u}}$. Then the following statements are equivalent:

(i) p is translational;

(ii) $\sum_{\underset{\sim}{u}\in P_k^{\ell}} a_{\underset{\sim}{u}}\underset{\sim}{x}^{\underset{\sim}{u}} = 0$ for each $\underset{\sim}{v} \in P_{k-1}^{\ell}$;

(iii) The vector $[a_{\underset{\sim}{u}}]$ of coefficients of p is in the null space of the matrix A(k-1,k).

Proof: This is an immediate consequence of the Lemma provided $k \geq 1$. For k=0, the space $\mathbb{F}_0^{\ell}[\underset{\sim}{x}]$ consists of the constant polynomials, and P_{-1}^{ℓ} is empty. In this case, condition (ii) holds vacuously and the Theorem reduces to the obvious result that all constant polynomials are translational. QED

Remark (2.8): (i) We do not consider $k > n\ell$ since the space $\mathbb{F}_k^\ell[\underset{\sim}{x}]$ contains only the zero polynomial in this case.

(ii) The matrix elements of $A(k-1,k)$ $(1 \leq k \leq n\,\ell)$ are easily determined. In fact, if $\underset{\sim}{v} \in P_{k-1}^\ell$ and $\underset{\sim}{u} \in P_k^\ell$, then

$$\binom{\underset{\sim}{u}}{\underset{\sim}{v}} = \begin{cases} u_j, & \text{if } \underset{\sim}{u} = \underset{\sim}{v} + \underset{\sim}{e}_j \ (1 \leq j \leq n) \\ 0, & \text{otherwise}. \end{cases}$$

[Here, $\underset{\sim}{e}_j$ is the standard basis vector all of whose components are 0 except for the jth component which is 1.] This fact along with condition (iii) of the Theorem may be used to develop an algorithm for constructing (a basis for) the Selberg polynomials.

Theorem (2.9): [Characterization of the Spaces T^ℓ, T_k^ℓ]

(i) T^ℓ has the direct sum subspace decomposition

$$T^\ell = \bigoplus_{0 \leq 2k \leq n\ell} T_k^\ell .$$

Moreover,

$$\dim T_k = |P_k^\ell|$$

where k is the largest integer satisfying $2k \leq n\ell$.

(ii) $T_k^\ell = N(k-1,k)$, the null space of $A(k-1,k)$. Moreover,

$$\dim T_k^\ell = \begin{cases} |P_k^\ell| - |P_{k-1}^\ell| & \text{if } 0 \leq 2k \leq n\ell \\ 0 & \text{if } 2k > n\ell . \end{cases}$$

The proof of this Theorem rests on the following result which will be proved in stages below.

Theorem (2.10): Let k be an integer.

(i) If $0 \leq 2k \leq n\,\ell$, then $A(k-1,k)$ is onto.

(ii) If $2k < n\,\ell$, then A(k-1,,) is one-to-one.

<u>Proof of Theorem (2.9)</u>: We prove (ii) first and then (i).

(ii) We conclude from Theorem (2.7) that $T_k^\ell = N(k-1, k)$. Thus, from Theorem (2.10) we obtain

$$\dim T_k^\ell = \dim N(k-1,k)$$

$$= \begin{cases} \dim \mathbb{F}_k^\ell[\underset{\sim}{x}] - \dim \mathbb{F}_{k-1}^\ell[\underset{\sim}{x}] & \text{if } 0 \le 2k \le n\ell \\ 0 & \text{if } 2k > n\ell \end{cases}$$

$$= \begin{cases} |P_k^\ell| - |P_{k-1}^\ell| & \text{if } 0 \le 2k \le n\ell \\ 0 & \text{if } 2k > n\ell . \end{cases}$$

(i) From condition (iv) of Theorem (2.3) we know that

$T = \underset{k}{\oplus} T_k^\ell$. Therefore, from (ii) we get $T = \underset{0 \le 2k \le n\ell}{\oplus} T_k^\ell$

and

$$\dim T = \sum_{0 \le 2k \le n\ell} \dim T_k^\ell$$

$$= \sum_{0 \le 2k \le n\ell} (|P_k^\ell| - |P_{k-1}^\ell|) \quad .$$

This sum telescopes to yield the result since $|P_{-1}^\ell| = 0$. QED

The remainder of this section is devoted to showing that A(k-1,k) is onto when $0 \le 2k \le n\ell$ and one-to-one when $2k > n\ell$. In fact, we will prove the following generalization of Theorem (2.10).

<u>Theorem (2.11)</u>: Let i,j be integers.

(i) If $0 \le \bar{j} \le i \le j \le n\ell$, then A(i,j) is one-to-one.

(ii) If $0 \le i \le j \le \bar{i} \le n\ell$, then A(i,j) is onto.

In particular, if $0 \le \bar{j} \le j \le n\ell$ then $A(\bar{j},j)$ is invertible. [Here, $\bar{i} \equiv n\ell - i$.]

Theorem (2.10) follows easily upon setting i=k-1, j=k. Moreover, each operator A(i,j) with $0 \leq i \leq j \leq n\ell$ is characterized in Theorem (2.11). The appearance of the reciprocal index $\overline{i} = n\ell - i$ stems from the use of the reciprocal operators R_k in the proof.

Definition (2.12): The reciprocal operator R: $\mathbb{F}^{\ell}[\underset{\sim}{x}] \to \mathbb{F}^{\ell}[\underset{\sim}{x}]$ is given by

$$(Rp)(\underset{\sim}{x}) \equiv (x_1 \ldots x_n)^{\ell}\; p(\frac{1}{x_1}, \ldots, \frac{1}{x_n}) .$$

For each integer k satisfying $0 \leq k \leq n\ell$, the map R_k: $\mathbb{F}^{\ell}_{k}[\underset{\sim}{x}] \to \mathbb{F}^{\ell}_{n\ell-k}[\underset{\sim}{x}]$ denotes the restriction of R to $\mathbb{F}^{\ell}_{k}$.

It is easily verified that each R_k is an isomorphism whose inverse is $R_{\overline{k}}$. Consequently, R is also an isomorphism as well as its own inverse. Finally, we will use an inner product on $\mathbb{F}^{\ell}[\underset{\sim}{x}]$ in which $R_{\overline{k}}$ is the adjoint of R_k and in which the monomials $\underset{\sim}{x}^{\underset{\sim}{u}}$ are orthogonal.

Definition (2.13): For $p,q \in \mathbb{F}^{\ell}[\underset{\sim}{x}]$, set

$$\langle p,q\rangle \equiv \frac{1}{\ell!} \sum_{\underset{\sim}{u} \in P^{\ell}} \partial^{\underset{\sim}{u}} p(\underset{\sim}{0})\; \partial^{\underset{\sim}{\ell}-\underset{\sim}{u}}(R_q)(\underset{\sim}{0}) .$$

Theorem (2.14): If $p(\underset{\sim}{x}) = \sum_{\underset{\sim}{u} \in P^{\ell}} a_{\underset{\sim}{u}} \underset{\sim}{x}^{\underset{\sim}{u}}$ and $q(\underset{\sim}{x}) = \sum_{\underset{\sim}{u} \in P^{\ell}} b_{\underset{\sim}{u}} \underset{\sim}{x}^{\underset{\sim}{u}}$, then

$$\langle p,q\rangle = \sum_{\underset{\sim}{u}} \binom{\underset{\sim}{\ell}}{\underset{\sim}{u}}^{-1} a_{\underset{\sim}{u}} b_{\underset{\sim}{u}} .$$

In particular, <,> is an inner product on $\mathbb{F}^{\ell}[\underset{\sim}{x}]$ in which monomials are orthogonal. In fact,

$$\langle \underset{\sim}{x}^{\underset{\sim}{u}}, \underset{\sim}{x}^{\underset{\sim}{v}}\rangle = \begin{cases} 0 & \text{if } \underset{\sim}{u} \neq \underset{\sim}{v} \\ \binom{\underset{\sim}{\ell}}{\underset{\sim}{u}}^{-1} & \text{if } \underset{\sim}{u} = \underset{\sim}{v} \end{cases} .$$

Proof: We need check only the final formula since <,> is bilinear. Now,

$$\partial^{\alpha} x^{u} = \begin{cases} \alpha!\binom{u}{\alpha} x^{u-\alpha}, & \text{if } \alpha \le u \\ 0, & \text{otherwise,} \end{cases}$$

and

$$R\, x^{v} = x^{\ell - v},$$

so that for any u, v, α in P^{ℓ} we obtain

$$\partial^{\alpha}(x^{u})\partial^{\ell-\alpha}(R\, x^{v})$$

$$= \begin{cases} \alpha!\binom{u}{\alpha}(\ell-\alpha)!\binom{\ell-v}{\ell-\alpha} x^{u-v} & \text{if } v \le \alpha \le u \\ 0 & \text{otherwise} \end{cases}$$

Thus, all terms in $\langle x^{u}, x^{v}\rangle$ are 0 unless $u=v$. In this case, the one non-zero term arises from $\alpha=u$, and we get

$$\langle x^{u}, x^{u}\rangle = \frac{1}{\ell!}\, u!(\ell-u)! = \binom{\ell}{u}^{-1}. \qquad \text{QED}$$

Theorem (2.15) (i): For $0 \le k \le n\ell$, $R_k^* = R_{\overline{k}} = R_k^{-1}$.

Proof: It suffices to show that

$$\langle R_k x^{u}, x^{v}\rangle = \langle x^{u}, R_{\overline{k}} x^{v}\rangle$$

for any $u \in P_k^{\ell}$, $v \in P_{\overline{k}}^{\ell}$. But

$$\langle R_k x^{u}, x^{v}\rangle = \langle x^{\ell-u}, x^{v}\rangle = \begin{cases} \binom{\ell}{u}^{-1} & \text{if } u+v=\ell \\ 0 & \text{otherwise} \end{cases}$$

and

$$\langle x^{u}, R_{\overline{k}} x^{v}\rangle = \langle x^{u}, x^{\ell-v}\rangle = \begin{cases} \binom{\ell}{u}^{-1} & \text{if } u+v=\ell \\ 0 & \text{otherwise.} \end{cases} \qquad \text{QED}$$

Theorem (2.15) (iii): For $0 \leq i \leq j \leq k \leq n\ell$,

$$A(i,j)A(j,k) = \binom{k-i}{k-j} A(i,k) .$$

Proof: This follows from the definition since

$$A(i,j)A(j,k) = \frac{1}{(j-i)!} \frac{1}{(k-j)!} \partial^{j-i} \partial^{k-j}$$

$$= \frac{1}{(j-i)!} \frac{1}{(k-j)!} \partial^{k-i}$$

$$= \binom{k-i}{k-j} \frac{1}{(k-i)!} \partial^{k-i}$$

$$= \binom{k-i}{k-j} A(k,i) .$$

QED

Theorem (2.15) (ii): For $0 \leq i \leq j \leq n\ell$,

$$A(i,j)^* = R_{\bar{j}} A(\bar{j},\bar{i}) R_i .$$

Proof: We need show only

$$\langle R_{\bar{j}} A(\bar{j},\bar{k}) R_i \, \underset{\sim}{x}^{\underset{\sim}{v}}, \underset{\sim}{x}^{\underset{\sim}{u}} \rangle = \langle \underset{\sim}{x}^{\underset{\sim}{v}}, A(i,j) \underset{\sim}{x}^{\underset{\sim}{u}} \rangle$$

for every $\underset{\sim}{v} \in P_i^{\ell}$, $\underset{\sim}{u} \in P_j^{\ell}$. First, we use Lemma (2.6) to obtain

$$\langle A(i,j) \underset{\sim}{x}^{\underset{\sim}{u}}, \underset{\sim}{x}^{\underset{\sim}{v}} \rangle = \langle \underset{\sim}{x}^{\underset{\sim}{v}}, A(i,j) \underset{\sim}{x}^{\underset{\sim}{u}} \rangle$$

$$= \langle \underset{\sim}{x}^{\underset{\sim}{v}}, \sum_{\underset{\sim}{\alpha} \in P_i^{\ell}} \binom{\underset{\sim}{u}}{\underset{\sim}{\alpha}} \underset{\sim}{x}^{\underset{\sim}{\alpha}} \rangle$$

$$= \sum_{\underset{\sim}{\alpha} \in P_i^{\ell}} \binom{\underset{\sim}{u}}{\underset{\sim}{\alpha}} \langle \underset{\sim}{x}^{\underset{\sim}{v}}, \underset{\sim}{x}^{\underset{\sim}{\alpha}} \rangle$$

$$= \binom{\underset{\sim}{u}}{\underset{\sim}{v}} / \binom{\underset{\sim}{\ell}}{\underset{\sim}{v}} .$$

Therefore,

$$\langle R_{\bar{j}} A(\bar{j},\bar{i})\, R_i\, x^{\underset{\sim}{v}}_{\sim},\, x^{\underset{\sim}{u}}_{\sim}\rangle = \langle A(\bar{j},\bar{i})\, R_i\, x^{\underset{\sim}{v}}_{\sim},\, R_j\, x^{\underset{\sim}{u}}_{\sim}\rangle$$

$$= \langle A(\bar{j},\bar{i}) x^{\bar{\underset{\sim}{v}}}_{\sim},\, x^{\bar{\underset{\sim}{u}}}_{\sim}\rangle$$

$$= \binom{\bar{\underset{\sim}{v}}}{\bar{\underset{\sim}{u}}} \Big/ \binom{\underset{\sim}{\ell}}{\bar{\underset{\sim}{u}}}$$

$$= \frac{(\underset{\sim}{\ell}-\underset{\sim}{v})!}{(\underset{\sim}{\ell}-\underset{\sim}{u})!(\underset{\sim}{u}-\underset{\sim}{v})!} \cdot \frac{(\underset{\sim}{\ell}-\underset{\sim}{u})!\underset{\sim}{u}!}{\underset{\sim}{\ell}!}$$

$$= \frac{\underset{\sim}{u}!}{(\underset{\sim}{u}-\underset{\sim}{v})!\underset{\sim}{v}!} \cdot \frac{(\underset{\sim}{\ell}-\underset{\sim}{v})!\underset{\sim}{v}!}{\underset{\sim}{\ell}!}$$

and the two operators are equal. QED

<u>Lemma (2.16):</u>

(i) If $0 \le v,u$, then $\binom{x-u}{v} = \sum_{k=0}^{v} (-1)^k \binom{x-k}{v-k}\binom{u}{k}$.

Here, if r and s are integers, then $\binom{x-s}{r}$ denotes the polynomial

$$\binom{x-s}{r} \equiv \begin{cases} \dfrac{(x-s)(x-s-1)\dots(x-s-(r-1))}{r!} & \text{if } r \ge 1 \\ 1 & \text{if } r = 0 \\ 0 & \text{if } r < 0\,. \end{cases}$$

(ii) If $0 \le i,j \le n\ell$, and if $\underset{\sim}{v} \in P^{\ell}_i$, $\underset{\sim}{u} \in P^{\ell}_j$, then

$$\binom{\underset{\sim}{\ell}-\underset{\sim}{u}}{\underset{\sim}{v}} = \sum_{\underset{\sim}{\beta}<\underset{\sim}{v}} (-1)^{|\beta|} \binom{\underset{\sim}{\ell}-\underset{\sim}{\beta}}{\underset{\sim}{v}-\underset{\sim}{\beta}}\binom{\underset{\sim}{u}}{\underset{\sim}{\beta}} = \sum_{\underset{\sim}{\beta}<\underset{\sim}{v}} (-1)^{|\underset{\sim}{\beta}|} \binom{\underset{\sim}{\ell}}{\underset{\sim}{v}}\binom{\underset{\sim}{v}}{\underset{\sim}{\beta}}\binom{\underset{\sim}{u}}{\underset{\sim}{\beta}}\Big/\binom{\underset{\sim}{\ell}}{\underset{\sim}{\beta}}\,.$$

<u>Proof</u>: (i) If v=0, both sides are 1. If u=0, then both sides are $\binom{x}{v}$. We proceed by induction on u simultaneously for all v. So assume the formula holds for some $u \ge 0$ for all $v \ge 0$. Using the identity

$$\binom{x}{r} = \binom{x-1}{r} + \binom{x-1}{r-1}$$

which is valid for all r, we obtain

$$\binom{x-(u+1)}{u} = \binom{x-u-1}{v} = \binom{x-u}{v} - \binom{x-u-1}{v-1} = \binom{x-u}{v} - \binom{(x-1)-u}{v-1} .$$

Since the formula is correct for v=0 (and any u), we may assume $v \geq 1$. Thus,

$$\binom{x-(u+1)}{v} = \sum_{k=0}^{v} (-1)^k \binom{x-k}{v-k}\binom{u}{k} - \sum_{m=0}^{v-1} (-1)^m \binom{x-1-m}{v-1-m}\binom{u}{m}$$

$$= \sum_{k=0}^{v} (-1)^k \binom{x-k}{v-k}\binom{u}{k} - \sum_{k=1}^{v} (-1)^{k-1} \binom{x-k}{v-k}\binom{u}{k-1}$$

$$= \sum_{k=0}^{v} (-1)^k \binom{x-k}{v-k} \left[\binom{u}{k} + \binom{u}{k-1}\right] \quad \left[\text{since } \binom{u}{-1} = 0\right]$$

$$= \sum_{k=0}^{v} (-1)^k \binom{x-k}{v-k}\binom{u+1}{k}$$

and the induction is complete.

(ii) If $0 \leq v,u \leq \ell$, then replacing x by ℓ in (i) gives $\binom{\ell-u}{v} = \sum_{k=0}^{v} (-1)^k \binom{\ell-k}{v-k}\binom{u}{k}$.

Multiplying the n equalities obtained by replacing u,v by u_r, v_r $(1 \leq r \leq n)$ yields the first equality. The second equality follows from the simple identity

$$\binom{\ell-k}{v-k} = \binom{\ell}{v}\binom{v}{k}/\binom{\ell}{k} .$$

QED

<u>Definition (2.17)</u>: For $0 \leq i \leq j \leq n\ell$, the maps

$$H_i(j): \mathbb{F}_j^{\ell}[\underset{\sim}{x}] \to \mathbb{F}_j^{\ell}[\underset{\sim}{x}] ,$$

$$N_j(i): \mathbb{F}_i^{\ell}[\underset{\sim}{x}] \to \mathbb{F}_i^{\ell}[\underset{\sim}{x}] ,$$

are given by

$$H_i(j) = A(i,j)^*A(i,j),$$

$$N_j(i) = \sum_{k=0}^{i} [(-1)^k/\binom{j-k}{i-k}]H_k(i).$$

These maps are clearly hermitian.

Theorem (2.18):

(i) If $0 \le i \le j, \bar{j} \le n\ell$, then

$$A(i,\bar{j})\, R_j = N_j(i)A(i,j).$$

(ii) If $0 \le \bar{j} \le j \le n\ell$, then

$$N_j(\bar{j})A(\bar{j},j) = R_j.$$

In particular, $A(\bar{j},j)$ is invertible.

Proof: (i) We show $\langle x^{\underset{\sim}{v}}, A(i,\bar{j})R_j\, x^{\underset{\sim}{u}}\rangle = \langle x^{\underset{\sim}{v}}, N_j(i)A(i,j)x^{\underset{\sim}{u}}\rangle$ for each $v \in P_i^\ell$, $u \in P_j^\ell$. In fact,

$$\langle x^{\underset{\sim}{v}}, A(i,\bar{j})R_j\, x^{\underset{\sim}{u}}\rangle = \langle x^{\underset{\sim}{v}}, A(i,\bar{j})x^{\overline{\underset{\sim}{u}}}\rangle$$

$$= \binom{\overline{\underset{\sim}{u}}}{\underset{\sim}{v}}/\binom{\ell}{\underset{\sim}{v}} \qquad \text{[Lemma (2.6)]}$$

$$= \sum_{\underset{\sim}{\beta} \le \underset{\sim}{v}} (-1)^{|\underset{\sim}{\beta}|} \binom{\underset{\sim}{v}}{\underset{\sim}{\beta}}\binom{\underset{\sim}{u}}{\underset{\sim}{\beta}}/\binom{\ell}{\underset{\sim}{\beta}}$$

$$= \sum_{k=0}^{i} (-1)^k \sum_{\underset{\sim}{\beta} \in P_k^\ell} \binom{\underset{\sim}{v}}{\underset{\sim}{\beta}} \langle x^{\underset{\sim}{\beta}}, A(k,j)x^{\underset{\sim}{u}}\rangle$$

[any term not satisfying $\underset{\sim}{\beta} \le \underset{\sim}{v}$ is 0]

$$= \sum_{k=0}^{i} (-1)^k \langle A(k,i)x^{\underset{\sim}{v}}, A(k,j)x^{\underset{\sim}{u}}\rangle$$

$$= \sum_{k=0}^{i} (-1)^k \langle A(k,i)x^{\underset{\sim}{v}}, \binom{j-k}{i-k}^{-1} A(k,i)A(i,j)x^{\underset{\sim}{u}}\rangle$$

[by Theorem (2.15) (iii)]

$$= \sum_{k=0}^{i} [(-1)^k/\binom{j-k}{i-k}] \langle A(k,i)^*A(k,i)\underset{\sim}{x}^{v}, A(i,j)\underset{\sim}{x}^{u}\rangle$$

$$= \langle N_j(i)\underset{\sim}{x}^{v}, A(i,j)\underset{\sim}{x}^{u}\rangle$$

$$= \langle \underset{\sim}{x}^{v}, N_j(i)A(i,j)\underset{\sim}{x}^{u}\rangle$$

[since $N_j(i)$ is hermitian]. So the matrices are equal.

(ii) The formula is obtained from the formula in part (i) by replacing i by $\bar{j}$ and noting that $A(\bar{j},\bar{j})$ is the identity map. [Recall Definition (2.5i).] The invertibility of $A(\bar{j},j)$ now follows from the fact that R_j is an isometry (Theorem (2.15i)). QED

The proof of Theorem (2.11) is now elementary. In fact, if $\bar{j} \leq i \leq j$, then

$$A(\bar{j},i)A(i,j) = \binom{j-\bar{j}}{j-i} A(\bar{j},j).$$

Hence, $A(i,j)$ is one-to-one since $A(\bar{j},j)$ is invertible. If $i \leq j \leq \bar{i}$, then

$$A(i,j)\ A(j,\bar{i}) = \binom{\bar{i}-i}{\bar{i}-j} A(i,\bar{i}) .$$

Hence, $A(i,j)$ is onto for a similar reason.

<u>Corollary (2.19)</u>: If $0 \leq i \leq j \leq \bar{i} \leq n\ell$, then $N_j(i)$ is an isomorphism of $\mathbb{F}_i^{\ell}[\underset{\sim}{x}]$ with

$$N_j(i)^{-1} = N_{\bar{j}}(i) .$$

<u>Proof</u>: The hypotheses imply $i \leq \bar{j} \leq \bar{i}$, so that $A(i,\bar{j})$ is onto by Theorem (2.11). Therefore, the formula

$$(*) \qquad A(i,\bar{j})R_j = N_j(i)A(i,j)$$

from Theorem (2.18) shows that $N_j(i)$ is onto since R_j is onto by Theorem (2.15). Consequently, $N_j(i)$ is an isomorphism. Multiplying (*) on the left by $N_{\bar{j}}(i)$ gives

$$\begin{aligned} N_{\bar{j}}(i)N_j(i)A(i,j) &= N_{\bar{j}}(i)A(i,\bar{j})R_j \\ &= (A(i,j)R_{\bar{j}})R_j \\ &= A(i,j) \end{aligned}$$

since $R_{\bar{j}}$ is the inverse of R_j. But A(i,j) is onto (again by Theorem (2.11)), so $N_{\bar{j}}(i)N_j(i)$ is the identity. QED

Section 3: Reciprocal Translational Polynomials and the Subspace Decompositions

The fact that the reciprocal operators R_k are used in the previous section to help in the characterization of ℓ-bounded translational polynomials leads us to expect that there is a connection between such polynomials and reciprocal polynomials. In fact, one of the main results of this section (Theorem (3.3)) is that, aside from some obvious necessary conditions, homogeneous polynomials are reciprocal if and only if they are translational. To prove this, we will obtain an orthogonal subspace decomposition of $\mathbb{F}^\ell[\underset{\sim}{x}]$ using the spaces $\mathcal{C}(i,j)$, $\mathcal{N}(i,j)$ (see Definition (2.5)), and then we will consider in detail the actions of the operators $A(i,j)$ on this orthogonal decompsition. We end the section with some additional analysis of the operators $H_i(j)$ (see Definition (2.17)). However, we begin the section with some elementary results on reciprocal polynomials.

Definition (3.1): The polynomial p is $(+\ell)$ - *reciprocal* if

$$(Rp)(\underset{\sim}{x}) \equiv p(\underset{\sim}{x}) .$$

It is $(-\ell)$ - *reciprocal* if

$$(Rp)(\underset{\sim}{x}) \equiv -p(\underset{\sim}{x}) .$$

In either case, p is called reciprocal. The vector space of $(+\ell)$ - reciprocal (resp. $(-\ell)$ - reciprocal) polynomials is denoted $R^{+\ell}$ (resp. $R^{-\ell}$).

Since Rp is a polynomial if and only if p is ℓ-bounded, we restrict our attention to the spaces $\mathbb{F}^\ell[\underset{\sim}{x}]$ and $\mathbb{F}^\ell_k[\underset{\sim}{x}]$ $(0 \le k \le n\ell)$. Naturally, $R_k^{+\ell}$ denotes the vector spaces of k-homogeneous, $(+\ell)$ - reciprocal polynomials, and $R_k^{-\ell}$ has a similar meaning.

<u>Theorem (3.2)</u>: [Characterization of Reciprocal Polynomials]

Suppose $p \in \mathbb{F}^{\ell}[\underset{\sim}{x}]$ and p has the homogeneous decomposition $p = \sum_{k=0}^{n\ell} p_k$.

Then the following statements are equivalent:

(i) p is $(+\ell)$ - reciprocal (resp. $(-\ell)$ - reciprocal).

(ii) For $0 \le k \le n\ell$, $Rp_k = p_{n\ell-k}$ (resp. $Rp_k = -p_{n\ell-k}$).

(iii) For $0 \le k \le n\ell$, $R_k p_k = p_{n\ell-k}$ (resp. $R_k P_k = -P_{n\ell-k}$).

(iv) If $p = \sum_{\underset{\sim}{u} \in P^{\ell}} a_{\underset{\sim}{u}} \underset{\sim}{x}^{\underset{\sim}{u}}$, then $a_{\bar{\underset{\sim}{u}}} = a_{\underset{\sim}{u}}$ (resp. $a_{\bar{\underset{\sim}{u}}} = -a_{\underset{\sim}{u}}$).

Moreover, if $p = \sum_{\underset{\sim}{u} \in P_k^{\ell}} a_{\underset{\sim}{u}} \underset{\sim}{x}^{\underset{\sim}{u}}$, then p is $(+\ell)$ - reciprocal (resp. $(-\ell)$ - reciprocal) if and only if both $2k = n\ell$ and $a_{\bar{\underset{\sim}{u}}} = a_{\underset{\sim}{u}}$ (resp. $a_{\bar{\underset{\sim}{u}}} = -a_{\underset{\sim}{u}}$).

<u>Proof</u>: This is a straight forward application of the appropriate definitions and the fact that $Rp = \sum Rp_k$ is again a homogeneous decomposition. QED

This leads immediately to the

<u>Theorem (3.3)</u>: [Characterization of the Spaces $(R^{\pm}, R_k^{\pm}]$

(i) $R^{+\ell}$, $R^{-\ell}$ are isomorphic to the following direct sums:

$$R^{\pm\ell} \cong \begin{cases} \bigoplus_{\substack{j \\ 0 \le 2j < n\ell}} \mathbb{F}_j^{\ell}[\underset{\sim}{x}] & \text{, if } n\ell \text{ is odd,} \\ \left(\bigoplus_{j=0}^{k-1} \mathbb{F}_j^{\ell}[\underset{\sim}{x}]\right) \oplus R_k^{\pm\ell}, & \text{if } n\ell = 2k \text{ is even.} \end{cases}$$

(ii) Suppose $0 \le k \le n\ell$. Then

$$\dim R_k^{\pm\ell} = \begin{cases} 0 & \text{if } 2k \ne n\ell \\ \tfrac{1}{2}|P_k^{\ell}| & \text{if } 2k = n\ell,\ \ell \text{ odd} \\ \tfrac{1}{2}(|P_k^{\ell}| \pm 1) & \text{if } 2k = n\ell,\ \ell \text{ even.} \end{cases}$$

We are now in a position to state the main result of this section; however, as in Section 2, we will carry out the proof in several stages.

Theorem (3.4): Suppose p is a non-zero polynomial in $\mathcal{T}_k^\ell$. Then p is reciprocal if and only if $2k=n\ell$ (i.e., $k=\bar{k}$). Moreover, in this case,

$$Rp = (-1)^k p ,$$

so that $\mathcal{T}_k^\ell$ consists entirely of $(+\ell)$ - reciprocal (resp. $(-\ell)$ - reciprocal) polynomials if k is even (resp. odd). In other words, if k is an integer satisfying $0 \le k \le n\ell$, then

$$R_k^{\pm\ell} = (0) \qquad \text{if } 2k \neq n\ell,$$

$$R_k^{\pm\ell} \cap \mathcal{T} = \begin{cases} \mathcal{T}_k^\ell & \text{if } 2k=n\ell,\ k \text{ even} \\ (0) & \text{if } 2k=n\ell,\ k \text{ odd}, \end{cases}$$

$$R_k^{-\ell} \cap \mathcal{T} = \begin{cases} (0) & \text{if } 2k=n\ell,\ k \text{ even} \\ \mathcal{T}_k^\ell & \text{if } 2k=n\ell,\ k \text{ odd}. \end{cases}$$

Moreover, the non-zero spaces have dimension

$$\dim \mathcal{T}_k^\ell = |\mathcal{P}_k^\ell| - |\mathcal{P}_{k-1}^\ell| .$$

The first step towards the proof is a study of the spaces

$$\mathcal{C}(i,j) \equiv \text{range } (A(i,j)) \subseteq \mathbb{F}_i^\ell[\underset{\sim}{x}] ,$$

$$\mathcal{N}(i,j) \equiv \text{nullspace } (A(i,j)) \subseteq \mathbb{F}_j^\ell[\underset{\sim}{x}]$$

of Definition (2.5). We will also study the orthogonal complements $\mathcal{C}(i,j)^\perp$, $\mathcal{N}(i,j)^\perp$ taken in the respective spaces $\mathbb{F}_i^\ell[\underset{\sim}{x}]$, $\mathbb{F}_j^\ell[\underset{\sim}{x}]$ with respect to the inner product of Definition (2.13).

The second step consists of obtaining subspace decompositions of $\mathcal{C}(i,j)$ and $\mathcal{N}(i,j)$ as orthogonal direct sums. In the process, we will see how the operators $A(i,j)$ act on the summands, and Theorem (3.4) will follow. The section will be concluded with a detailed analysis of these actions.

Theorem (3.5): [Properties of $\mathcal{C}(i,j)$, $\mathcal{N}(i,j)$]

(i) If $0 \le \bar{j} \le i \le j \le n\ell$, then

$$\dim \mathcal{C}(i,j) = |P_j^\ell| , \quad \dim \mathcal{C}(i,j)^\perp = |P_i^\ell| - |P_j^\ell| ,$$

$$\dim \mathcal{N}(i,j) = 0 , \quad \dim \mathcal{N}(i,j)^\perp = |P_j^\ell| .$$

(ii) If $0 \le i \le j \le \bar{i} \le n\ell$, then

$$\dim \mathcal{C}(i,j) = |P_i^\ell| , \quad \dim \mathcal{C}(i,j)^\perp = 0 ,$$

$$\dim \mathcal{N}(i,j) = |P_j^\ell| - |P_i^\ell| , \quad \dim \mathcal{N}(i,j)^\perp = |P_i^\ell| .$$

Moreover, if $0 \le \bar{j} \le i \le j \le n\ell$, then

$$\mathcal{C}(i,j) = \mathcal{N}(\bar{j},i)^\perp .$$

Proof: The following formulas from linear algebra are standard. If A is a linear map from the vector space U to the vector space V, then

$$\dim U = \dim \text{range } (A) + \dim \text{nullspace } (A).$$

If U is an inner product space and W is a subspace, then

$$\dim W + \dim W^\perp = \dim U.$$

The dimension formulas in (i) and (ii) now follow from Theorem (2.11) since $A(i,j)$ is one-to-one in case (i) and onto in case (ii).

To show that $\mathcal{C}(i,j) = \mathcal{N}(\bar{j}-i)^\perp$, we note that the dimensions are equal when $\bar{j} \le i \le j$. Thus, it suffices to show that $\mathcal{C}(i,j) \subseteq \mathcal{N}(\bar{j},i)^\perp$. So suppose $A(i,j)p \in \mathcal{C}(i,j)$ and $q \in \mathcal{N}(\bar{j},i)$. Then

$$\begin{aligned}\langle A(i,j)p,\ q\rangle &= \langle p,\ A(i,j)^{*}q\rangle \\ &= \langle p,\ R_{\bar{j}}A(\bar{j},\bar{i})R_{i}q\rangle \qquad \text{(Theorem (2.15))} \\ &= \langle p,\ R_{j}\ N_{i}(\bar{j})A(\bar{j},i)q\rangle \quad \text{(Theorem (2.18))} \\ &= 0\end{aligned}$$

since $A(\bar{j},i)q = 0$, and the spaces are equal. QED

We are now ready to introduce the subspace which comprise the orthogonal direct sums.

Definition (3.6): Suppose j,k are integers with $\bar{k} \le k$, $j \le k$. Then the subspaces $V(j,k)$ are given by

$$V(j,k) \equiv A(j,k)\ N(\bar{k}-1,k).$$

Lemma (3.7): [Properties of $V(j,k)$]

Suppose $\bar{k} \le k$. Then

(i) $V(n\ell,n\ell) = \mathbb{F}^{\ell}_{n\ell}[\underset{\sim}{x}]$,

$V(k,k) = N(\bar{k}-1,k)$,

$V(j,k) = (0)$ for $j < \bar{k}$ or $k > n\ell$.

(ii) $V(j,k) \subseteq \mathbb{F}^{\ell}_{j}[\underset{\sim}{x}]$ for $j \le k$.

(iii) $\dim V(j,k) = |P^{\ell}_{k}| - |P^{\ell}_{k+1}|$ for $\bar{k} \le j \le k$.

(iv) $A(i,j)V(j,k) = V(i,k)$ for $i \le j \le k$,

In particular, $A(i,j)V(j,k) = (0)$ for $i < \bar{k}$.

(v) $V(j,k) = C(j,k) \cap C(j,k+1)^{\perp}$

$= N(\bar{k},j)^{\perp} \cap N(\bar{k}-1,j)$ for $\bar{k} \le j \le k$.

(vi) $V(i,j) \perp V(i,k)$ for $0 \le \bar{k} < \bar{j} \le i \le j < k \le n\ell$.

Proof: (i) Since $A(k,k)$ is the identity on $\mathbb{F}_k^\ell[\underset{\sim}{x}]$, we have $V(k,k) = N(\bar{k}-1,k)$. For $k=n\ell$, this becomes $V(n\ell,n\ell) = N(-1,n\ell) = \mathbb{F}_{n\ell}^\ell[\underset{\sim}{x}]$ since $\partial^{n\ell+1}\mathbb{F}_{n\ell}^\ell[\underset{\sim}{x}] = (0)$. If $j < \bar{k}$, then $V(j,k) = A(j,k)N(\bar{k}-1,k) = \partial^{k-j}\, N(\bar{k}-1,k) = (0)$ since $k-j \geq k-(\bar{k}-1)$. Also, $N(\bar{k}-1,k) = (0)$ if $k > n\ell$.

(ii) If $j \leq k$, then $V(j,k) = \partial^{k-j}\, N(\bar{k}-1,k) \subseteq \partial^{k-j}\, \mathbb{F}_k^\ell[\underset{\sim}{x}] \subseteq \mathbb{F}_j^\ell[\underset{\sim}{x}]$.

(iii) If $\bar{k} \leq j \leq k$, then $A(j,k)$ is one-to-one by Theorem (2.11) so that $\dim V(j,k) = \dim N(\bar{k}-1,k) = |P_k^\ell| - |P_{\bar{k}-1}^\ell|$ (by Theorem (3.5)) $= |P_k^\ell| - |P_{k+1}^\ell|$ since $\overline{\bar{k}-1} = k+1$.

(iv) $A(i,j)V(j,k) = A(i,j)A(j,k)N(\bar{k}-1,k) = \text{const. } A(i,k)N(\bar{k}-1,k) = V(i,k)$. By part (i), this is (0) for $i < \bar{k}$.

(v) Suppose $\bar{k} \leq j \leq k$. Then $V(j,k) = A(j,k)N(\bar{k}-1,k) \subseteq N(\bar{k}-1,j) = C(j,k+1)^{\perp}$ by Theorem (3.5). Since $V(j,k) \subseteq C(j,k)$ and $C(j,k+1) \subseteq C(j,k)$ (Theorem (3.5)), we get $V(j,k) \subseteq C(j,k) \cap C(j,k+1)^{\perp} = C(j,k) \ominus C(j,k+1)$. But the dimensions here are equal:

$$\dim V(j,k) = |P_k^\ell| - |P_{k+1}^\ell| ,$$

$$\dim C(j,k) \ominus C(j,k+1) = |P_k^\ell| - |P_{k+1}^\ell| .$$

So $V(j,k) = C(j,k) \cap C(j,k+1)^{\perp}$. The formula $V(j,k) = N(\bar{k},j)^{\perp} \cap N(\bar{k}-1,j)$ follows now from the identity $C(j,k) = N(\bar{k},j)^{\perp}$ (Theorem (3.5)).

(vi) This follows easily from the inclusions $V(i,j) \subseteq C(i,j+1)^{\perp}$ and $V(i,k) \subseteq C(i,k) \subseteq C(i,j+1)$ (Theorem (3.5)). QED

Theorem (3.8): [Orthogonal Subspace Decompositions]

The following subspace decompositions are orthogonal direct sums.

(i) $\mathbb{F}_i^\ell[\underset{\sim}{x}] = \bigoplus_{\substack{k \\ \bar{k} \leq i \leq k}} V(i,k)$ for $0 \leq i \leq n\ell$.

(ii) $\mathcal{C}(i,j) = \bigoplus\limits_{\substack{k \\ \bar{k} \le i \le j \le k}} \mathcal{V}(i,k)$ for $0 \le i \le j \le n\ell$.

(iii) $\mathcal{N}(i,j) = \bigoplus\limits_{\substack{k \\ i < \bar{k} \le j \le k}} \mathcal{V}(j,k)$ for $0 \le i \le j \le n\ell$.

Proof: The proofs are all similar; we prove only (i). By Lemma (3.7), we have $\mathcal{V}(i,k) \subseteq \mathbb{F}^{\ell}_{i}[\underset{\sim}{x}]$. So $\oplus\, \mathcal{V}(i,k) \subseteq \mathbb{F}^{\ell}_{i}[\underset{\sim}{x}]$. It suffices to show that the dimensions agree. Now,

$$\dim \left(\bigoplus\limits_{\substack{k \\ \bar{k} \le i \le k}} \mathcal{V}(i,k) \right) = \sum\limits_{\substack{k \\ \bar{k} \le i \le k}} \dim \mathcal{V}(i,k)$$

$$= \sum_{\bar{k} \le i \le k} \left(|P^{\ell}_{k}| - |P^{\ell}_{k+1}| \right)$$

$$= \begin{cases} \sum\limits_{k=i}^{n\ell} \left(|P^{\ell}_{k}| - |P^{\ell}_{k+1}| \right) & \text{if } \bar{i} \le i\,, \\ \sum\limits_{k=\bar{i}}^{n\ell} \left(|P^{\ell}_{k}| - |P^{\ell}_{k+1}| \right) & \text{if } i \le \bar{i}\,, \end{cases}$$

$$= \begin{cases} |P^{\ell}_{i}| & \text{if } \bar{i} \le i\,, \\ |P^{\ell}_{\bar{i}}| & \text{if } i \le \bar{i}\,. \end{cases}$$

Since $|P^{\ell}_{\bar{i}}| = |P^{\ell}_{i}|$, the dimension is $|P^{\ell}_{i}|$ in either case. Since $\dim \mathbb{F}^{\ell}_{i}[\underset{\sim}{x}] = |P^{\ell}_{i}|$, the spaces are equal. QED

We are now in a position to prove Theorem (3.4). In fact, all that needs to be shown is that the restriction of $(-1)^k R_k$ to $\mathcal{T}^{\ell}_{k} = \mathcal{N}(k-1,k)$ is the identity when $k=\bar{k}$. This is generated in part (ii) of the next result.

<u>Theorem (3.9)</u>: Suppose $0 \le \bar{k} \le i \le k \le n\ell$. Then

(i) The restriction of R_i to $\mathcal{V}(i,k)$ is an isometric isomorphism onto $\mathcal{V}(\bar{i},k)$.

(ii) The restriction of $(-1)^{\bar{k}} R_{\bar{k}} A(\bar{k},k)$ to $\mathcal{V}(k,k)$ is the identity.

<u>Proof</u>: (i) It suffices to show that $R_i\, \mathcal{C}(i,j) = \mathcal{C}(\bar{i},j)$. In fact, if this were true, then we would have

$$\bigoplus_{\substack{k \\ \bar{k}\le \bar{i} \le k}} \mathcal{V}(\bar{i},k) = \mathcal{C}(\bar{i},k) \qquad \text{[by Theorem (3.8ii)]}$$

$$= R_i\, \mathcal{C}(i,k) \qquad \text{[to be proven]}$$

$$= R_i \Big[\bigoplus_{\substack{k \\ \bar{k}\le i \le k}} \mathcal{V}(i,k)\Big] \qquad \text{[by Theorem (3.8ii)]}$$

$$= \bigoplus_{\substack{k \\ \bar{k}\le i \le k}} R_i\, \mathcal{V}(i,k) \qquad [R_i \text{ is an isometry}]\,.$$

$$= \bigoplus_{\substack{k \\ \bar{k}\le \bar{i} \le k}} R_i\, \mathcal{V}(i,k)\,.$$

But the spaces $\mathbb{F}_i^{\ell}[\underset{\sim}{x}]$ $(0 \le i \le n\ell)$ are orthogonal, and $\mathcal{V}(\bar{i},k) \subseteq \mathbb{F}_{\bar{i}}^{\ell}[\underset{\sim}{x}]$, $R_i\, \mathcal{V}(i,k) \subseteq R_i\, \mathbb{F}_i^{\ell}[\underset{\sim}{x}] = \mathbb{F}_{\bar{i}}^{\ell}[\underset{\sim}{x}]$. So $\mathcal{V}(\bar{i},k) = R_i\, \mathcal{V}(i,k)$ since the sums above are orthogonal direct sums.

We prove now the contention that $R_i \mathcal{C}(i,j) = \mathcal{C}(\bar{i},j)$. By Theorem (2.15ii) we have $R_i A(i,j) = A(\bar{j},\bar{i})^* R_j$, so that

$$R_i \mathcal{C}(i,j) = R_i\, A(i,j)\, \mathbb{F}_j^{\ell}[\underset{\sim}{x}] = A(\bar{j},\bar{i}) R_j\, \mathbb{F}_j^{\ell}[\underset{\sim}{x}]$$

$$= A(\bar{j},\bar{i})^*\, \mathbb{F}_{\bar{j}}^{\ell}[\underset{\sim}{x}] = \mathcal{C}(A(\bar{j},\bar{i})^*)$$

$$= \mathcal{N}(\bar{j},\bar{i})^{\perp} = \mathcal{C}(\bar{i},j) \qquad \text{[Theorem (3.5)]}.$$

(ii) Suppose $0 \leq \bar{k} \leq k \leq n\ell$. Then $R_k = N_k(\bar{k})A(\bar{k},k)$ by Theorem (2.18ii). Therefore, for $p \in \mathcal{V}(k,k)$ we obtain

$$\begin{aligned} R_k p &= N_k(\bar{k})A(\bar{k},k)p \\ &= \sum_{j=0}^{\bar{k}} [(-1)^j / \tbinom{k-j}{k-j}] A(j,\bar{k})^* A(j,\bar{k}) A(\bar{k},k)p \\ &= (-1)^{\bar{k}} A(\bar{k},k)p \end{aligned}$$

since $A(j,\bar{k})A(\bar{k},k)p = 0$ for $j < \bar{k}$. Consequently,

$$p = R_{\bar{k}} R_k p = (-1)^{\bar{k}} R_{\bar{k}} A(\bar{k},k)p \,. \qquad \text{QED}$$

In the remainder of this section we describe the action of the operators $A(i,j)$ on the spaces $\mathcal{V}(j,k)$ and conclude from this the structure of the operators $H_i(j)$ on $\mathbb{F}_j^{\ell}[\underset{\sim}{x}]$.

<u>Theorem (3.11)</u>: [Action of $A(i,j)$: $\mathcal{V}(j,k) \to \mathcal{V}(i,k)$]

Suppose $0 \leq \bar{k} \leq i \leq j \leq k \leq n\ell$. Then

$$\langle A(i,j)p,\ A(i,j)p\rangle = \binom{k-i}{k-j}\binom{j-\bar{k}}{i-\bar{k}} \langle p,p\rangle$$

for all $p \in \mathcal{V}(j,k)$.

<u>Proof</u>: We consider first the case $j=k$. So suppose $p \in \mathcal{V}(k,k)$ and $0 \leq \bar{k} \leq i \leq k \leq n\ell$. Then

$$\begin{aligned} \langle A(i,k)p, A(i,k)p\rangle &= \langle p, A(i,k)^* A(i,k)p\rangle \\ &= \langle p, R_{\bar{k}} A(\bar{k},\bar{i}) R_i A(i,k)p\rangle && \text{[Theorem (2.15ii)]} \\ &= \langle p, R_{\bar{k}} N_i(\bar{k}) A(\bar{k},i) A(i,k)p\rangle && \text{[Theorem (2.18i)]} \\ &= \binom{k-\bar{k}}{k-i} \langle p, R_{\bar{k}} N_i(\bar{k}) A(\bar{k},k)p\rangle && \text{[Theorem (2.15iii)]} \end{aligned}$$

$$= (-1)^{\overline{k}} \binom{k-\overline{k}}{k-i} \langle p, R_{\overline{k}} A(\overline{k},k)p \rangle \qquad \text{[Lemma (3.10]}$$

$$= \binom{k-\overline{k}}{k-i} \langle p,p \rangle . \qquad \text{[Theorem (3.9)].}$$

So the result is true when k=j.

To prove the general result, assume $p \in V(j,k)$. Then $p = A(j,k)q$ for some $q \in V(k,k)$. Therefore,

$$\langle p,p \rangle = \langle A(j,k)q,\ A(j,k)q \rangle$$

$$= \binom{k-\overline{k}}{k-j} \langle q,q \rangle$$

by the first part of the proof, and so

$$\langle A(i,j)p,\ A(i,j)p \rangle = \langle A(i,j)A(j,k)q,\ A(i,j)A(j,k)q \rangle$$

$$= \binom{k-i}{k-j}^2 \langle A(i,k)q,\ A(i,k)q \rangle \qquad \text{[Theorem (2.15iii)]}$$

$$= \binom{k-i}{k-j}^2 \binom{k-\overline{k}}{k-i} \langle q,q \rangle \qquad \text{[the case k=j]}$$

$$= \left\{ \binom{k-i}{k-j}^2 \binom{k-\overline{k}}{k-i} \Big/ \binom{k-\overline{k}}{k-j} \right\} \langle p,p \rangle$$

$$= \binom{k-i}{k-j} \binom{j-\overline{k}}{i-\overline{k}} \langle p,p \rangle \quad . \qquad \text{QED}$$

We digress slightly to discuss properties of maps which have the same behavior exhibited in the previous Theorem. Recall that a linear map A from the vector space V to itself is scalar if $A=\alpha I$ for some scalar α. To generalize this notion to a linear map A between two vector spaces V and W, we require that they be inner product spaces.

<u>Definition (3.12)</u>: Suppose V and W are inner product spaces. The linear transformation A: $V \to W$ is called *scalar* (with respect to the corresponding inner products) if there is a scalar α so that for all $v \in V$,

$$<Av,Av>_W = \alpha < v,v>_V$$

where $<\cdot,\cdot>_W$ and $<\cdot,\cdot>_V$ are the inner products on W and V respectively.

It is easy to see that if A: $V \to W$ is scalar, then either $\alpha=0$ and A=0, or $\alpha>0$ and A is one-to-one. Unfortunately, this notion of a scalar map is not very restrictive in the following sense: if A is any one-to-one linear map between vector spaces, it is possible to determine inner products on the spaces with respect to which A will be scalar. On the other hand, in any given problem the knowledge that a map is scalar helps to determine its structure.

We give now several equivalent characterizations of scalar maps.

<u>Theorem (3.13)</u>: [Characterizations of Scalar Maps]

Suppose V and W are inner product spaces and A: $V \to W$ is a linear transformation. Then each of the following conditions are equivalent:

(i) There is a scalar α so that

$$<Av,Av> = \alpha<v,v> \qquad \text{for all } v \in V.$$

(ii) There is a scalar α so that

$$<Au,Av> = \alpha<u,v> \qquad \text{for all } u,v \in V.$$

(iii) $<u,v> = 0$ implies $<Au,Av> = 0$ for all $u,v \in V$.

(iv) A*Av is a multiple of v for each $v \in V$.

(v) There is a scalar α so that $A^*A = \alpha I$. That is A*A: $V \to V$ is scalar in the original sense.

Moreover, the scalar α in (i), (ii) and (v) is the same.

Proof: We will show (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(i). The polarization identity shows that (i)⇒(ii) with the same α. The implication (ii)⇒(iii) is trivial. Now assume (iii) is true and $v \in V$. If $v=0$, then A^*Av is clearly a multiple of v. So assume $v \neq 0$. Let

$$w \equiv A^*Av - \frac{\langle A^*Av,v\rangle}{\langle v,v\rangle} v .$$

Direct calculation shows $\langle w,v\rangle = 0$. On the other hand, if $\langle u,v\rangle = 0$, then from (iii) we get

$$\begin{aligned}\langle w,u\rangle &= \langle A^*Av,u\rangle - \frac{\langle A^*Av,v\rangle}{\langle v,v\rangle}\langle v,u\rangle \\ &= \langle Av,Au\rangle - 0 \\ &= 0 .\end{aligned}$$

Therefore, w is orthogonal to any vector in V so that $w=0$. Hence, (iii)⇒(iv). To prove that (iv) implies (v), we will show more generally that if $B: V \to V$ is a linear map so that Bv is a multiple of v for each $v \in V$, then $B=\alpha I$ for some scalar α. So let v be a nonzero element of V, and define α by $Bv=\alpha v$. If $u \in V$ and u is a multiple of v, then clearly $Bu=\alpha u$. On the other hand if $u \in V$ and u is independent of v with $Bu=\beta u$ and $B(u+v) = \gamma(u+v)$, then

$$\begin{aligned}\alpha v + \beta u &= Bv + Bu = B(v+u) = \gamma(v+u) \\ &= \gamma v + \gamma u\end{aligned}$$

so that $\alpha=\gamma=\beta$. Hence, $B=\alpha I$. Finally, it is clear that (v) implies (i) with the same α. QED

By Theorem (3.11) we see that the maps $A(i,j): V(j,k) \to V(i,k)$ are scalar. Therefore, we can apply characterization (v) above to the maps $H_i(j) = A(i,j)^*A(i,j): \mathbb{F}_j^\ell[\underset{\sim}{x}] \to \mathbb{F}_j^\ell[\underset{\sim}{x}]$ to determine their structure completely.

Theorem (3.14): [The Operators $H_i(j)$]

Suppose $0 \leq i \leq j \leq n\ell$.

(i) For $0 \leq \bar{k} \leq i \leq j \leq k \leq n\ell$, the restriction of $H_i(j)$ to $V(j,k)$ is scalar with multiple $\binom{k-i}{k-j}\binom{j-\bar{k}}{i-\bar{k}}$.

In particular, the eigenvalues of $H_i(j)$ are $\binom{k-i}{k-j}\binom{j-\bar{k}}{i-\bar{k}}$ with multiplicity $|P_k^\ell| - |P_{k+1}^\ell|$.

(ii) If $0 \leq k \leq j$, then

$$H_i(j)\, H_k(j) = H_k(j)\, H_i(j) .$$

Proof: We know that $H_i(j): \mathbb{F}_j^\ell[\underset{\sim}{x}] \to \mathbb{F}_j^\ell[\underset{\sim}{x}]$ and $\mathbb{F}_j^\ell[\underset{\sim}{x}] = \bigoplus_{\substack{k \\ \bar{k}<j<k}} V(j,k)$ by Theorem (3.8i). Both results then follow from Theorems (3.11) and (3.13).

QED

Section 4: Selberg Polynomials and Tactical Decompositions

We have characterized the polynomials which satisfy the first three of the four conditions imposed by Selberg. In this section we complete the characterization by handling the condition of symmetry. This will require the partitioning of the matrix A(k-1,k) into a "tactical decomposition" (see Definition (4.4)). A related matrix B(k-1,k) is determined whose null space consists of the vectors of coefficients of Selberg polynomials. The relation between A and B will allow us to determine the dimension of this null space.

Definition (4.1): The polynomial $p \in \mathbb{F}[\underset{\sim}{x}]$ is *symmetric* if

$$(\sigma p)(\underset{\sim}{x}) = p(\underset{\sim}{x})$$

for all permutations $\sigma \in \pi_n$ where

$$(\sigma p)(\underset{\sim}{x}) \equiv p(\sigma \underset{\sim}{x}) \equiv p(x_{\sigma(1)}, \dots, x_{\sigma(n)}) .$$

The vector space of symmetric polynomials is denoted Sym and is spanned by the basis of polynomials

$$\left\{ \sum_{\underset{\sim}{\alpha} \in O(\underset{\sim}{u})} \underset{\sim}{x}^{\underset{\sim}{\alpha}} : \underset{\sim}{u} \in \vec{P} \right\}$$

where $\vec{P}$ denotes the set of *ordered partitions* (Appendix 2)

$$\vec{P} \equiv \{\underset{\sim}{u} \in P : u_1 \geq u_2 \geq \dots \geq u_n\} .$$

and $O(\underset{\sim}{u})$ is the *orbit* of $\underset{\sim}{u}$

$$O(\underset{\sim}{u}) \equiv \{\sigma \underset{\sim}{u} : \underset{\sim}{u} \in \pi_n\} .$$

We think of $\vec{P}$ as an index set for Sym. The ℓ-bounded and k-homogeneous subspace of Sym are denoted as usual by Sym^ℓ, Sym_k, Sym_k^ℓ and are indexed by the sets $\vec{P}^\ell$, $\vec{P}_k$, $\vec{P}_k^\ell$, respectively.

The *symmetric Selberg space* S_k^ℓ is given by

$$S_k^\ell \equiv T_k^\ell \cap R \cap \mathrm{Sym} .$$

By Theorem (3.4) we have:

If $2k \neq n\ell$, then $S_k^\ell = (0)$.

If $2k = n\ell$, then $S_k^\ell = T_k^\ell \cap \mathrm{Sym}$.

Moreover, if k is even, then S_k^ℓ consists of $(+\ell)$ - reciprocal polynomials, while if k is odd, then S_k^ℓ consists of $(-\ell)$ - reciprocal polynomials. Consequently, we need to determine which polynomials in T_k^ℓ are also symmetric.

<u>Theorem (4.2)</u>: [Characterization of S_k^ℓ]

Suppose $2k=n\ell$ and $p \in \mathrm{Sym}_k^\ell$, say $p = \sum_{\underset{\sim}{u} \in P_k^\ell} a_{\underset{\sim}{u}} x^{\underset{\sim}{u}}$. Then $p \in S_k^\ell$ if and only if

$$\sum_{\underset{\sim}{u} \in \vec{P}_k^\ell} b_{\underset{\sim}{v}\underset{\sim}{u}}\, a_{\underset{\sim}{u}} = 0 \qquad \text{for all } v \in \vec{P}_{k-1}^\ell$$

where

$$b_{\underset{\sim}{v}\underset{\sim}{u}} = \sum_{\underset{\sim}{\alpha} \in O(\underset{\sim}{u})} \binom{\underset{\sim}{\alpha}}{\underset{\sim}{v}} .$$

<u>Proof</u>: Assume $p \in \mathrm{Sym}_k^\ell$. By Theorems (2.7) and (3.4), p is translational and reciprocal (hence in S_k^ℓ) if and only if

$$\sum_{\underset{\sim}{u} \in P_k^\ell} \binom{\underset{\sim}{u}}{\underset{\sim}{v}} a_{\underset{\sim}{u}} = 0 \qquad \text{for all } \underset{\sim}{v} \in P_{k-1}^\ell .$$

But for $\sigma \in \pi_n$ we have $\sigma P_k^\ell = P_k^\ell$, $\binom{\sigma w}{\sigma v} = \binom{w}{v}$, and $a_{\sigma w} = a_w$ so that

$$\sum_{u \in P_k^\ell} \binom{u}{\sigma v} a_u = \sum_{u \in \sigma P_k^\ell} \binom{u}{\sigma v} a_u$$

$$= \sum_{w \in P_k^\ell} \binom{\sigma w}{\sigma v} a_{\sigma w} \qquad (\text{with } u = \sigma w)$$

$$= \sum_{w \in P_k^\ell} \binom{w}{v} a_w .$$

Thus, $p \in S_k^\ell$ if and only if

$$\sum_{u \in \vec{P}_k^\ell} \binom{u}{v} a_u = 0 \qquad \text{for all } v \in \vec{P}_{k-1}^\ell .$$

However,

$$\sum_{u \in P_k^\ell} \binom{u}{v} a_u = \sum_{u \in \vec{P}_k^\ell} \Big(\sum_{\alpha \in O(u)} \binom{\alpha}{v} a_\alpha \Big)$$

$$= \sum_{u \in \vec{P}_k^\ell} \Big(\sum_{\alpha \in O(u)} \binom{\alpha}{v} \Big) a_u \qquad (\text{since } a_\alpha = a_u)$$

$$= \sum_{u \in P_k^\ell} b_{vu} a_u ,$$

and the result follows. QED

This result leads us to consider the matrices $B(i,j)$ for $i \leq j$ whose rows and columns are indexed by $\vec{P}_i^\ell$ and $\vec{P}_j^\ell$, respectively, and whose (v,u)-element is given by

$$b_{vu} \equiv \sum_{\alpha \in O(u)} \binom{\alpha}{v} .$$

From Theorem (4.2) we see that S_k^ℓ is determined in an obvious way from the null-space of $B(k-1,k)$. The main result is

Theorem (4.3): [Dimension of S_k^ℓ]
If $2k=n\ell$, then $B(k-1,k)$ has full row rank. In particular,

$$\dim S_k^\ell = |\vec{P}_k^\ell| - |\vec{P}_{k-1}^\ell| .$$

As in the previous sections, this result is proven in stages. We will use the following notion.

Definition (4.4): Let $A=[A_{ij}]$ $(1\le i\le s,\ 1\le j\le t)$ be a partitioned matrix. The partition is a *tactical decomposition* if each A_{ij} has constant row sum b_{ij} and constant column sum c_{ij}. The $s\cdot t$ matrices $B\equiv[b_{ij}]$ and $C\equiv[c_{ij}]$ are the *row-* and *column-sum matrices* of the decomposition.

Lemma (4.5): Let $[A_{ij}]$ be an $s\cdot t$ tactical decomposition of A.
(i) If A has full row rank, then so does B.
(ii) If A has full column rank, then so does C.
Note: A has full row rank if A^T is one-to-one, and it has full column rank if A is one-to-one.

Proof of Lemma (4.5): Let $[A_{ij}]$ be an $s\cdot t$ tactical decomposition of A with row-sum matrix B and column-sum matrix C. We prove only (ii); (i) follows by taking transposes since $[A_{ji}^T]$ is a $t\cdot s$ tactical decomposition of A^T with row-sum matrix C^T and column-sum matrix B^T.

Let A_{ij} be $e_i\cdot f_j$. Adding the elements of A_{ij} by rows and by columns gives

$$e_i\, b_{ij} = c_{ij}\, f_j \qquad \text{for } 1\le i\le s,\ 1\le j\le t$$

or

$$ER = CF$$

where $E\equiv \operatorname{diag}(e_1,\dots,e_s)$ and $F\equiv \operatorname{diag}(f_1,\dots,f_t)$. Since E and F are invertible, C is one-to-one if and only if B is. But $B[b_1,\dots b_t]^T = 0$

if and only if

$$A[\underbrace{b_1,\ldots,b_1}_{f_1},\ \underbrace{b_2,\ldots,b_2}_{f_2},\ldots,\ \underbrace{b_t,\ldots,b_t}_{f_t}]^T = 0 .$$

It follows easily that if A is one-to-one, then so is B (and hence C).

QED

To complete the proof of Theorem (4.3), we need show only that B(k-1,k) is the row-sum matrix of a tactical decomposition of A(k-1,k). The point about full row rank will follow from Theorem (3.5ii) (which shows A(k-1,k) has full row rank for $k=\bar{k}$) by applying Lemma (4.5).

To partition A(i,j), we order the elements of $\vec{\mathcal{P}}^{\ell}_{j}$ and $\vec{\mathcal{P}}^{\ell}_{i}$ lexicographically, and then we order the orbits $O(\underset{\sim}{u})$ and $O(\underset{\sim}{v})$ ($\underset{\sim}{u} \in \vec{\mathcal{P}}^{\ell}_{j}$, $\underset{\sim}{v} \in \vec{\mathcal{P}}^{\ell}_{i}$) lexicographically. We obtain

$$A(i,j) \equiv [A_{\underset{\sim}{v}\underset{\sim}{u}}]$$

where, for $\underset{\sim}{u} \in \vec{\mathcal{P}}^{\ell}_{j}$ and $\underset{\sim}{v} \in \vec{\mathcal{P}}^{\ell}_{i}$, $A_{\underset{\sim}{v}\underset{\sim}{u}} \equiv A_{\underset{\sim}{v}\underset{\sim}{u}}(i,j)$ denotes the block of A(i,j) whose rows are indexed by $O(\underset{\sim}{v})$ and whose columns are indexed by $O(\underset{\sim}{u})$.

<u>Theorem (4.6)</u>: [Tactical Decomposition of A(i,j)]

The partition $A(i,j) = [A_{\underset{\sim}{v}\underset{\sim}{u}}(i,j)]$ is a tactical decomposition whose row-sum matrix is B(i,j).

<u>Proof</u>: A typical row sum of $A_{\underset{\sim}{v}\underset{\sim}{u}}$ is

$$\sum_{\underset{\sim}{\alpha} \in O(\underset{\sim}{u})} \binom{\underset{\sim}{\alpha}}{\sigma \underset{\sim}{v}} = \sum_{\underset{\sim}{\alpha} \in \sigma O(\underset{\sim}{u})} \binom{\underset{\sim}{\alpha}}{\sigma \underset{\sim}{v}}$$

$$= \sum_{\underset{\sim}{\beta}\in O(\underset{\sim}{u})} \binom{\sigma\underset{\sim}{\beta}}{\sigma\underset{\sim}{v}} \quad \text{(with } \alpha=\sigma\underset{\sim}{\beta}\text{)}$$

$$= \sum_{\underset{\sim}{\beta}\in O(\underset{\sim}{u})} \binom{\underset{\sim}{\beta}}{\underset{\sim}{v}} \; b_{\underset{\sim}{v}\underset{\sim}{u}} \; .$$

The column sums are handled similarly. QED

Appendix 1: Notation

Throughout this paper n and ℓ denote fixed integers satisfying $n \geq 2$ and $\ell \geq 1$.

I. n-tuples and operations on them

(1) $\underset{\sim}{0} = (0, 0, \ldots, 0)$

$\underset{\sim}{1} = (1, 1, \ldots, 1)$

(2) $\underset{\sim}{u} = (u_1, u_2, \ldots, u_n)$

$\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$

(3) $t\underset{\sim}{x} = (tx_1, tx_2, \ldots, tx_n)$

$t\underset{\sim}{1} = (t, t, \ldots, t) \equiv \underset{\sim}{t}$

(4) $\overline{\underset{\sim}{u}} = \underset{\sim}{\ell} - \underset{\sim}{u} = (\ell-u_1, \ell-u_2, \ldots, \ell-u_n)$

(5) $|\underset{\sim}{u}| = u_1 + u_2 + \ldots + u_n$

(6) $\underset{\sim}{u}! = u_1!\, u_2! \ldots u_n!$ (! denotes factorial)

(7) $\binom{\underset{\sim}{u}}{\underset{\sim}{v}} = \binom{u_1}{v_1}\binom{u_2}{v_2}\cdots\binom{u_n}{v_n}$ ($\binom{u}{v}$ denotes a binomial coefficient)

(8) $\underset{\sim}{x}^{\underset{\sim}{u}} = x_1^{u_1}\, x_2^{u_2}\, x_2^{u_2} \ldots x_n^{u_n}$.

II. Polynomial Properties

All polynomials are in $\mathbb{F}[\underset{\sim}{x}] = \mathbb{F}[x_1, x_2, \ldots, x_n]$ (see Subsection III below) unless otherwise specified. Let $p \in \mathbb{F}[\underset{\sim}{x}]$ and k, ℓ be nonnegative integers. Then

(1) p is *ℓ-bounded* if the largest power to which any x_i occurs in p is at most ℓ.

(2) p is *translational* if

$$p(\underset{\sim}{x} + \underset{\sim}{1}) = p(\underset{\sim}{x}) .$$

(3) p is *k-homogeneous* if

$$p(t\underset{\sim}{x}) = t^k p(\underset{\sim}{x}) .$$

(4) p is *ℓ^+-reciprocal* if

$$(x_1 x_2 \cdots x_n)^\ell \, p(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}) = p(\underset{\sim}{x}) .$$

p is *ℓ^--reciprocal* if

$$(x_1 x_2 \cdots x_n)^\ell \, p(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}) = -p(\underset{\sim}{x}) .$$

In either case, p is *ℓ-reciprocal*.

(5) p is *symmetric*$^+$ if

$$p(\sigma(\underset{\sim}{x})) = p(\underset{\sim}{x}) \quad \text{for all } \sigma \in \pi_n .$$

p is *symmetric*$^-$ if

$$p(\sigma(\underset{\sim}{x})) = (-1)^\sigma p(\underset{\sim}{x}) \quad \text{for all } \sigma \in \pi_n .$$

In either case, p is *symmetric*. [Here, $(-1)^\sigma$ is +1 (resp. -1) if σ is an even (resp. odd) permutation.]

III. Spaces and Sets

(1) $\mathbb{F}[x_1, \ldots, x_m]$ denotes the vector space over the real field $\mathbb{F}$ of all polynomials in m variables with coefficients in $\mathbb{F}$.

($\mathbb{F}[\underset{\sim}{x}] = \mathbb{F}[x_1, \ldots, x_m]$.)

(2) $\mathbb{F}^{\ell}[x_1, \ldots, x_m]$ denotes the subspace of $\mathbb{F}[x_1, \ldots, x_m]$ consisting of all ℓ-bounded polynomials.

(3) $\mathbb{F}_k[x_1, \ldots, x_m]$ denotes the subspace of $\mathbb{F}[x_1, \ldots, x_m]$ consisting of all k-homogeneous polynomials.

(4) $\mathbb{F}_k^{\ell}[x_1, \ldots, x_m]$ denotes the subspace of $\mathbb{F}[x_1, \ldots, x_m]$ consisting of all ℓ-bounded, k-homogeneous polynomials.

(5) Let Q be a subspace of $F[x_1, \ldots, x_m]$. Then the subspace 2^{ℓ}, 2_k, Q_k of Q are defined by

$$Q^{\ell} = Q \cap \mathbb{F}^{\ell}[x_1, \ldots, x_m],$$

$$Q_k = Q \cap \mathbb{F}_k[x_1, \ldots, x_m],$$

$$Q_k^{\ell} = Q \cap F_k^{\ell}[x_1, \ldots, x_m].$$

(6) $\mathcal{T}$ denotes the subspace of $\mathbb{F}[\underset{\sim}{x}]$ consisting of all translational polynomials. ($\mathcal{T}^{\ell}$, $\mathcal{T}_k^{\ell}$, $\mathcal{T}_k^{\ell}$.)

(7) R denotes the subspace of $\mathbb{F}[\underset{\sim}{x}]$ consisting of all ℓ-reciprocal polynomials. (R^{ℓ}, R_k, R_k^{ℓ}.)

(8) $\mathcal{S}$ denotes the subspace of $\mathbb{F}[\underset{\sim}{x}]$ consisting of all symmetric polynomials. ($\mathcal{S}^{\ell}$, $\mathcal{S}_k$, $\mathcal{S}_k^{\ell}$.)

(9) $\mathcal{C}(i,j)$ denotes the range of $A(i,j)$ for $0 \leq i \leq j \leq n\ell$.

(10) $\mathcal{N}(i,j)$ denotes the nullspace of $A(i,j)$ for $0 \leq i \leq j \leq n\ell$

(11) $\mathcal{V}(n\ell,n\ell) = \mathbb{F}^{\ell}_{n\ell}[\underset{\sim}{x}]$.

$\mathcal{V}(k,k) = \mathcal{C}(k,k+1)^{\perp}$ for $\bar{k} \leq k < n\ell$.

$\mathcal{V}(i,j) = A(i,j)\mathcal{V}(j,j)$ for $0 \leq i < j \leq n\ell$, $\bar{j} \leq j$.

(12) Let ε be a subset of $\mathbb{Z}^n = \{\underset{\sim}{u} = (u_1, \ldots, u_n): u_i \in \mathbb{Z}$ for $i \leq i \leq n\}$. Then the subsets ε^{ℓ}, ε_k, ε_k^{ℓ} of ε are defined by

$\varepsilon^{\ell} = \{\underset{\sim}{u} \in \varepsilon: u_i \leq \ell$ for $i \leq i \leq n\ell\}$,

$\varepsilon_k = \{\underset{\sim}{u} \in \varepsilon: |\underset{\sim}{u}| = u_1 + u_2 + \ldots + u_n = k\}$,

$\varepsilon_k^{\ell} = \{\underset{\sim}{u} \in \varepsilon: |\underset{\sim}{u}| = k$ and $u_i \leq \ell$ for $i \leq i \leq n\}$.

(13) $P = \{\underset{\sim}{u} \in \mathbb{Z}^n: u_i \geq 0$ for all $i\}$. $(P^{\ell}, P_k, P_k^{\ell})$ [These are the unordered partitions.]

(14) $\vec{P} = \{\underset{\sim}{u} \in \mathbb{Z}^n: u_1 \geq u_2 \geq \ldots u_n \geq 0\}$. $(\vec{P}^{\ell}, \vec{P}_k, \vec{P}_k^{\ell}.)$ [These are the ordered partitions.]

(15) π_n denotes the group of permutations of the set $\{1,2,\ldots,n\}$.

IV. Operators

(1) $\partial_i = \dfrac{\partial}{\partial x_i}$ for $1 \le i \le n$.

(2) $\underset{\sim}{\partial} = (\partial_1, \partial_2, \ldots, \partial_n)$.

(3) $\partial = \partial_1 + \partial_2 + \ldots + \partial_n$.

(4) $\underset{\sim}{\partial}^{\underset{\sim}{u}} = \partial_1^{u_1} \partial_2^{u_2} \ldots \partial_n^{u_n}$, $\underset{\sim}{u} = (u_1, \ldots u_n)$ in P.

(5) $A(i,j)$: $\mathbb{F}_j[\underset{\sim}{x}] \to \mathbb{F}_i[\underset{\sim}{x}]$, for $i \le j$, is defined by

$$A(i,j)p = \frac{1}{(j-i)!} \partial^{j-i} p .$$

(6) R: $\mathbb{F}^{\ell}[\underset{\sim}{x}] \to \mathbb{F}^{\ell}[\underset{\sim}{x}]$ is defined by

$$(Rp)(\underset{\sim}{x}) = (x_1 x_2 \ldots x_n)^{\ell}\, p\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right).$$

R_k: $\mathbb{F}_k^{\ell}[\underset{\sim}{x}] \to \mathbb{F}_{n\ell-k}^{\ell}[\underset{\sim}{x}]$ is the restriction of R to $\mathbb{F}_k^{\ell}[\underset{\sim}{x}]$.

(7) $<,>$ is the inner product on $\mathbb{F}[\underset{\sim}{x}]$ defined by

$$<p,q> = \frac{1}{\underset{\sim}{\ell}!} \sum_{\underset{\sim}{u} \in P} \underset{\sim}{\partial}^{\underset{\sim}{u}} p(\underset{\sim}{0})\, \overline{\underset{\sim}{\partial}^{\underset{\sim}{u}} \underset{\sim}{q}}(\underset{\sim}{0}) .$$

(8) S denotes the orthogonal complement (with respect to $<,>$) in $\mathbb{F}_k[\underset{\sim}{x}]$ of the subset S of $\mathbb{F}_k[\underset{\sim}{x}]$.

(9) $A(i,j)^*$: $\mathbb{F}_i[\underset{\sim}{x}] \to \mathbb{F}_j[\underset{\sim}{x}]$ is the adjoint (with respect to $<,>$) of $A(i,j)$: $\mathbb{F}_j[\underset{\sim}{x}] \to \mathbb{F}_i[\underset{\sim}{x}]$.

(10) $H_k(i)$: $\mathbb{F}_i[\underset{\sim}{x}] \to \mathbb{F}_i[\underset{\sim}{x}]$, for $0 \le k \le i \le n\ell$, is defined by

$$H_k(i) = A(k,i)^* A(k,i) .$$

(11) $N_j(i): \mathbb{F}_i[\underset{\sim}{x}] \to \mathbb{F}_i[\underset{\sim}{x}]$, for $0 \le i \le j \le n\ell$, is defined by

$$N_j(i) = \sum_{k=0}^{i} \frac{(-1)^k}{\binom{j-k}{i-k}} H_k(i).$$

(12) $\sigma\underset{\sim}{x} = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for $\sigma \in \pi_n$.

(13) $(\sigma p)(\underset{\sim}{x}) = p(\sigma\underset{\sim}{x})$ for $p \in \mathbb{F}[\underset{\sim}{x}]$, $\sigma \in \pi_n$.

V. Miscellaneous

(1) $\overline{k} = n\ell - k$

(2) $|\varepsilon|$ denotes the cardinality of the set ε.

Appendix 2: Homogeneous and Symmetric Polynomials

The purpose of this appendix is to summarize some basic facts about polynomials and to set some notation.

Recall that the vector space over the field $\mathbb{F}$ of all polynomials in n variables with coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}[\underset{\sim}{x}]$ (where $\underset{\sim}{x} = (x_1,\ldots,x_n)$). A basis for the space is given by the monomials

$$\underset{\sim}{x}^{\underset{\sim}{u}} \equiv x_1^{u_1} \ldots x_n^{u_n}$$

where $\underset{\sim}{u} = (u_1, \ldots, u_n)$ is a multi-index in the index set

$$P \equiv \{(u_1, \ldots, u_n): u_i \in \mathbb{Z} , u_i \geq 0 \ (1 \geq i \geq n)\} .$$

Thus, a typical polynomial $p(\underset{\sim}{x})$ in $\mathbb{F}[\underset{\sim}{x}]$ has a unique representation

$$p(\underset{\sim}{x}) = \sum_{\underset{\sim}{u} \in P} a_{\underset{\sim}{u}} \underset{\sim}{x}^{\underset{\sim}{u}}$$

where all but finitely many of the coefficients $a_{\underset{\sim}{u}}$ are 0. The spaces $\mathbb{F}_k[\underset{\sim}{x}]$, $\mathbb{F}^{\ell}[\underset{\sim}{x}]$, and $\mathbb{F}_k^{\ell}[\underset{\sim}{x}]$ are the subspaces of $\mathbb{F}[\underset{\sim}{x}]$ obtained by restricting the multi-indices to lie in the index sets

$$P_k \equiv \{\underset{\sim}{u} \in P : |\underset{\sim}{u}| \equiv u_1 + \ldots + u_n = k\} ,$$

$$P^{\ell} \equiv \{\underset{\sim}{u} \in P : u_i \leq \ell \ (1 \leq i \leq n)\},$$

$$P_k^{\ell} \equiv P_k \cap P^{\ell},$$

respectively. The polynomials in $\mathbb{F}^{\ell}[\underset{\sim}{x}]$ are called *ℓ-bounded*. We note that $P_k^{\ell} = \emptyset$ if $2k > n\ell$.

Definition: Let k be a non-negative integer. The polynomial p in $\mathbb{F}[\underset{\sim}{x}]$ is k-*homogeneous* (or homogeneous of degree k) if

$$p(\alpha \underset{\sim}{x}) = \alpha^k \, p(\underset{\sim}{x}) .$$

Theorem: Let p be a polynomial in $\mathbb{F}[\underset{\sim}{x}]$, say $p = \sum_{\underset{\sim}{u} \in P} a_{\underset{\sim}{u}} \underset{\sim}{x}^{\underset{\sim}{u}}$. Then

(1) p is k-homogeneous if and only if p is in $\mathbb{F}_k[\underset{\sim}{x}]$.

(2) p has a unique decomposition

$$p = \sum_{k \geq 0} p_k$$

where each p_k is k-homogeneous. In fact,

$$p_k = \sum_{\underset{\sim}{u} \in P_k} a_{\underset{\sim}{u}} \underset{\sim}{x}^{\underset{\sim}{u}} .$$

We call this decomposition the *homogeneous decomposition* of p.

(3) p is in $\mathbb{F}^{\ell}[\underset{\sim}{x}]$ if and only if each p_k is in $\mathbb{F}_k^{\ell}[\underset{\sim}{x}]$.

Theorem: Let k, ℓ be non-negative integers. Then

(1) $\dim \mathbb{F}_k[\underset{\sim}{x}] = |P_k| = \binom{n+k-1}{k-1}$.

(2) $\dim \mathbb{F}^{\ell}[\underset{\sim}{x}] = |P^{\ell}| = (\ell+1)^n$.

(3) $\dim \mathbb{F}_k^{\ell}[\underset{\sim}{x}] = |P_k^{\ell}|$.

In particular, $\mathbb{F}_k^{\ell}[\underset{\sim}{x}]$ contains only the zero polynomial if $2k > n\ell$.

Symmetric and Alternating Polynomials

To each permutation σ in π_n (the group of permutations on $\{1,2,\ldots,n\}$) and each polynomial p in $\mathbb{F}[\underset{\sim}{x}]$, we can associate a new polynomial σp in $\mathbb{F}[\underset{\sim}{x}]$ defined by

$$(\sigma p)(\underset{\sim}{x}) \equiv p(\sigma \underset{\sim}{x})$$

where

$$\sigma \underset{\sim}{x} \equiv (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) .$$

This operation can clearly be restricted to each of the subspaces $\mathbb{F}_k[\underset{\sim}{x}]$, $\mathbb{F}^\ell[\underset{\sim}{x}]$, $\mathbb{F}_k^\ell[\underset{\sim}{x}]$.

<u>Definition</u>: The polynomial p in $\mathbb{F}[\underset{\sim}{x}]$ is *symmetric* if

$$(\sigma p)(\underset{\sim}{x}) = p(\underset{\sim}{x})$$

for each σ in π_n. The polynomial is *alternating* if

$$(\sigma p)(\underset{\sim}{x}) = (-1)^\sigma p(\underset{\sim}{x}) .$$

Here, $(-1)^\sigma$ is +1 (resp. -1) if σ is an even (resp. odd) permutation.

<u>Theorem</u>: Let p be a polynomial in $\mathbb{F}[\underset{\sim}{x}]$, say $p = \sum_{\underset{\sim}{u} \in P} a_{\underset{\sim}{u}} \underset{\sim}{x}^{\underset{\sim}{u}}$.

(1) p is symmetric (resp. alternating) if and only if each p_k in the homogeneous decomposition of p is symmetric (resp. alternating).

(2) p is symmetric if and only if $a_{\sigma \underset{\sim}{u}} = a_{\underset{\sim}{u}}$ for each σ in π_n .

(3) p is alternating if and only if $a_{\sigma \underset{\sim}{u}} = (-1)^\sigma a_{\underset{\sim}{u}}$ for each σ in π_n .

<u>Theorem</u>: Let Sym (resp. Alt) denote the subspace of $\mathbb{F}[\underset{\sim}{x}]$ consisting of symmetric (resp. alternating) polynomials. Then

(1) A basis for Sym is the set of monomials with multi-indices in the index set

$$\vec{P} \equiv \{\underset{\sim}{u} \in P: u_1 \geq u_2 \geq \ldots \geq u_n\} .$$

(2) A basis for Alt is the set of monomials with multi-indices in the index set

$$\overset{>}{\overline{\overline{P}}} \equiv \{\underset{\sim}{u} \in P: u_1 > u_2 > \ldots > u_n\} .$$

Similar results hold for Sym_k, Sym^ℓ, Sym_k^ℓ and alt_k, alt^ℓ, alt_k^ℓ .

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