OPTIMAL WEIGHING DESIGNS WITH A STRING PROPERTY

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0. ABSTRACT

We consider the usual (spring balance) weighing design set-up with the design matrix having a string property meaning thereby that in every row of it, there is exactly one run of 1's (the rest of the elements being 0's). We have investigated some interesting features of such matrices and used them in deriving various optimality results.

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1. INTRODUCTION

Weighing problems were first posed and discussed by Hotelling (1944). Over the past three decades, various aspects of such problems have been extensively studied. We refer to Raghavarao (1971) for all the basic results on this topic.* An important point to be noted is that "The designs are applicable to a great variety of problems of measurement, not only of weights, but of lengths, voltages and resistances, concentrations of chemicals in solutions, in fact any measurements such that the measure of a combination is a known linear function of the separate measures with numerically equal coefficients." (Mood (1946)).

In this paper we study some aspects of spring balance weighing designs with a "string property" meaning thereby that in every row of the "design matrix", there is exactly one run of 1's (the rest of the elements being 0's). Such a situation arises, for example, while measuring distances among a set of fixed objects along a line. To be specific, suppose there are $(n+1)$ objects, serially numbered $1, 2, \ldots, n+1$; their positions are fixed along a line and $\underline{\theta} = (\beta_1, \beta_2, \ldots, \beta_n)'$ is the vector parameter of (unknown) consecutive distances among them. Suppose we want to estimate $\underline{\theta}$ (as a whole or some functions of it) by undertaking N measuring operations in each of which we can measure the distance between any two objects along the line. Clearly, this problem fits into the framework of spring balance weighing designs with a string property. Under this set-up, we intend to discuss here various optimality results. The special case of $N = n$ is extensively studied in Sections 2 and 3. The case of $N > n$ seems to be complicated and is still under investigation.

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*(Vide also Banerjee (1975)).

Assume that the recorded observations follow the standard regression model:

$$\underline{Y}(N\times 1) = X(N\times n)\underline{\theta}(n\times 1) + \underline{\epsilon}(N\times 1),\ E(\underline{\epsilon}) = \underline{0},\ E(\underline{\epsilon}\underline{\epsilon}') = \sigma^2 I_N .$$

Here $X(N\times n)$ is a (0,1) design matrix with the string property. In Section 2, we suggest, for the particular case of $N = n$, Φ_p-optimal designs for the two familiar problems of inferring on $\underline{\theta}$ (as a whole) and $\underline{\eta} = \Gamma\,\underline{\theta}$ where $\Gamma(\overline{n-1}\times n)$ is the lower submatrix of an orthogonal matrix with the first row vector as $(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n})$. Incidentally, some interesting features of the (0,1) matrices with the string property have been observed and made use of in deducing the optimality results. Section 3 deals with other aspects of inference and derivation of corresponding optimal designs, again under the special case of $N = n$. Some concluding remarks regarding the case of $N > n$ are given in Section 4.

2. OPTIMAL DESIGNS

Throughout, our investigation relates to the case of $N = n$, i.e., to square matrix $X(n\times n)$. We refer to Kiefer (1975) for the notions of various optimality criteria. Our problem is to infer about $\underline{\theta}$ and $\underline{\eta} = \Gamma\,\underline{\theta}$ where $\bar{O}(n\times n) = \binom{\underline{1}'/\sqrt{n}}{\Gamma}$ is orthogonal. Here $\underline{1} = (1\,1\ldots 1)'$ of appropriate order. An appeal to the universal optimality criterion for full rank models (Kiefer (1975), Sinha and Mukerjee (1982)) would settle the problem of inferring on $\underline{\theta}$ provided $X_o = I_n$ (the identity matrix) would maximize $tr(X'X)$ among all design matrices X's in such a setting. It is known, however, that as a matter of fact, the situation is completely reverse in such a set-up. Even then, it will follow that X_o is Φ_p-optimal (vide Kiefer (1975)) for all $p \ge o$ - it being <u>unique</u> whenever $p > o$ ($p = o$ corresponds to the D-optimality criterion). As regards inference on $\underline{\eta}$, the result is yet stronger e.g., X_o is <u>uniquely</u> Φ_p-optimal for <u>all</u> $p \ge o$.

We now proceed in a systematic manner to develop the tools for establishing the above mentioned results. The key papers are Kiefer (1958, 1975). We observe the following:

<u>Lemma 2.1</u> If and only if rank $(X(n \times n)) = n$, η is estimable.

<u>Proof</u>. Easy and hence omitted.

<u>Lemma 2.2</u> Det.$(X) = \pm 1$ or 0 according as X is of full rank or not.

<u>Proof</u>. See Appendix.

<u>Lemma 2.3</u> $X_o(= I_n)$ is D-optimal (for inferring on θ) and, moreover, uniquely Φ_p-optimal for all $p > o$.

<u>Proof</u>. The proof is easy. It is based on the fact stated in Lemma 2.2 above and the results pertaining to D-optimality and Φ_p-optimality as discussed in Kiefer (1958, 1975). We omit the details.

<u>Lemma 2.4</u> For a square full rank (0,1) matrix X with the string property, the inverse matrix X^{-1} exhibits the following structure:

(i) The elements in X^{-1} are essentially $(0, \pm 1)$ and, in each column, the non-zero elements occur with alternate signs.

(ii) The column-totals (through all or any subset of consecutive rows) are essentially $(0, \pm 1)$ - not all being 0's .

(iii) If a certain column-total is +1(-1), the first non-zero entry in that column is +1(-1); if a certain column-total is 0, the first non-zero entry in that column is either +1 or -1 .

<u>Proof</u>. See Appendix.

<u>Lemma 2.5</u> Whatever the design matrix X(of full rank), $\underline{1}'(X'X)^{-1}\underline{1} \leq n$ with "=" for $X = X_o$ (and, possibly, for other matrices as well).

<u>Proof</u>. Use of Lemma 2.4 readily settles this.

<u>Theorem 2.1</u> X_o is <u>uniquely</u> D-optimal for inferring on $\underline{\eta} = \Gamma \underline{\theta}$.

<u>Proof</u>. Clearly, $D(\hat{\eta}) = \sigma^2\{\Gamma(X'X)^{-1}\Gamma'\}$ so that it is a question of minimizing $|\Gamma(X'X)^{-1}\Gamma'|$. Now, in view of Lemma 2.2, we deduce $1 = |\bar{O}(X'X)^{-1}\bar{O}'| \leq \{\underline{1}'(X'X)^{-1}\underline{1}/n\}\{|\Gamma(X'X)^{-1}\Gamma'|\}$ $\leq \{|\Gamma(X'X)^{-1}\Gamma'|\}$ (by Lemma 2.5) so that $|D(\hat{\eta})|]_X \geq |D(\hat{\eta})|]_{X_o}$ and "=" holds iff (i) $\Gamma'(X'X)^{-1}\underline{1} = \underline{0}$ and (ii) $\underline{1}'(X'X)^{-1}\underline{1} = n$ hold simultaneously. However, (i) $\Rightarrow (X'X)^{-1}\underline{1} \propto \underline{1}$ i.e., $(X'X)\underline{1} \propto \underline{1}$. Set now $(X'X)\underline{1} = k\underline{1}$ for some k(which is necessarily a positive integer). This yields $X\underline{1} = k(X')^{-1}\underline{1}$ which is impossible unless $(X')^{-1}\underline{1} = \underline{1}$ (in view of Lemma 2.4). This means $X\underline{1} = k\underline{1}$ for some positive integer k i.e., X has constant row-sums. Certainly, now, an $n \times n$ matrix with the string property can have at the most $(n - k + 1)$ linearly independent rows with each row-sum equal to k. Hence $k = 1$ and, further, because of full-rank, $X = X_o$ (up to a row permutation). This settles the claim.

Using the notions of Φ_p-optimality criteria we readily have the following

<u>Corollary 2.1</u> X_o is <u>uniquely</u> Φ_p-optimal for all $p \geq o$.

<u>Proof</u>. It is enough to note that $D(\hat{\eta})]_{X_o}$ is a multiple of the identity. Theorem 2.1 now justifies the statement.

3. FURTHER INFERENTIAL ASPECTS

Suppose we are interested in estimating parameters of the form $\theta(i,j) = \sum_i^j \beta_r$, $1 \le i \le j \le n$, retaining estimability of the individual lengths. In other words, this means that our interest is to estimate the length(s) between set(s) of pairs of points. Of particular interest is the problem of estimating the total length (case of $i = 1$, $j = n$). (*Vide* Banejee (1975), Sinha (1971, 1972), Panda (1976) for various aspects of this problem in the framework of spring balance weighing designs)). The results in Section 2 are useful to settle this problem completely. We again take $N = n$ here.

To start with, let us fix (i,j), $1 \le i \le j \le n$. We write $\theta(i,j) = \underline{\gamma}_{ij}' \underline{\theta}$ where $\underline{\gamma}_{ij} = (0\ 0 \ldots 1\ 1 \ldots 1\ 0 \ldots 0)'$ with 1's in the positions i through j . (Note that $\underline{\gamma}_{ii}$ is well-defined for every i). We have $V(\hat{\theta}(i,j)) = \sigma^2\ \underline{\gamma}_{ij}'\ (X'X)^{-1}\ \underline{\gamma}_{ij} \geq \sigma^2$ in view of the property (ii) of X^{-1} exhibited in Lemma 2.4. Here "=" holds iff $\underline{\gamma}_{ij}' = \underline{\gamma}_{hh}' X$ for some combination of X and h . It is easy to verify that given any (i,j) - we can have a choice of X for every h. (It is enough to choose the hth row of X as $\underline{\gamma}_{ij}'$). This settles the question for any single parametric function whatsoever. Likewise, minimum possible variance (σ^2) is attainable for each component of a simultaneous inference problem on $\{\underline{\gamma}_{i_1,j_1}, \underline{\gamma}_{i_2,j_2}, \ldots ,\underline{\gamma}_{i_k,j_k}\}$ provided they are linearly independent (since individual estimability of the β_i's is to be ensured using just n measuring operations). These results are highly interesting and are peculiar too to this set-up! (It may be recalled that the <u>best</u> unbiased spring balance weighing design for estimating the <u>total</u> weight of a set of $n(\geq 3)$ objects in exactly n weighing operations is provided by

$X_{oo} = \underline{\gamma}_{1n}\underline{\gamma}_{1n}' - I_n$ and the minimum variance is given by $n\sigma^2/(n-1)^2$ (Sinha (1971, 1972)). This quantity is smaller than σ^2 and X_{oo} does not enjoy the string property. We refer to Sinha (1971, 1972), Panda (1976) and Swamy (1980) for further aspects of these problems under a restricted set-up. Analogous results are derivable in the present framework as well).

4. CONCLUDING REMARKS

(a) Suppose we want a D-optimum design for estimating $\underline{\theta}$ using N measuring operations. This time we take $N > n$. Let $X(N \times n)$ be a full rank (0,1) design matrix with the string property. Denote by $\mathbf{C}$ the class of all such matrices for given $N > n \geq 3$. The algebraic problem is to maximize $|X'X|$ over $X \in \mathbf{C}$. (The same problem under the usual chemical balance set-up has been extensively studied in the literature and the latest available results are in Galil and Kiefer (1980)).

Expanding $|X'X|$ into a sum of $\binom{N}{n}$ squares of $n \times n$ determinants (Cauchy-Binet expansion) and using the Lemma 2.2, one achieves $|X'X| \leq \binom{N}{n} \; \forall \; X \in \mathbf{C}$. However, this bound is attained only for $N = n+1$. The design matrix $X^* = \begin{bmatrix} I_n \\ \hline 1\,1 \ldots 1 \end{bmatrix}$ attains this bound but this is not unique. For example,

$$X^{**} = \begin{bmatrix} 1 & 1 & 0 & 0 & . & . & 0 & 0 \\ 0 & 1 & 1 & 0 & . & . & 0 & 0 \\ \\ 0 & 0 & 0 & 0 & . & . & 1 & 1 \\ 1 & 1 & 1 & 1 & . & . & 1 & 0 \\ 0 & 1 & 1 & 1 & . & . & 1 & 1 \end{bmatrix}$$

also attains the bound.

(b) In general, the problem reduces to getting a design matrix $X^*(N \times n)$ (inside the class $\mathbf{C}$) which possesses maximum number of non-singular square $(n \times n)$ submatrices. The problem seems to be difficult. In the particular case of $N=5$, $n=3$, we made an exhaustive computer search and observed that there are essentially four different matrices X^* which yield the same result e.g., $|X^{*\prime}X^*| = 8 < 10 = \binom{5}{3}$. The matrices are shown below:

$$X_1^* = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ \hline 1&1&0 \\ 0&1&1 \end{bmatrix}, \quad X_2^* = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ \hline 1&1&0 \\ 0&1&1 \\ 1&1&1 \end{bmatrix}, \quad X_3^* = \begin{bmatrix} 1&0&0 \\ 0&0&1 \\ \hline 1&1&0 \\ 0&1&1 \\ 1&1&1 \end{bmatrix}, \quad X_4^* = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ \hline 1&1&0 \\ 0&1&1 \\ 1&1&1 \end{bmatrix}$$

This leads us to the <u>belief</u> that in general, the optimum design X^* might include I_n, the identity matrix, as a submatrix. We are unable to prove this right now. However, for $N=n+2$, taking the form of X^* as $X^* = (I_n|\underline{\alpha}|\underline{\beta})'$, we deduced the 'optimal' forms of $\underline{\alpha}$ and $\underline{\beta}$ (both having string structure) which maximize $|X^{*\prime}X^*|$. The result is given below:

Value of n	**Structures of $\underline{\alpha}$ and $\underline{\beta}$ for 'optimal' X^*** (1 2 ...m m+1 m+2 ... 2m 2m+1 2m+2 ... 3m 3m+1 3m+2)	**Value of** $\lvert X^{*\prime}X^*\rvert$
$n=3m$	$\underline{\alpha}'$ = (1 1 .... 1 0 0 ...0) $\underline{\beta}'$ = (0 0 ...0 1 1 ... .1)	$\frac{(n+1)(n+3)}{3}$
$n=3m+1$	$\underline{\alpha}'$ = (1 1 .... 1 1 0 ... 0) $\underline{\beta}'$ = (0 0 ...0 1 1 ... . 1)	$\frac{(n+2)^2}{3}$
$n=3m+2$	$\underline{\alpha}'$ = (1 1 .... 1 1 0 .. 0) $\underline{\beta}'$ = (0 0 .... 0 1 ... . 1)	$\frac{(n+1)(n+3)}{3}$

(c) The above 'optimal' structure of X^* may now be generalized for values of $N > n+2$ but $N \leq 2n$. We simply maintain the 'staircase' pattern suggested above for $N = n+2$. We present the following table of 'optimal' values of $|X^{*\prime}X^*|$ for X^* obeying the 'staircase' pattern. We consider only $N \leq 2n$.

N \ n	3	4	5
3	1	-	-
4	4	1	-
5	8	5	1
6	8	12	6
7	-	21	16
8	-	16	33
9	-	-	55
10	-	-	32

As for example, consider N=9, n=5. Our X^* is

$$X^* = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

(d) The general case is still under investigation.

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APPENDIX

1. Proof of Lemma 2.2

Let X be a (0,1) square non-singular matrix with the stated string property. By elementary operations (of row permutation and row differencing), one can reduce it to an identity matrix. Hence the result.

2. Proof of Lemma 2.4

The proof is by induction on n, the order of the square matrix X. The result is easily verified for $n = 1,2$. Assume the properties to hold good for all such matrices of order $\leq n$ and let X_{n+1} be a full-rank (0,1) square matrix of order $(n+1)$ with the stated string property. Since the properties (claimed) relate to the columns of X_{n+1}^{-1}, we may conveniently convert X_{n+1} to X_{n+1}^{*} (by necessary row permutations):

$$X_{n+1}^{*} = \left[\begin{array}{c|ccc|c} 1 & 1 & 1 & \cdots & . \\ \hline 1 & & & & \vdots \\ \vdots & & \cdots & & . \\ \vdots & & & & 1 \\ \hline . & . & \cdots & 1 & 1 \end{array}\right] \qquad (1)$$

Here the first (last) row of X_{n+1}^{*} has the smallest string-length among all rows starting (ending) with 1 in the first (last) position and these rows are arranged in increasing order of the string lengths from the top (bottom). Non-singularity of X_{n+1} guarantees existence of these two rows in X_{n+1}^{*} as distinct from one another. We will now deduce all the structural properties of X_{n+1}^{*} by induction argument. If the first column of X_{n+1}^{*} contains $(1+t)$ 1's, we pre-multiply X_{n+1}^{*} by

$$P\ (\overline{n+1} \times \overline{n+1}) = \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{1 \quad 0 \quad \cdots \quad 0}^{t+1}} & \\ -1 & 1 & 0 & \cdots 0 & \\ -1 & & 1 & \cdot & 0 \\ \vdots & & & \cdot & \\ -1 & & \cdots & 1 & \\ \hline \multicolumn{4}{c|}{0} & I_{n-t} \end{array}\right]$$

and convert X^{*}_{n+1} to one with the first column-vector as $(1\ 0\ 0\ \ldots\ 0)'$. This yields

$$X^{*-1}_{n+1} = \left[\begin{array}{c|c} 1 & 11 \ldots \quad \ldots\ 00 \\ \hline 0 & \\ 0 & \\ \vdots & A_n \\ 0 & \end{array}\right]^{-1} P\ . \tag{2}$$

Certainly, A_n is a (0,1) matrix with the string property and, hence, all the structural results apply to A_n^{-1}. Further, the effect of P would be to change only the first column-vector of the matrix on which it acts. Let $(1|\underline{\beta}')$ be the first-row vector of X^{*}_{n+1}. Then X^{*-1}_{n+1} has the representation

$$X^{*-1}_{n+1} = \left[\begin{array}{c|c} 1 & -\underline{\beta}'\ A_n^{-1} \\ \hline \underline{0} & A_n^{-1} \end{array}\right] P\ . \tag{3}$$

This will lead to the structural results for <u>all</u> the columns of the above matrix <u>except</u> for the first column as the following arguments will demonstrate. Suppose $\underline{\beta}$ has unity in the first s positions $(s \geq 0)$. Then $\underline{\beta}'\ A_n^{-1}$ is a vector comprising of the <u>first</u> s row-sums of A_n^{-1}. Consider a specific column in A_n^{-1}. If the column-total is 1(-1), then the partial sums from the top are 1 or 0(-1 or 0) and hence it will contribute -1 or 0(1 or 0) to the vector $-\underline{\beta}'A_n^{-1}$ and the result will be in accordance with the structure. The same is true for other columns having the totals in A_n^{-1} as zero. Finally, to deduce the structural results for the first column of X_{n+1}^{*-1}, we refer to (1) and this time pre-multiply X_{n+1}^{*} by a matrix $Q(\overline{n+1}\times\overline{n+1})$ of the form

$$Q = \left[\begin{array}{c|ccccc} I & & & 0 & & \\ \hline 0 & 1 & 0 & \cdots & & -1 \\ & & 1 & 0 & & -1 \\ & & 0 & \ddots & & \vdots \\ & & & & 1 & -1 \\ & & & & & 1 \end{array}\right]$$

so that X_{n+1}^{*} changes to

$$QX_{n+1}^{*} = \left[\begin{array}{c|c} & 0 \\ & 0 \\ B_n & 0 \\ & \vdots \\ & 0 \\ \hline 0\ 0\ 0 \ldots 1 1 1 & 1 \end{array}\right] \qquad (4)$$

where B_n is again a (0,1) matrix of order n with the string property. Let $(\underline{y}'|1)$ be the last row-vector in the representation of QX_{n+1}^{*} in (4).

Then

$$X_{n+1}^{*-1} = \left[\begin{array}{c|c} B_n^{-1} & \underline{0} \\ \hline -\underline{y}'B_n^{-1} & 1 \end{array}\right] Q \qquad (5)$$

and, hence, by induction argument, we are through in regard to <u>all</u> the elements in the first column of X_{n+1}^{*-1} <u>except</u> for the last element (in the (n+1,1)th position). If $\underline{y}$ contains 1's in the last r positions ($r \geq 0$), then $\underline{y}'B_n^{-1}$ is a vector consisting of the last r row-sums in B_n^{-1}. Now we argue as follows as regards the first column of B_n^{-1}. If this column total is 1(-1), then the partial totals from the bottom are 1 or 0 (-1 or 0) and hence the last element in the first column of X_{n+1}^{*-1} is -1 or 0 (1 or 0) and this is in accordance with the structure indicated. Similar considerations apply if this column total is zero. Hence the claim.

Remarks

1. It would be interesting to check if a non-singular matrix with the properties described in the statement of the theorem has for its inverse a (0,1) matrix with a string property. It turns out that the inverse matrix does not necessarily have the elements as (0,1) alone. Here is an example.

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

However, we strongly feel that the following conjecture should prove valid:

If a $(0,\pm1)$ non-singular matrix enjoys the properties described in the statement of the theorem and if its inverse is again a matrix with elements $(0,\pm1)$, then the non-zero elements in any row of the inverse matrix are of the same sign (either all 1's or all -1's) and there is a permutation of the columns of the inverse matrix leading to the string property (in the sense that every row vector will contain exactly one run of 1's or -1's).

2. It may be noted that the (0,1) matrices with the string property constitute special types of "totally unimodular" matrices. Various properties of unimodular matrices have been discussed and made use of in "Integer Programming" problems. (Vide, for example, Garfinkel and Nemhauser (1972)).