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TESTING FOR THE SAME ORDERING IN SEVERAL GROUPS OF MEANS

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I. INTRODUCTION

In many situations an experimenter is interested in questions regarding the ordering of some parameters. Interest may focus on the largest or smallest parameter. This interest is due, at least in part, to the fact that orderings often determine which populations are "best" or "worst" in some sense.

Over the past thirty years, many statistical procedures have been developed to answer questions regarding the ordering of parameters. Some of the early work of Bechhofer [2] dealt with completely ranking parameters and finding the largest parameter. These and other methods of selecting the largest or smallest parameters and ranking parameters are discussed in Gupta and Panchapakesan [7] which also includes a fairly complete bibliography.

Much of the literature regarding testing hypotheses about ordered parameters falls into the "isotonic regression" literature. An early book involving testing hypotheses about ordered parameters is that of Barlow, Bartholomew, Bremner, and Brunk [1]. Later papers regarding hypothesis testing include Robertson and Wegman [9] and Dykstra and Robertson [5].

In this paper we consider a hypothesis testing problem regarding ordered parameters which does not fall in the isotonic regression area. We consider a problem in which there are two groups of normal means, $\mu_{11},\ldots,\mu_{1K}$, and $\mu_{21},\ldots,\mu_{2K}$. The question we consider is, "Are these two groups in the same order?". Specifically we consider testing

$$H_0: (\mu_{1i} - \mu_{1j})(\mu_{2i} - \mu_{2j}) \leq 0 \quad \text{for some } 1 \leq i < j \leq K$$

versus

$$H_1: (\mu_{1i} - \mu_{1j})(\mu_{2i} - \mu_{2j}) > 0 \quad \text{for every } 1 \leq i < j \leq K .$$

The alternative states that the two groups of means are in exactly the same order. The multiplicative way of writing H_0 and H_1 is convenient notationally. But note that it might be more explanatory to write H_1 as this: There exists a permutation $(\pi(1),\ldots,\pi(K))$ of $(1,\ldots,K)$ such that $\mu_{1\pi(1)} > \cdots > \mu_{1\pi(K)}$ and $\mu_{2\pi(1)} > \cdots > \mu_{2\pi(K)}$. This choice of null and alternative hypotheses is appropriate if one wishes to demonstrate that the two groups are in the same order and is going to follow the usual practice of controlling the Type I error probability at a small value.

An example in which the experimenter might wish to test H_0 versus H_1 is the following. An agricultural researcher might be interested in K varieties of a crop which are to be grown in two dissimilar regions. The researcher might hypothesize that despite the dissimilarities, the rankings of the mean yields for the K varieties will be the same in the two regions. That is, the same variety will have the highest mean yield, the same variety will have the second highest mean yield, etc. To confirm his hypothesis, he might test H_0 versus H_1.

In Section 2, the assumptions of our model are presented. The size α likelihood ratio test (LRT) of H_0 versus H_1 is derived in Section 3. The LRT's for two related problems, testing whether several (more than two) groups of means are in the same order and testing whether a group of means is in a specified order, are presented in Section 4.

II. NOTATION AND ASSUMPTIONS

We will consider testing H_0 versus H_1 based on the following sample. Let X_{rst}; $r = 1,2$; $s = 1,\ldots,K$; $t = 1,\ldots,n_{rs}$; be independent normal observations. The mean and variance of X_{rst} are μ_{rs} and σ^2, respectively. Let $\bar{X}_{rs} = \sum_{t=1}^{n_{rs}} X_{rst}/n_{rs}$ be the sample mean from the (r,s) population. Let $SS_X = \sum_r\sum_s\sum_{t=1}^{n_{rs}} (X_{rst} - \bar{X}_{rs})^2$ (we will use the notation $\sum_r$ for $\sum_{r=1}^{2}$ and $\sum_s$ for $\sum_{s=1}^{K}$ throughout). Let $N = \sum_r\sum_s n_{rs}$ denote the total sample size. Then $S^2 = SS_X/(N-2K)$ is the usual pooled unbiased estimate of σ^2 and S^2 is independent of the sample means.

For every $r = 1,2$ and $1 \leq i < j \leq K$, define the statistic

$$T_{ij}^r = \frac{\bar{X}_{ri} - \bar{X}_{rj}}{S(1/n_{ri} + 1/n_{rj})^{1/2}} \quad . \tag{2.1}$$

This is the usual t-statistic for testing hypotheses like H_0^*: $\mu_{ri} \leq \mu_{rj}$. If T_{ij}^1 and T_{ij}^2 are both large positive numbers, this might be taken as evidence that $\mu_{1i} > \mu_{1j}$ and $\mu_{2i} > \mu_{2j}$. Similarly if T_{ij}^1 and T_{ij}^2 are both small negative numbers this might be taken as evidence that $\mu_{1i} < \mu_{1j}$ and $\mu_{2i} < \mu_{2j}$. It will be seen in Section 3 that the LRT of H_0 versus H_1 is based on this reasoning.

Consider the hypothesis H_0^{ij}: $(\mu_{1i} - \mu_{1j})(\mu_{2i} - \mu_{2j}) \leq 0$. This hypothesis states that the ith and jth means are not in the same order in the two sets. We shall use the convention that H_0, H_1, and H_0^{ij} refer to both the statements about the parameters and the subsets of the parameter space defined by the statements. Using this convention we can write that

$$H_0 = \bigcup_{1\leq i<j\leq K} H_0^{ij} \quad . \tag{2.2}$$

Hypotheses such as H_0 which can be conveniently expressed as unions have been considered in Gleser [6], Berger [3], and Berger and Sinclair [4].

The LRT statistic for H_0 is defined as

$$\lambda(\underset{\sim}{x}) = \frac{\sup_{H} L(\underset{\sim}{x};\underset{\sim}{\mu},\sigma)}{\sup_{H_0} L(\underset{\sim}{x};\underset{\sim}{\mu},\sigma)} \tag{2.3}$$

where $\underset{\sim}{x}$ is the sample of N observations, $\underset{\sim}{\mu}$ is the vector of 2K population means, L is the likelihood function from the normal model and $H = H_0 \cup H_1$. A LRT of H_0 versus H_1 is any test which rejects H_0 for large values of λ, that is, any test with a rejection region of the form $\{\underset{\sim}{x}: \lambda(\underset{\sim}{x}) \geq c\}$ where $c \geq 1$ is a specified constant.

The LRT statistic for H_0^{ij} is defined as

$$\lambda^{ij}(\underset{\sim}{x}) = \frac{\sup_{H} L(\underset{\sim}{x};\underset{\sim}{\mu},\sigma)}{\sup_{H_0^{ij}} L(\underset{\sim}{x};\underset{\sim}{\mu},\sigma)} \; . \tag{2.4}$$

Because of (2.2), it clear that $\lambda(\underset{\sim}{x}) = \min_{1\leq i<j\leq K} \lambda^{ij}(\underset{\sim}{x})$. Thus a LRT of H_0 versus H_1 is any test which rejects H_0 if

$$\min_{1\leq i<j\leq K} \lambda^{ij}(\underset{\sim}{x}) \geq c \tag{2.5}$$

where $c \geq 1$ is a specified constant. In the next section, (2.5) will be used to derive a simple form for the LRT and determine critical values so that the LRT has a specified size.

III. DERIVATION OF THE LRT

In this section we prove the following two theorems which describe the LRT for H_0 versus H_1.

THEOREM 3.1. *The test which rejects* H_0 *in favor of* H_1 *iff, for every* $1 \leq i < j \leq K$, *either* $\min(T^1_{ij}, T^2_{ij}) \geq t$ *or* $\max(T^1_{ij}, T^2_{ij}) \leq -t$ *where* t *is a positive constant is a* LRT *of* H_0 *versus* H_1.

THEOREM 3.2. *Let* $0 \leq \alpha < .5$. *If in Theorem* 3.1, $t = t_{\alpha,N-2K}$, *the upper* αth *percentile from a Student's* t *distribution with* N-2K *degress of freedom, then the resulting* LRT *is a size* α *test.*

LEMMA 3.1. *For any* $c > 1$ *and* $1 \leq i < j \leq K$, $\{\underset{\sim}{x}: \lambda^{ij}(\underset{\sim}{x}) \geq c\}$ *is the same set as* $\{\underset{\sim}{x}: \min(T^1_{ij}, T^2_{ij}) \geq t \text{ or } \max(T^1_{ij}, T^2_{ij}) \leq -t\}$ *where* $t = ((c^{2/N} - 1)(N - 2K))^{1/2} > 0$.

Proof. By standard analysis for LRT statistics for normal models, $\lambda^{ij}(\underset{\sim}{x}) = (\lambda^{ij}_*(\underset{\sim}{x}))^{N/2}$ where

$$\lambda^{ij}_*(\underset{\sim}{x}) = \frac{\inf_{H_0^{ij}} \sum_r \sum_s \sum_{t=1}^{n_{rs}} (x_{rst} - \mu_{rs})^2}{SS_x}$$

$$= 1 + \frac{\inf_{H_0^{ij}} \sum_r \sum_s n_{rs} (\bar{x}_{rs} - \mu_{rs})^2}{SS_x}$$

$$= 1 + \frac{\inf_{H_0^{ij}} \sum_r [n_{ri}(\bar{x}_{ri} - \mu_{ri})^2 + n_{rj}(\bar{x}_{rj} - \mu_{rj})^2]}{SS_x} \qquad (3.1)$$

Equality (3.1) is true since all the $n_{rs}(\bar{x}_{rs} - \mu_{rs})^2$ terms, except for $s = i,j$, can be set to zero by choosing $\mu_{rs} = \bar{x}_{rs}$. This choice is permissible since the μ_{rs}, $s \neq i,j$, are unconstrained by H_0^{ij}.

If $(\bar{x}_{1i} - \bar{x}_{1j})(\bar{x}_{2i} - \bar{x}_{2j}) \leq 0$, $\lambda^{ij}(\underset{\sim}{x}) = \lambda_*^{ij}(\underset{\sim}{x}) = 1$ because the infimum in (3.1) is zero. This infimum is attained at the parameter point with $\mu_{rs} = \bar{x}_{rs}$ $r = 1,2$, and $s = i,j$, a parameter point in H_0^{ij}. So a sample point with $(\bar{x}_{1i} - \bar{x}_{1j})(\bar{x}_{2i} - \bar{x}_{2j}) \leq 0$ is in neither of the sets specified in the Lemma.

Consider calculation of the infimum in (3.1) if $(\bar{x}_{1i} - \bar{x}_{1j})(\bar{x}_{2i} - \bar{x}_{2j}) > 0$. First suppose $\bar{x}_{1i} < \bar{x}_{1j}$ and $\bar{x}_{2i} < \bar{x}_{2j}$. Consider calculating the infimum in (3.1) on the set of μ's such that $\mu_{1i} \leq \mu_{1j}$ and $\mu_{2i} \geq \mu_{2j}$. This infimum is clearly attained at a point with $\mu_{1i} = \bar{x}_{1i}$ and $\mu_{1j} = \bar{x}_{1j}$. Thus the infimum is just

$$(3.2) \qquad \inf_{\mu_{2i} \geq \mu_{2j}} \; n_{2i}(\bar{x}_{2i} - \mu_{2i})^2 + n_{2j}(\bar{x}_{2j} - \mu_{2j})^2 \; .$$

Using the fact that $\bar{x}_{2i} < \bar{x}_{2j}$, it is easily verified that (3.2) is attained when $\mu_{2i} = \mu_{2j} = (n_{2i}\bar{x}_{2i} + n_{2j}\bar{x}_{2j})/(n_{2i} + n_{2j})$. Thus (3.2) is equal to $(\bar{x}_{2i} - \bar{x}_{2j})^2/(1/n_{2i} + 1/n_{2j})$. Now consider calculating the infimum in (3.1) on the set of $\underset{\sim}{\mu}$'s such that $\mu_{1i} \geq \mu_{1j}$ and $\mu_{2i} \leq \mu_{2j}$. The same reasoning as above yields that this infimum is $(\bar{x}_{1i} - \bar{x}_{1j})^2/(1/n_{1i} + 1/n_{1j})$. Since H_0^{ij} is the union of these two sets of $\underset{\sim}{\mu}$'s, the infimum in (3.1) is the minimum of these two infima. Thus we have that if $\bar{x}_{1i} < \bar{x}_{1j}$ and $\bar{x}_{2i} < \bar{x}_{2j}$,

$$(3.3) \qquad \lambda_*^{ij}(\underset{\sim}{x}) = 1 + \min_{r=1,2} \frac{(\bar{x}_{ri} - \bar{x}_{rj})^2}{SS_x(1/n_{ri} + 1/n_{rj})}$$

The calculation of $\lambda_*^{ij}(\underset{\sim}{x})$ for a sample point with $\bar{x}_{1i} > \bar{x}_{1j}$ and $\bar{x}_{2i} > \bar{x}_{2j}$ yields the same value of $\lambda_*^{ij}(\underset{\sim}{x})$ as in (3.3).

The condition that $(\bar{x}_{1i} - \bar{x}_{1j})(\bar{x}_{2i} - \bar{x}_{2j}) > 0$ is equivalent to the condition that T_{ij}^1 and T_{ij}^2 have the same sign. Thus the following inequalities are equivalent:

$$\lambda^{ij}(\underset{\sim}{x}) \geq c$$

$$\lambda_*^{ij}(\underset{\sim}{x}) \geq c^{2/N}$$

$$\min((T_{ij}^1)^2,(T_{ij}^2)^2) \geq (c^{2/N} - 1)(N - 2K) \text{ and } (\bar{x}_{1i} - \bar{x}_{1j})(\bar{x}_{2i} - \bar{x}_{2j}) > 0$$

$$\min(T_{ij}^1,T_{ij}^2) \geq t \quad \text{or} \quad \max(T_{ij}^1,T_{ij}^2) \leq -t \quad .$$

This proves Lemma 3.1.||

Proof of Theorem 3.1. Theorem 3.1 follows from Lemma 3.1, representation (2.5), and noting that any $t > 0$ corresponds to a $c > 1$ in (2.5).||

Theorem 3.2 describes how the critical value t should be chosen to yield a size α LRT. Lemma 3.2 will be used in the proof of Theorem 3.2.

LEMMA 3.2. *Let* $0 \leq \alpha < .5$, $1 \leq i < j \leq K$, *and* $\nu = N - 2K$. *Then*

$$(3.4) \quad \sup_{H_0^{ij}} P_{\underset{\sim}{\mu},\sigma}(\min(T_{ij}^1,T_{ij}^2) \geq t_{\alpha,\nu} \quad \text{or} \quad \max(T_{ij}^1,T_{ij}^2) \leq -t_{\alpha,\nu}) \leq \alpha \ .$$

Proof. Let $F(s;\sigma)$ denote the distribution function of S. Let $Y_r = (\bar{X}_{ri} - \bar{X}_{rj})/(1/n_{ri} + 1/n_{rj})^{\frac{1}{2}}$, $r = 1,2$. Then $T_{ij}^r = Y_r/S$. Furthermore Y_1, Y_2, and S are independent and Y_r is normal with mean $\xi_r = (\mu_{ri} - \mu_{rj})/(1/n_{ri} + 1/n_{rj})^{\frac{1}{2}}$ and variance σ^2. The probability in (3.4) is equal to

$$(3.5) \quad \int_0^\infty P_{\underset{\sim}{\xi},\sigma}(\min(Y_1,Y_2) \geq st_{\alpha,\nu} \quad \text{or} \quad \max(Y_1,Y_2) \leq -st_{\alpha,\nu})dF(s;\sigma) \ .$$

If $(\mu,\sigma) \in H_0^{ij}$ then either $\xi_1 \geq 0 \geq \xi_2$ or $\xi_1 \leq 0 \leq \xi_2$. In either case $\underset{\sim}{\xi}^* = (\xi_1 + \xi_2, 0)$ is majorized by $\underset{\sim}{\xi}$. Since the joint density of (Y_1, Y_2) is $g_\sigma(\underset{\sim}{y} - \underset{\sim}{\xi})$ where g_σ is a Schur concave function, and since the set $\{(y_1,y_2)\colon \min(y_1,y_2) \geq st_{\alpha,\nu}$ or $\max(y_1,y_2) \leq -st_{\alpha,\nu}\}$ is a Schur concave set, by Theorem 2.1 of Marshall and Olkin [8], the integrand in (3.5) is less than or equal to

$$P_{\underset{\sim}{\xi}^*,\sigma}(\min(Y_1,Y_2) \geq st_{\alpha,\nu} \text{ or } \max(Y_1,Y_2) \leq -st_{\alpha,\nu}) \tag{3.6}$$

if $(\underset{\sim}{\mu},\sigma) \in H_0^{ij}$. So the Lemma will be proved if we can show that (3.5), with $\underset{\sim}{\xi}$ replaced by $\underset{\sim}{\xi}^*$, is at most α.

$$\int_0^\infty P_{\underset{\sim}{\xi}^*,\sigma}(\min(Y_1,Y_2) \geq st_{\alpha,\nu} \text{ or } \max(Y_1,Y_2) \leq -st_{\alpha,\nu})dF(s;\sigma)$$

$$= \int_0^\infty P_{\underset{\sim}{\xi}^*,\sigma}(Y_1 \geq st_{\alpha,\nu}, Y_2 \geq st_{\alpha,\nu}) + P_{\underset{\sim}{\xi}^*,\sigma}(Y_1 \leq -st_{\alpha,\nu}, Y_2 \leq -st_{\alpha,\nu})dF(s;\sigma). \tag{3.7}$$

The fact that $st_{\alpha,\nu} \geq 0$ was used to obtain the equality. Since Y_1 and Y_2 are independent normal random variables and the mean of Y_2 is zero, (3.7) equals

$$\int_0^\infty P_{\underset{\sim}{\xi}^*,\sigma}(Y_1 \geq st_{\alpha,\nu}, Y_2 \geq st_{\alpha,\nu}) + P_{\underset{\sim}{\xi}^*,\sigma}(Y_1 \leq -st_{\alpha,\nu}, Y_2 \geq st_{\alpha,\nu})dF(s;\sigma)$$

$$\leq \int_0^\infty P_{\underset{\sim}{\xi}^*,\sigma}(Y_2 \geq st_{\alpha,\nu})dF(s;\sigma)$$

$$= P_{\underset{\sim}{\xi}^*,\sigma}(Y_2/S \geq t_{\alpha,\nu}) = \alpha.$$

The fact that $st_{\alpha,\nu} \geq 0$ was used to obtain the inequality. At $\underset{\sim}{\xi}^*$, Y_2/S has a central t distribution since the mean of Y_2 is zero. This yields the last equality. Thus Lemma 3.2 is proved.||

Proof of Theorem 3.2. First we will show that for any $(\underset{\sim}{\mu},\sigma) \in H_0$, the probability of rejecting H_0 is at most α. Fix $(\underset{\sim}{\mu},\sigma) \in H_0$. By (2.2), there is an i^* and j^* with $1 \le i^* < j^* \le K$, such that $(\underset{\sim}{\mu},\sigma) \in H_0^{i^*j^*}$. Then

$$\begin{aligned} &P_{\underset{\sim}{\mu},\sigma}(\text{reject } H_0) \\ &= P_{\underset{\sim}{\mu},\sigma}(\min(T^1_{ij},T^2_{ij}) \ge t_{\alpha,\nu} \text{ or } \max(T^1_{ij},T^2_{ij}) \le -t_{\alpha,\nu} \text{ for every } 1 \le i < j \le K) \\ &\le P_{\underset{\sim}{\mu},\sigma}(\min(T^1_{i^*j^*},T^2_{i^*j^*}) \ge t_{\alpha,\nu} \text{ or } \max(T^1_{i^*j^*},T^2_{i^*j^*}) \le -t_{\alpha,\nu}) \\ &\le \alpha, \end{aligned}$$

the last inequality being true by Lemma 3.2 since $(\underset{\sim}{\mu},\sigma) \in H_0^{i^*j^*}$.

Now we will show that there exists a sequence of parameter points in H_0 such that the probabilities of rejecting H_0 approach a limit which is at least α. This will complete the proof of Theorem 3.1. Let $\underset{\sim}{\mu}_n$ be a sequence of mean vectors (with elements μ_{rsn}) such that $\mu_{11n} = \mu_{12n}$ for all n, $\mu_{1sn} - \mu_{1(s+1)n} \to \infty$ as $n \to \infty$ for $s = 2,\ldots,K-1$, and $\mu_{2sn} - \mu_{2(s+1)n} \to \infty$ as $n \to \infty$ for $s = 1,\ldots,K-1$. This implies that for $r = 1,2$, and $1 \le i < j \le K$ (except if $r = 1$, $i = 1$, and $j = 2$) $\mu_{rin} - \mu_{rjn} \to \infty$ as $n \to \infty$. All such $\underset{\sim}{\mu}_n$ are in H_0 since $\mu_{11n} - \mu_{12n} = 0$. Fix $\sigma > 0$ and let $\nu = N-2K$. Then (all limits are as $n \to \infty$)

$$\begin{aligned} &\lim P_{\underset{\sim}{\mu}_n,\sigma}(\text{reject } H_0) \\ &\ge \lim P_{\underset{\sim}{\mu}_n,\sigma}(\min(T^1_{ij},T^2_{ij}) \ge t_{\alpha,\nu}\ ,\quad 1 \le i < j \le K) \\ &= \lim P_{\underset{\sim}{\mu}_n,\sigma}(T^r_{ij} \ge t_{\alpha,\nu}\ ,\quad r = 1,2,\quad 1 \le i < j \le K) \end{aligned}$$

$$= 1 - \lim P_{\underset{\sim}{\mu}_n,\sigma}\Big(\bigcup_{\substack{r=1,2 \\ 1\le i<j\le K}} \{T_{ij}^r < t_{\alpha,\nu}\}\Big)$$

$$(3.8) \qquad \ge 1 - \lim \sum_{\substack{r=1,2 \\ 1\le i<j\le K}} P_{\underset{\sim}{\mu}_n,\sigma}(T_{ij}^r < t_{\alpha,\nu}) \quad .$$

Since $\mu_{11n} = \mu_{12n}$, $P_{\underset{\sim}{\mu}_n,\sigma}(T_{12}^1 < t_{\alpha,\nu}) = 1-\alpha$ for every n. For any other choice of r, i, and j, since $\mu_{rin} - \mu_{rjn} \to \infty$, $\lim P_{\underset{\sim}{\mu}_n,\sigma}(T_{ij}^r < t_{\alpha,\nu}) = 0$. Thus the expression in (3.7) is α, as was to be shown. ||

IV. RELATED PROBLEMS

In this section we consider two problems which are similar to testing H_0 versus H_1. In each case we state a theorem which gives the LRT for the problem. The analyses are very similar to that in Section 3 and thus the proofs of the theorems are omitted.

4.1 *Testing whether More than Two Groups of Means Are in the Same Order*

This problem is the same as that considered in the previous section except that rather than two groups of means there are $R > 2$ groups of means. We are interested in whether all R groups are in exactly the same order. The means are denoted by μ_{rs}; r=1,...,R; s=1,...,K. We are interested in testing

$$H_0^R: (\mu_{ui} - \mu_{uj})(\mu_{vi} - \mu_{vj}) \leq 0 \quad \text{for at least one choice of } 1 \leq u < v \leq R \text{ and } 1 \leq i < j \leq K$$

versus

$$H_1^R: (\mu_{ui} - \mu_{uj})(\mu_{vi} - \mu_{vj}) > 0 \quad \text{for every } 1 \leq u < v \leq R \text{ and } 1 \leq i < j \leq K .$$

If $T_{ij}^r = (\bar{X}_{ri} - \bar{X}_{rj})/S_R(1/n_{ri} + 1/n_{rj})^{\frac{1}{2}}$, where S_R is the usual, pooled estimate of σ based on all the observations, then we have the following result.

THEOREM 4.1. *Let* $0 \leq \alpha < .5$. *The test which rejects* H_0^R *iff, for every* $1 \leq i < j \leq K$, *either* $\min(T_{ij}^1,\ldots,T_{ij}^R) \geq t_{\alpha,N-RK}$ *or* $\max(T_{ij}^1,\ldots,T_{ij}^R) \leq -t_{\alpha,N-RK}$ *is the size* α LRT *of* H_0^R *versus* H_1^R .

4.2 *Testing whether a Group of Means Is in a Specified Order*

We now consider the problem in which there is only one group of means, $\mu_1,\ldots,\mu_K$, of interest. We are interested in whether these means are in a specified order. Without loss of generality, we can assume the populations have been labeled so that the hypotheses of interest are these:

$$H_0^1\ \mu_{i+1} \le \mu_i \quad \text{for some} \quad i = 1,\ldots,K-1$$

$$H_1^1\ \mu_{i+1} > \mu_i \quad \text{for every} \quad i = 1,\ldots,K-1 \ .$$

If $T_i = (\bar{X}_{i+1} - \bar{X}_i)/S_1(1/n_i + 1/n_j)^{\frac{1}{2}}$, where S_1 is the usual, pooled estimate of σ based on the observations from the K populations, then we have the following result.

THEOREM 4.2. *Let* $0 \le \alpha < .5$. *The test which rejects* H_0^1 *iff* $\min(T_1,\ldots,T_{K-1}) \ge t_{\alpha,N-K}$ is the size α LRT *of* H_0^1 *versus* H_1^1.

Robertson and Wegman [9] consider a class of problems, one of which is testing the null hypothesis H_1^1 versus the alternative H_0^1. Thus they have the roles of the null and alternative reversed from our formulation. (They also have the equality signs in H_1^1 but this is unimportant. The same tests result in both formulations regardless of the location of the equality signs.) Our formulation is more appropriate if one wishes to control the probability of deciding the means are in the specified order when they are not so ordered.

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